Kolmogorov Spectrum in the Glasma

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based on the works with Francois Gelis

Talk Contents

Lecture Part I

Introduction to the Color Glass Condensate and the McLerran-Venugopalan (MV) model

- Kinematics in the high-energy QCD processes
- Gluon saturation and the MV model
- MV model setup for the heavy-ion collisions

Lecture Part II

Introduction to the Glasma and its instability and the resulting spectrum

- Notion of the Glasma and its instability
- Fluctuations, Cascade, and Kolmogorov spectrum

Part I

Introduction to the Color Glass Condensate and the McLerran-Venugopalan model

Kinematics in the high-energy QCD processes

Common Kinematic Variables

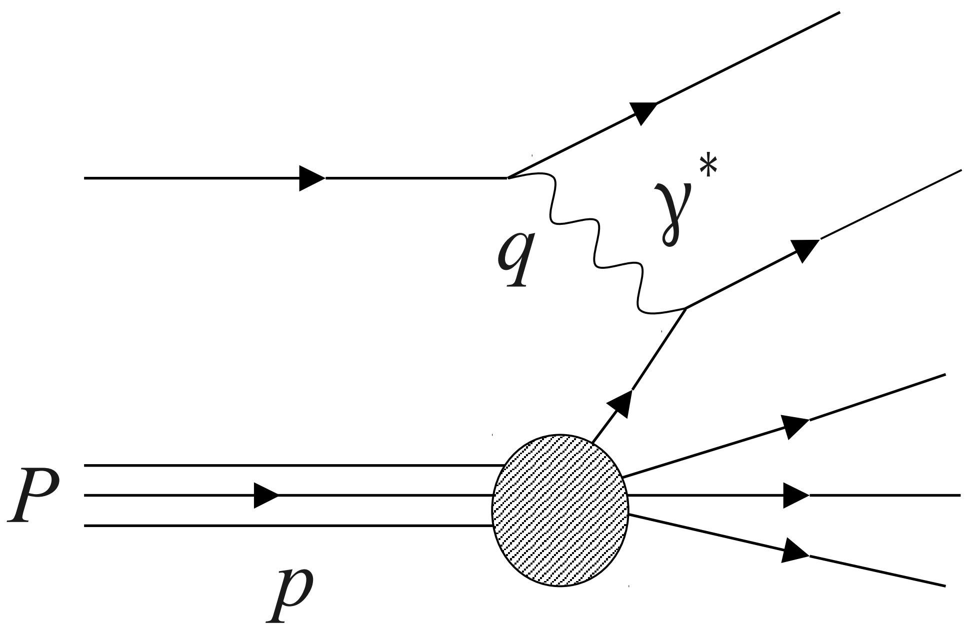
Virtuality Q

Bjorken's x

$$Q^2 = -q^2$$

$$x = \frac{Q^2}{2\,P \cdot q} = \frac{Q^2}{s + Q^2 - M^2}$$

$s = (P+q)^2$ in $\gamma^* p$ system



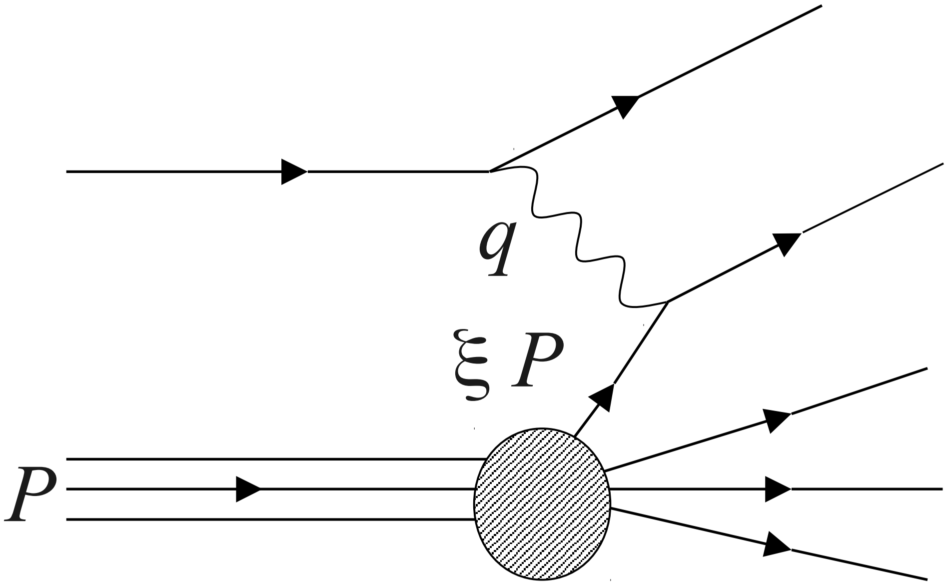
small-x $x << 1$

⬍

high energy $s >> Q^2$

Kinematics in the high-energy QCD processes

Elastic Parton Scattering



$$0 \simeq (\xi P + q)^2 \simeq 2\xi P \cdot q - Q^2$$

$$\rightarrow \quad \xi \simeq \frac{Q^2}{2P \cdot q} = x$$

Momentum Fraction

Light-cone Variables

$$x^{\pm} = \frac{1}{\sqrt{2}}(t \pm z), \quad p^{\pm} = \frac{1}{\sqrt{2}}(E \pm p^z)$$

x^+ : time $\qquad$ x^- : longitudinal coordinate

p^- : energy $\qquad$ p^+ : longitudinal momentum

Infinite Momentum Frames (IMF)

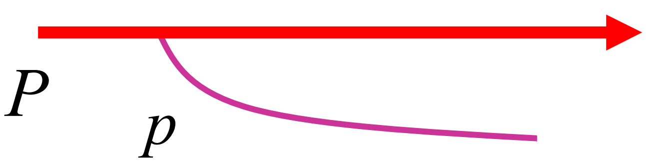
Frames with infinite P^+

$$x \approx \frac{p^+}{P^+}$$

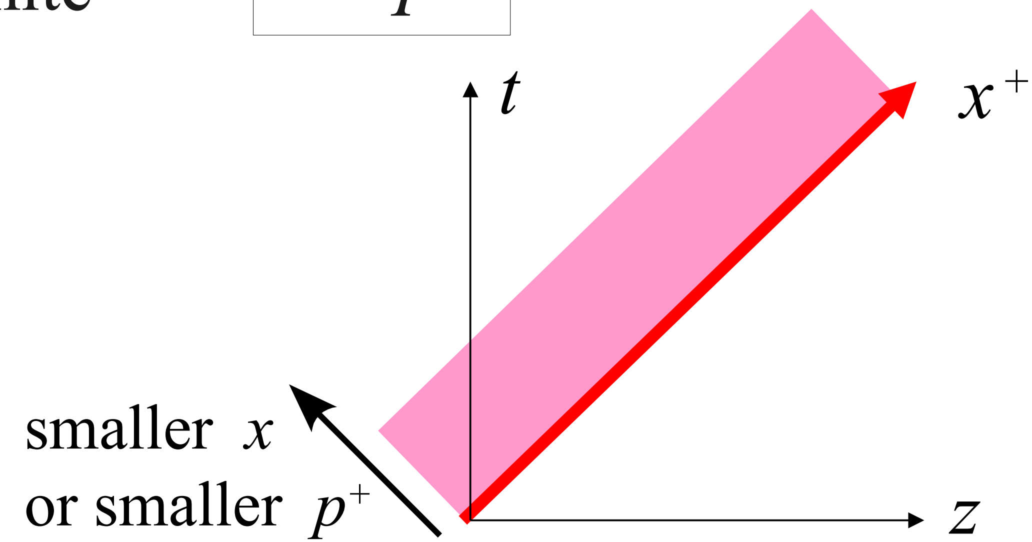
$E \approx P^z$ and P^+ are infinite

x^- : Lorentz contract

x^+ : time dilation



$$\Delta E \sim p^- = \frac{p_\perp^2}{2xP^+}$$



Partons live long in the IMFs → Parton Picture

Breit Frame

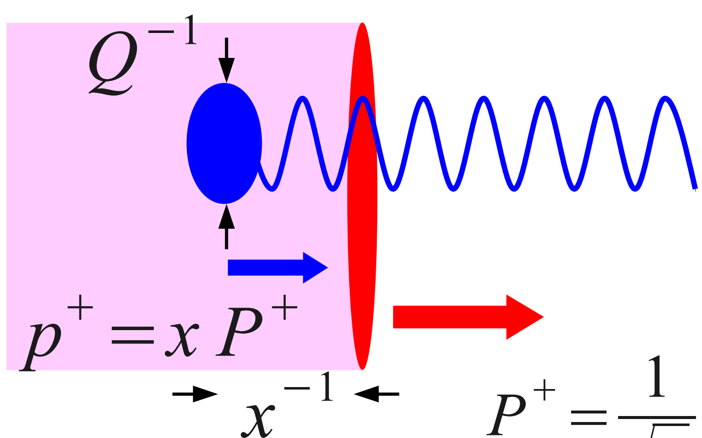
Physical Meaning of Two Variables

Transverse Momentum Q

Transverse size of partons (quark-antiquark ~ gluon)

Bjorken x

Longitudinal fraction of parton momentum



Breit frame (IMF)

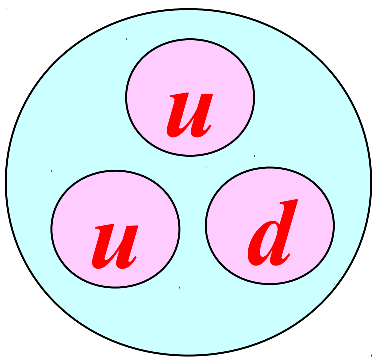
$$q^{\mu} = \left(q^0 \approx 0, q_{\perp}, 0\right)$$

$$Q^2 \approx q_{\perp}^2$$

$$P^+ = \frac{1}{\sqrt{2}}\left(P^0 + P^z\right) \sim \infty$$

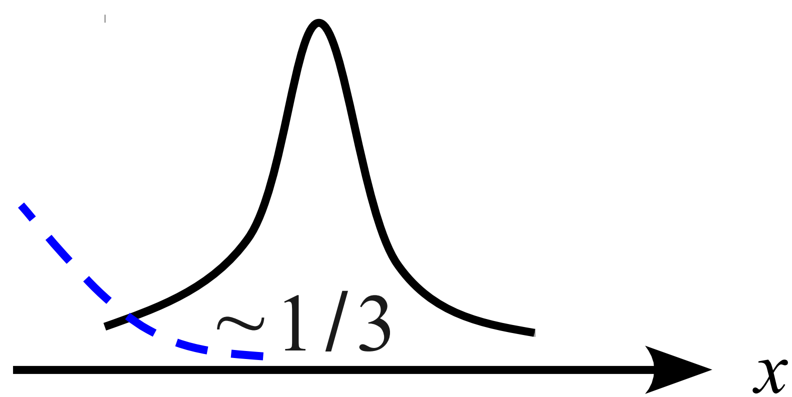
Parton Distribution Function

Valence and Sea Quarks and Gluons



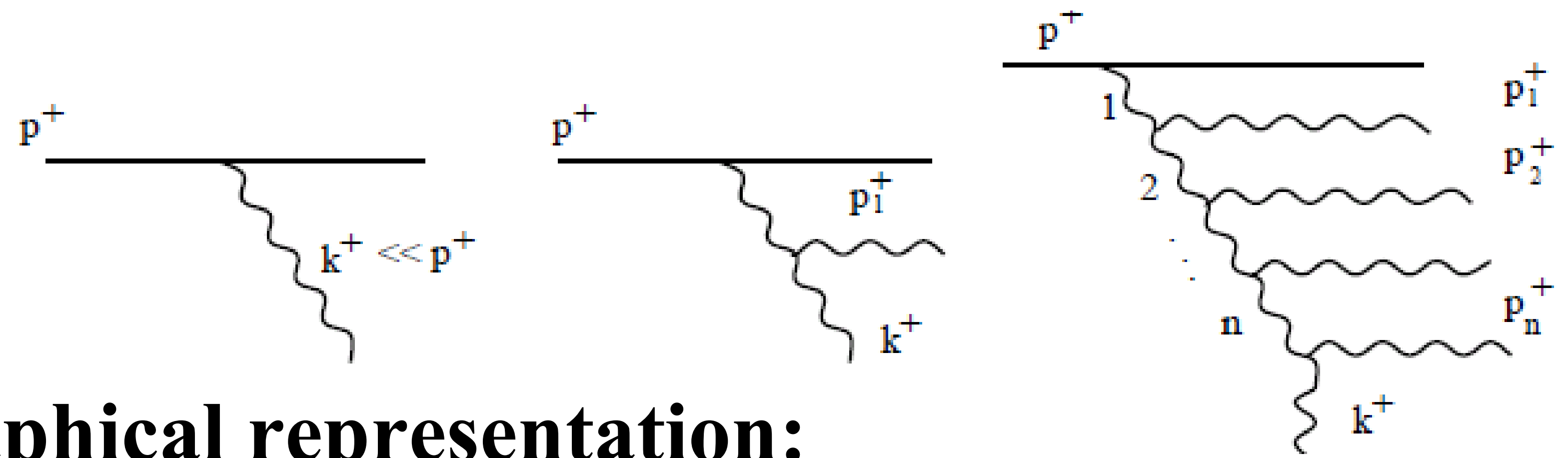
proton

valence quark constituent

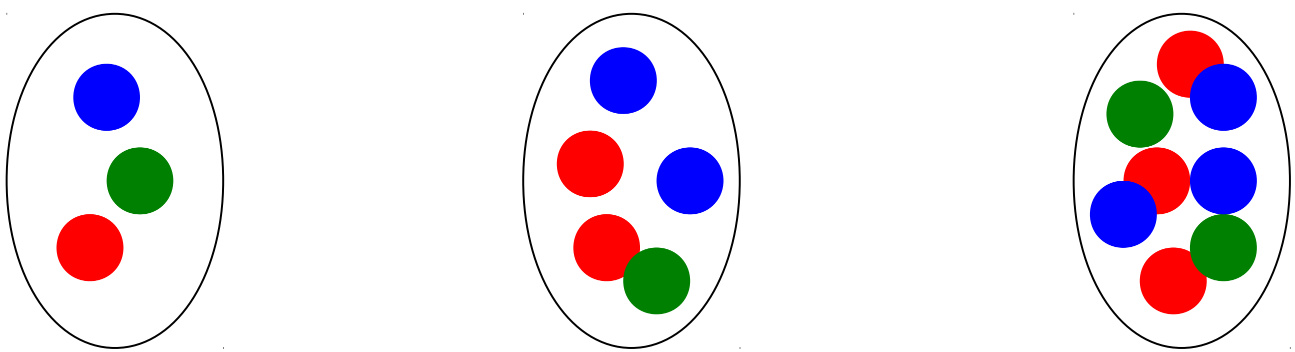


BFKL – Smaller x with Fixed Q²

Gluon increases with (nearly) fixed transverse area:



Graphical representation:

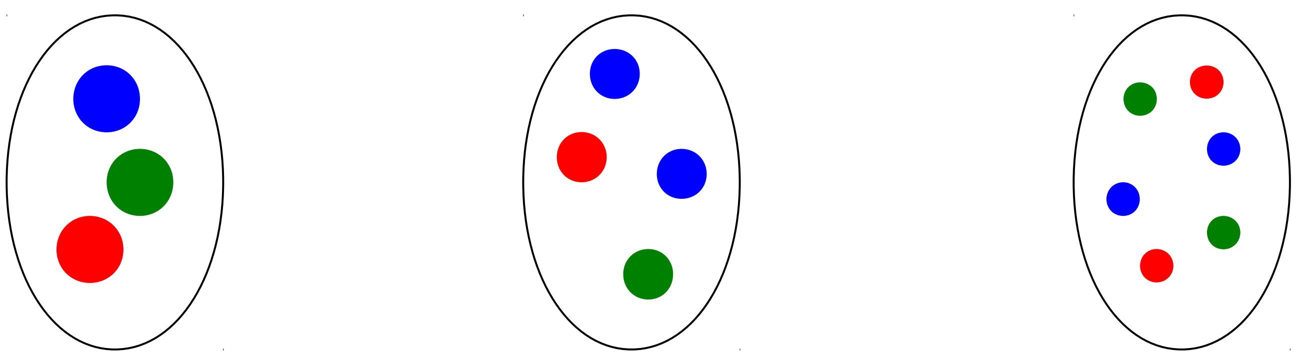


small-*x* → Dense Gluon Matter

DGLAP – Larger Q^2 with Fixed x

Gluon slowly increases with decreasing area:

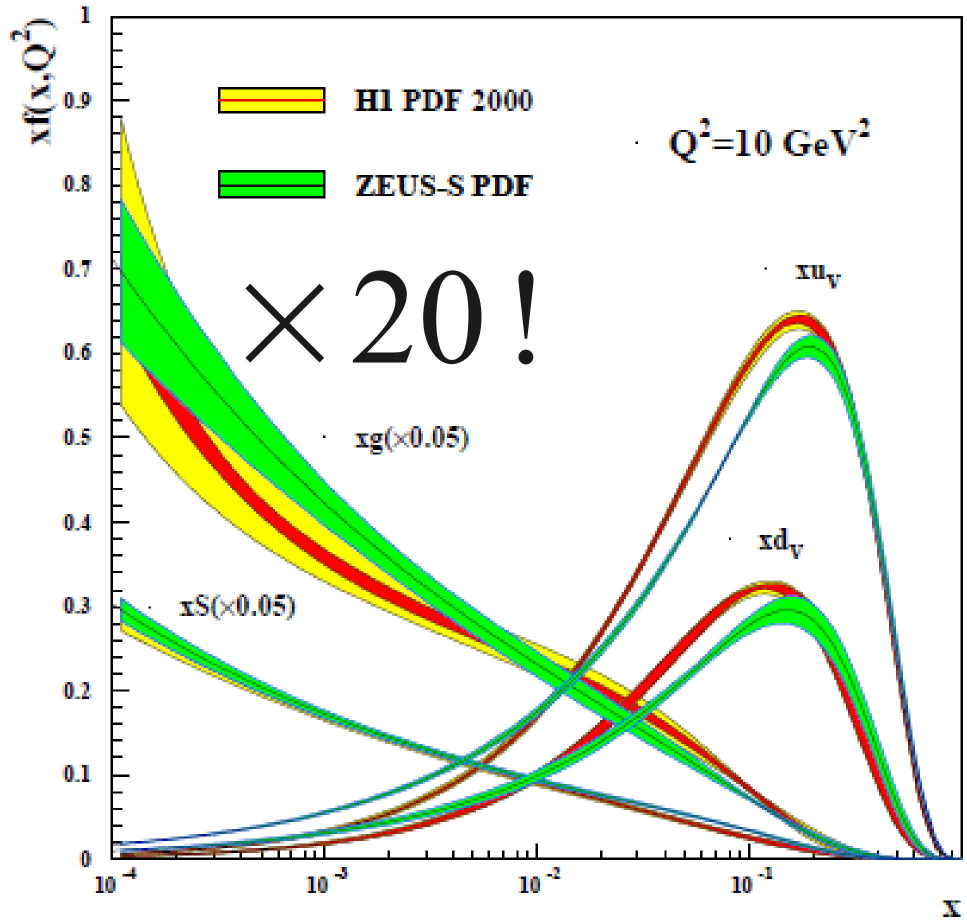
Graphical representation:



large *Q* → Dilute Gluon Matter

Data from HERA

Quantum Evolution of PDFs at fixed Q^2

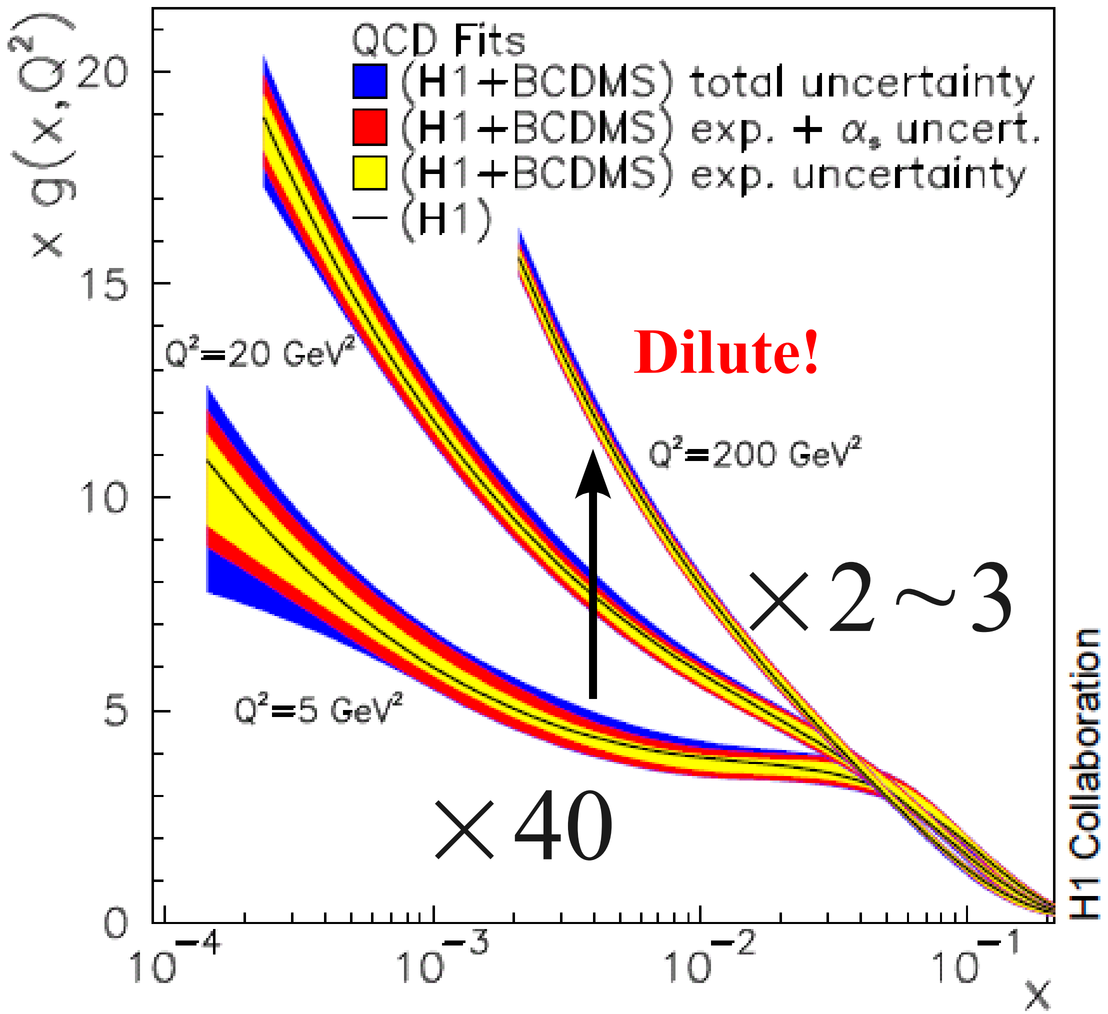


As x goes smaller than ~ 10^{-2} **gluon** is dominant.

High energy (large s and small t) processes are dominated by abundant **gluons**.

Data from HERA

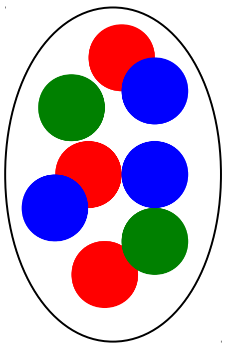
Quantum Evolution of PDFs at various Q^2



As Q^2 goes larger gluon grows slowly.

Saturation

Gluons eventually cover the transverse area:



$$\text{Area} \quad \sim \pi R^2 \sim 75\ \text{GeV}^{-2} \quad \text{(proton)}$$

$$\text{Crammed density} \quad \sim \frac{(N_c^2 - 1)\,Q^2}{\alpha_s N_c}\,\pi R^2 \sim 600$$

$$\left(\text{for } Q^2 = 1\ \text{GeV}^2\right)$$

Naive condition for saturation:

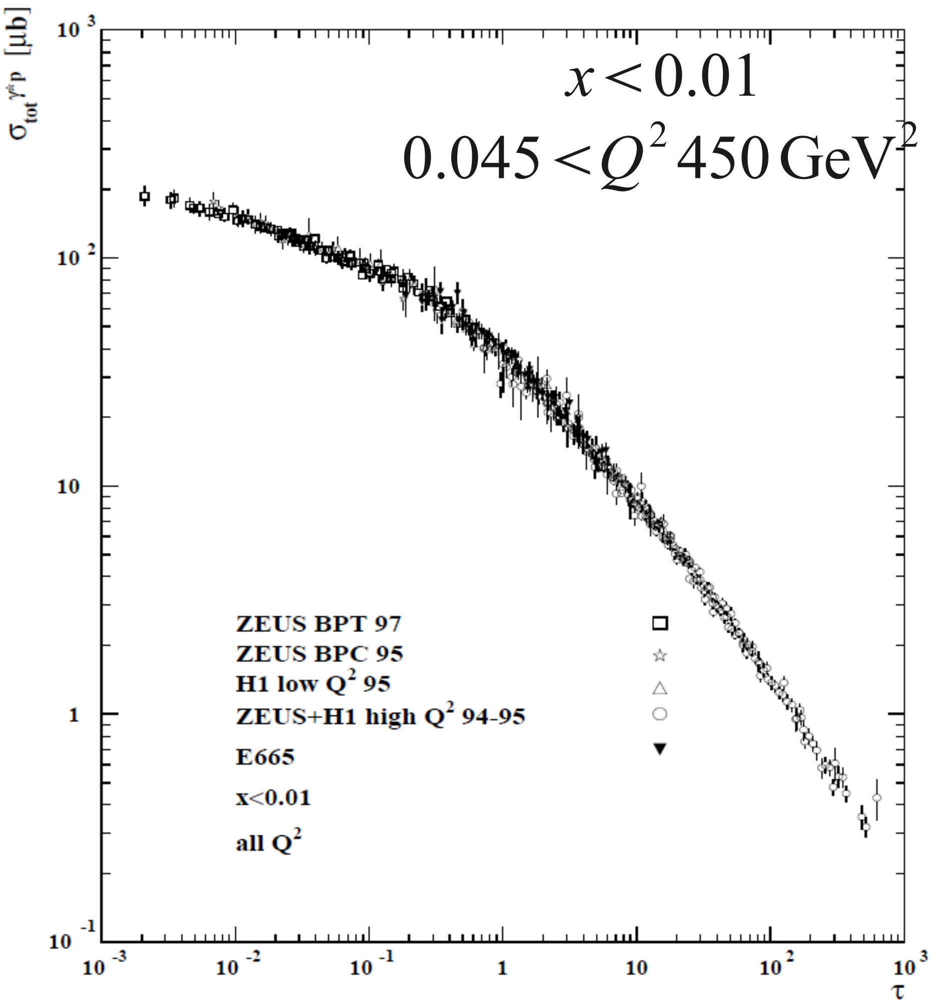
$$xg(x,Q) \Big/ (N_c^2 - 1)\,Q^2 \pi R^2 \sim \frac{1}{\alpha_s N_c} \sim 1$$

Overlapping Factor

No need to achieve such complete saturation

Scaling Behavior

Dipole Cross Section in a Saturation Model



$$\sigma_{\gamma^* p}(x, Q^2) \rightarrow \sigma_{\gamma^* p}(Q^2/Q_s^2(x))$$

$$Q_s^2(x) = Q_0^2 (x/x_0)^{-\lambda}$$

Golec-Biernat-Wuesthoff

Stasto-Golec-Biernat-Kwiecinski Plot

Geometric Scaling

Q_s as a function of x is fixed

$$Q_0 = 1\ \mathrm{GeV}$$

$$x_0 = 3.04 \times 10^{-4}$$

$$\lambda = 0.288$$

Saturation is sufficient for scaling, but not necessary to it.

Saturation?

Let us put some numbers:

$$x = 10^{-4} \;\rightarrow\; Q_s^2 = 1.38\,\mathrm{GeV}^2$$

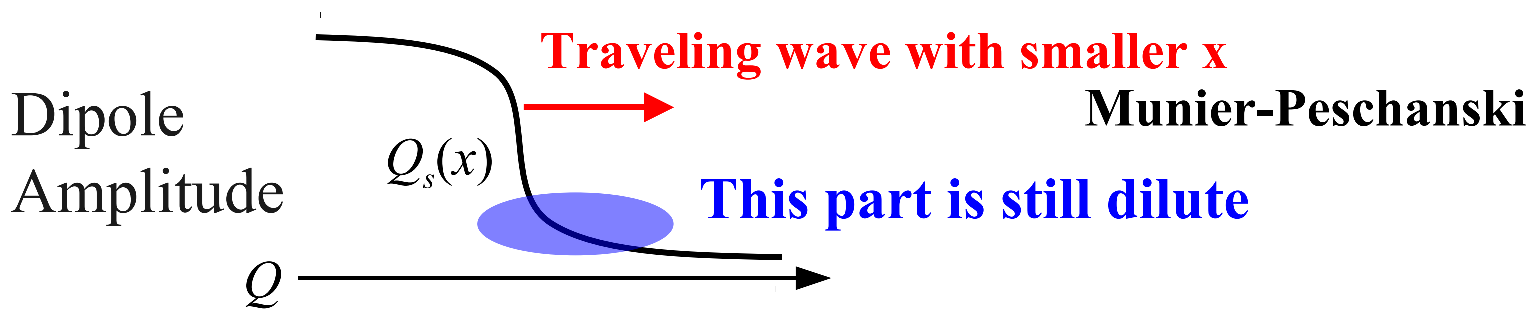
$$\frac{x\,g(x, Q_s)}{(N_c^2 - 1)\,Q_s^2 \pi R^2} \sim \frac{10}{8 \cdot 1.38 \cdot 75} \sim 0.01$$

No need to take it seriously
Don't have to realize saturation
for the saturation physics!

Scaling is consistent with pQCD:

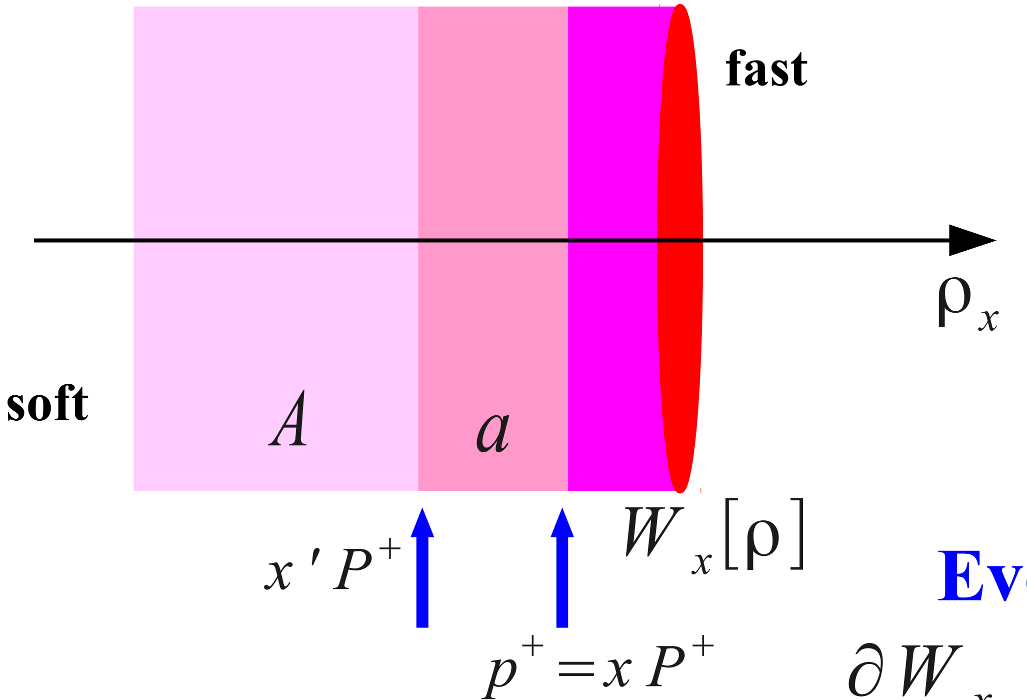
CGC = saturation + pQCD

BFKL (dilute regime) can fix the parameters:



Effective Theory of Saturation

Effective Theory at *x*
Integrate faster (larger *x*) degrees of freedom



Integration over *a*

$$\delta\rho_x \to \rho_x$$

Classical Weight

$$W_x[\rho] \to W_{x'}[\rho]$$

Evolution (RG) eq.

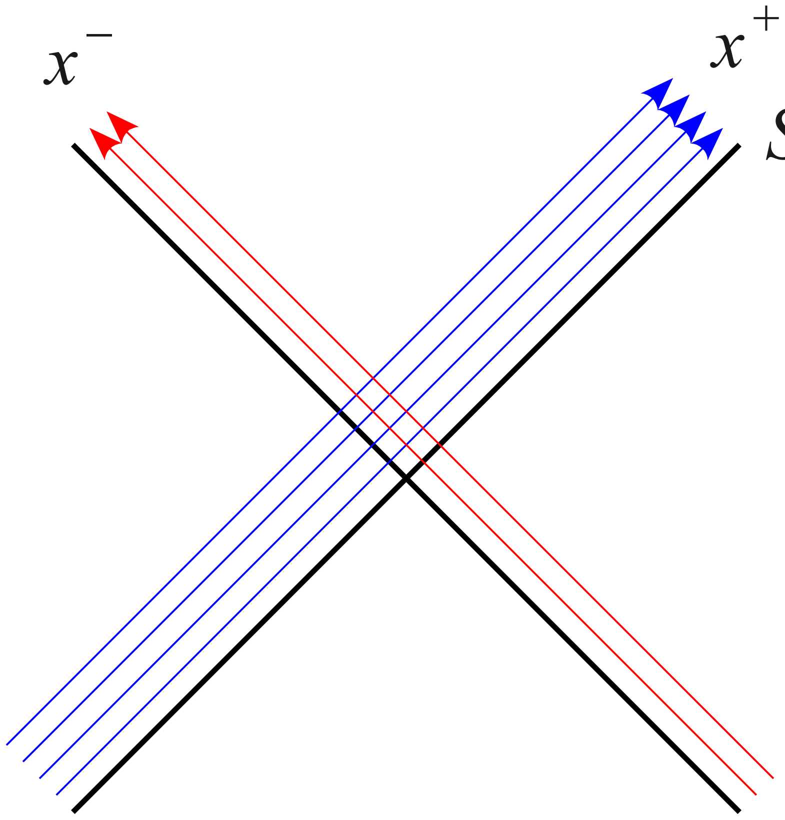
$$\frac{\partial W_x}{\partial x} = H[A,\rho] W_x \;\Leftrightarrow\; Q_s(x)$$

JIMWLK (BFKL) Equation

Kovner, McLerran, Weigert,
Iancu, Jalilian-Marian, Leonidov, ...

Scattering Problem

Scattering Amplitude in the Eikonal Approx.



$$S \sim \langle \sum_{\{\rho_t\}} W_x[\rho_t] \prod_{\{\rho_t\}} W \cdot \sum_{\{\rho_p\}} W_{x'}[\rho_p] \prod_{\{\rho_p\}} V \rangle$$

$$W(x_\perp) = \exp\left[ig \int dz^+ A^-(z^+, x_\perp) \right]$$

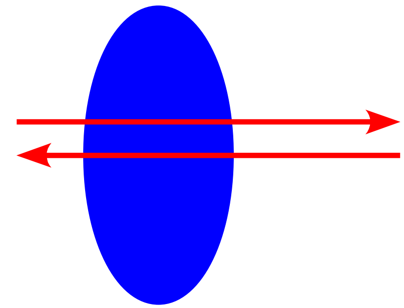
$$V(x_\perp) = \exp\left[ig \int dz^- A^+(z^-, x_\perp) \right]$$

In case of Dipole-CGC scattering

$$S \sim \langle \langle V(x_\perp) V^\dagger(y_\perp) \rangle \rangle_{\rho_t}$$

Stationary-Point Approximation

Dipole Scattering Amplitude



$$S \sim \langle \sum_{\{\rho_t\}} W_x[\rho_t] \prod_{\{\rho_t\}} W \cdot V(x_\perp)\, V^\dagger(y_\perp) \rangle$$

$$= \sum_{\{\rho_t\}} W_x[\rho_t] \int_{p^+ < xP^+} [DA]\, V(x_\perp)\, V^\dagger(y_\perp) \exp[iS_{\rm YM}[A] + iS_{\rm source}[\rho_t, W]]$$

$$= \langle\langle V(x_\perp)\, V^\dagger(y_\perp) \rangle\rangle_{\rho_t}$$

$$S_{\rm source} = \frac{i}{gN_c} \int d^4x\; {\rm tr}\big[\rho_t \ln W\big] \sim -\int d^4x\; \rho_t^a A_a^-$$

Easily solvable

$$\frac{\delta S_{\rm YM}}{\delta A_a^\mu}\Big|_{A=\mathcal{A}} = \delta^{\mu -} \rho_t$$

Large enough ρ_t → Stationary-point approx.

Color Glass Condensate

As a result of the stationary-point approx.

$$\langle\langle V(x_\perp)\, V^\dagger(y_\perp)\rangle\rangle_{\rho_t}$$
$$= \sum_{\{\rho_t\}} W_x[\rho_t] \int_{p^+ < xP^+} [DA]\, V(x_\perp)\, V^\dagger(y_\perp) \exp[iS_{\mathrm{YM}}[A] + iS_{\mathrm{source}}[\rho_t, W]]$$
$$\sim \sum_{\{\rho_t\}} W_x[\rho_t]\, V(x_\perp)\, V^\dagger(y_\perp)|_{A=\mathcal{A}[\rho_t]}$$

General expression:

$$\langle\langle \mathcal{O}[A]\rangle\rangle_{\rho_t} \sim \int D\rho_t\, W_x[\rho_t]\, \mathcal{O}[\mathcal{A}[\rho_t]]$$

Quantum corrections lead to $W_x \to W_{x+\delta x}$ i.e. small-x evolution

Dense-Dense Scattering (HIC)

Stationary-point is shifted:

$$S \sim \langle \sum_{\{\rho_t\}} W_x[\rho_t] \prod_{\{\rho_t\}} W \cdot \sum_{\{\rho_p\}} W'_{x'}[\rho_p] \prod_{\{\rho_p\}} V \rangle$$

$$= \sum_{\{\rho_t,\rho_p\}} W_x[\rho_t]\, W'_{x'}[\rho_p] \int [DA] \exp\left[iS_{\rm YM} + iS_{\rm source}[\rho_t, \rho_p, W, V] \right]$$

$$\sim \sum_{\{\rho_t,\rho_p\}} W_x[\rho_t] W'_{x'}[\rho_p] \int [DA] \exp\left[iS_{\rm YM} - i\int d^4x \left(\rho_t^a A_a^- + \rho_p^a A_a^+\right)\right]$$

Stationary-point approx. is made at

$$\left.\frac{\delta S_{\rm YM}}{\delta A_a^\mu}\right|_{A=\mathcal{A}} = \delta^{\mu -}\rho_t^a + \delta^{\mu +}\rho_p^a$$

Not solvable analytically

McLerran-Venugopalan (MV) Model

Gaussian Approximation:

McLerran-Venugopalan (1993)

$$W_x[\rho] = \exp\left[-\int d^3x \frac{|\rho(x)|^2}{2g^2\mu_x^2} \right]$$

μ_x is related to $Q_s(x)$

larger μ_x = larger ρ = dense gluons = larger Q_s

Once $\mathcal{A}$ is known, observables such as the energy density are calculable in the unit of μ (scaling)

$$\langle\langle \mathcal{O}[A] \rangle\rangle_{\rho_t} \sim \int D\rho_t W_x[\rho_t]\, \mathcal{O}[\mathcal{A}[\rho_t]]$$

$$\langle\langle \mathcal{O}[A] \rangle\rangle_{\rho_t,\rho_p} \sim \int D\rho_t\, D\rho_p W_x[\rho_t] W_{x'}[\rho_p]\, \mathcal{O}[\mathcal{A}[\rho_t,\rho_p]]$$

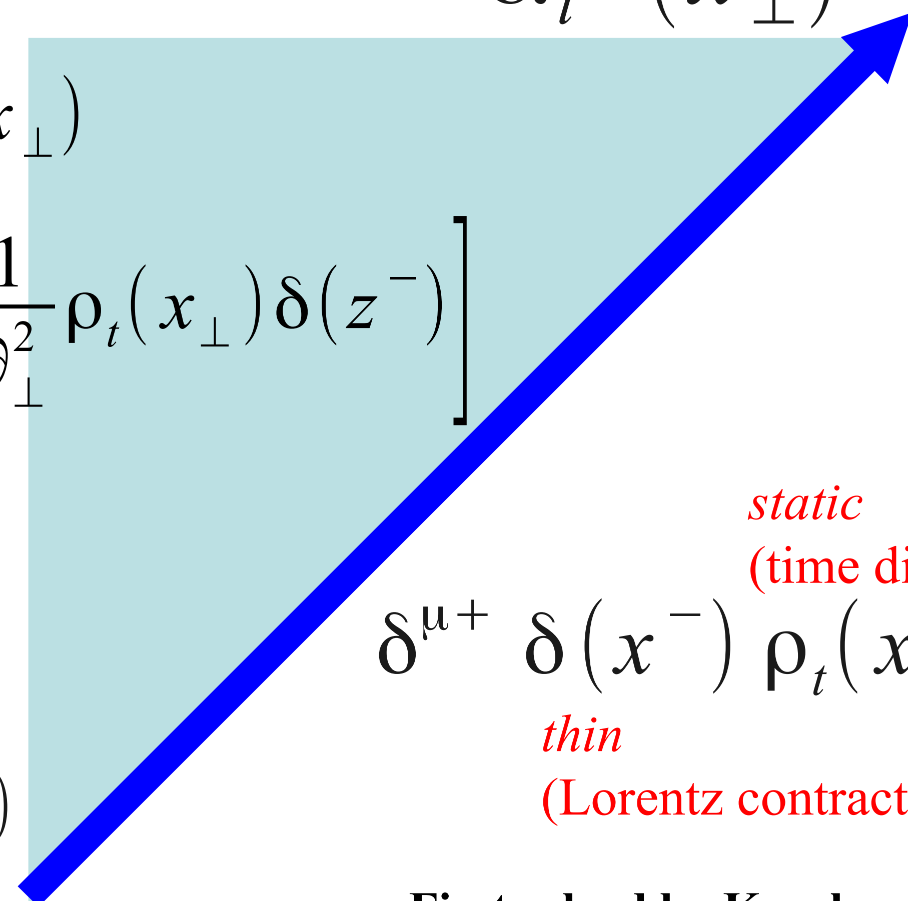
One Source Problem

One-source problem is solvable:

$$A^+ = A^- = 0 \quad \text{(gauge choice)}$$

$$A_i = \alpha_i^{(1)} = -\frac{1}{ig} V(x_\perp)\,\partial_i V^\dagger(x_\perp)$$

$$V^+(x_\perp) = P\exp\left[-ig\int dz^- \frac{1}{\partial_\perp^2}\rho_t(x_\perp)\,\delta(z^-)\right]$$



c.f. in EM

$$\partial_\perp^2 \phi = -\rho \quad (\text{Poisson eq})$$

$$\rightarrow \phi' = 0 \quad (\text{Gauge trans})$$

$$\rightarrow A'_i = \frac{1}{ie} e^{ie\phi}\,\partial_i e^{-ie\phi} \quad (= -\partial_i \phi)$$

First solved by Kovchegov

Relation between μ and Q_s

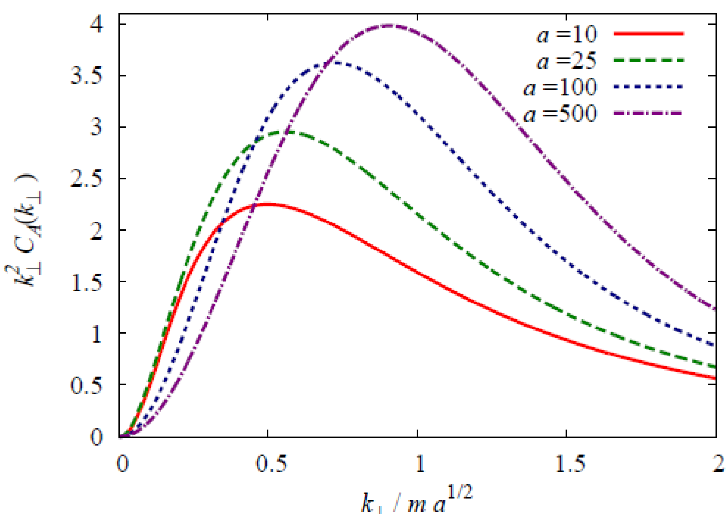
Rough Relationship (dipole amplitude)

$$Q_s^2 \sim (g^2\mu)^2 \ln[Q_s^2 a] \qquad (a : \text{infrared regulator})$$

It is extremely difficult to fix μ directly from this...

Gluon Density

$$\left\langle V_A^{\dagger ca}(x_\perp) V_A^{\dagger cb}(y_\perp) \right\rangle = \delta^{ab} \mu^2 C_{\text{adj}}(x_\perp - y_\perp)$$



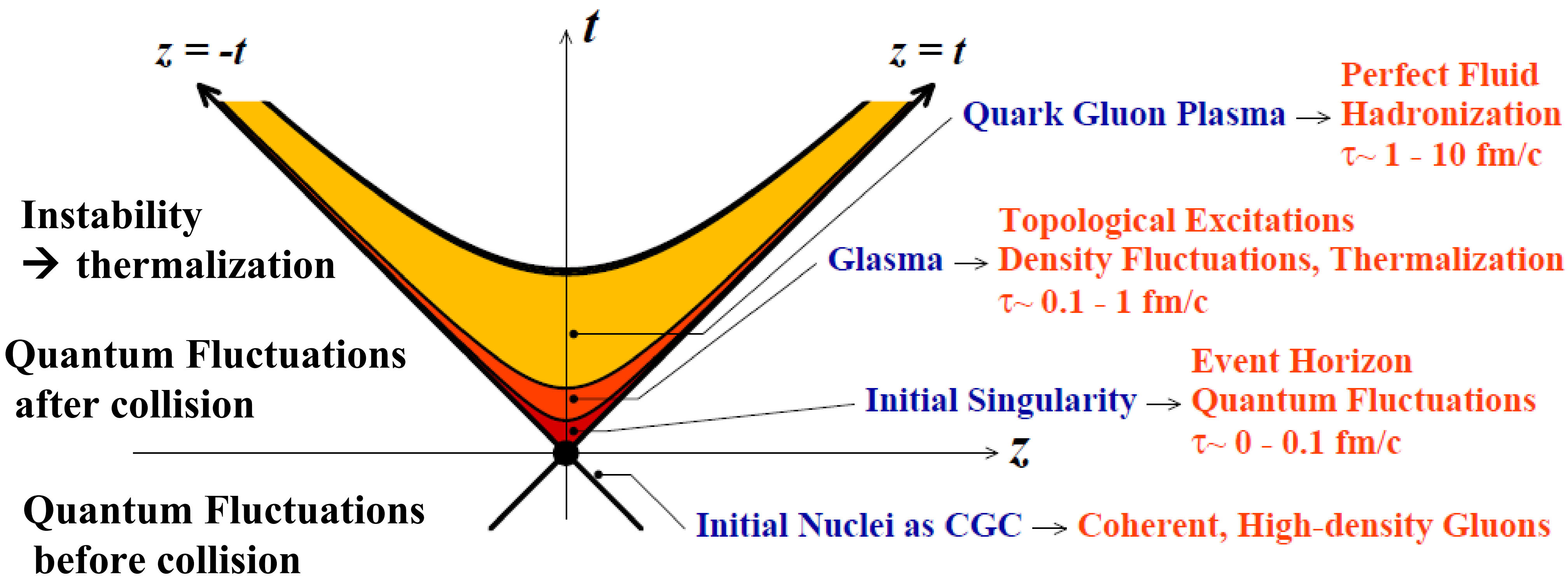
a	m	$g^2\mu_A$
10	$0.64Q_s$	$1.65Q_s$
25	$0.36Q_s$	$1.46Q_s$
100	$0.14Q_s$	$1.13Q_s$
500	$0.050Q_s$	$0.90Q_s$

Peak position fixes Q_s

Lappi (2007)
Fujii-KF-Hidaka (2008)

Heavy-Ion Collisions

Space-Time Evolution of the Little Bang



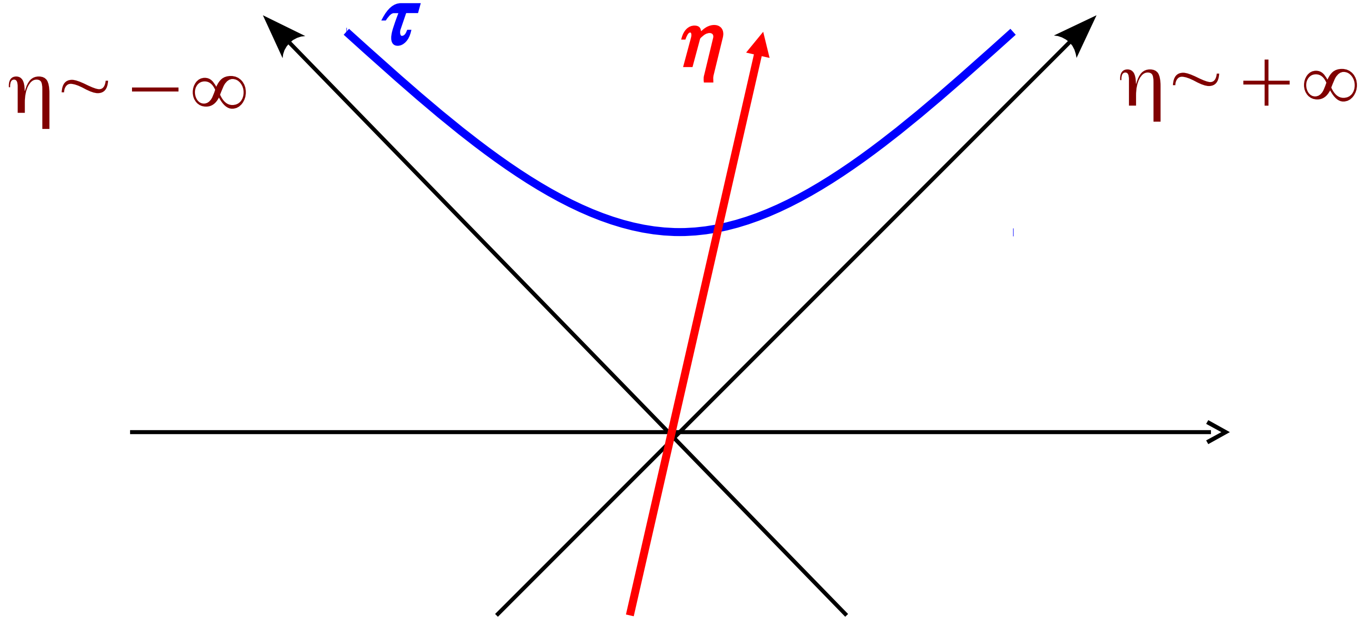
Fluctuations (seeds) → Instability → Thermalization

Lecture Part II

Bjorken (Expanding) Coordinates

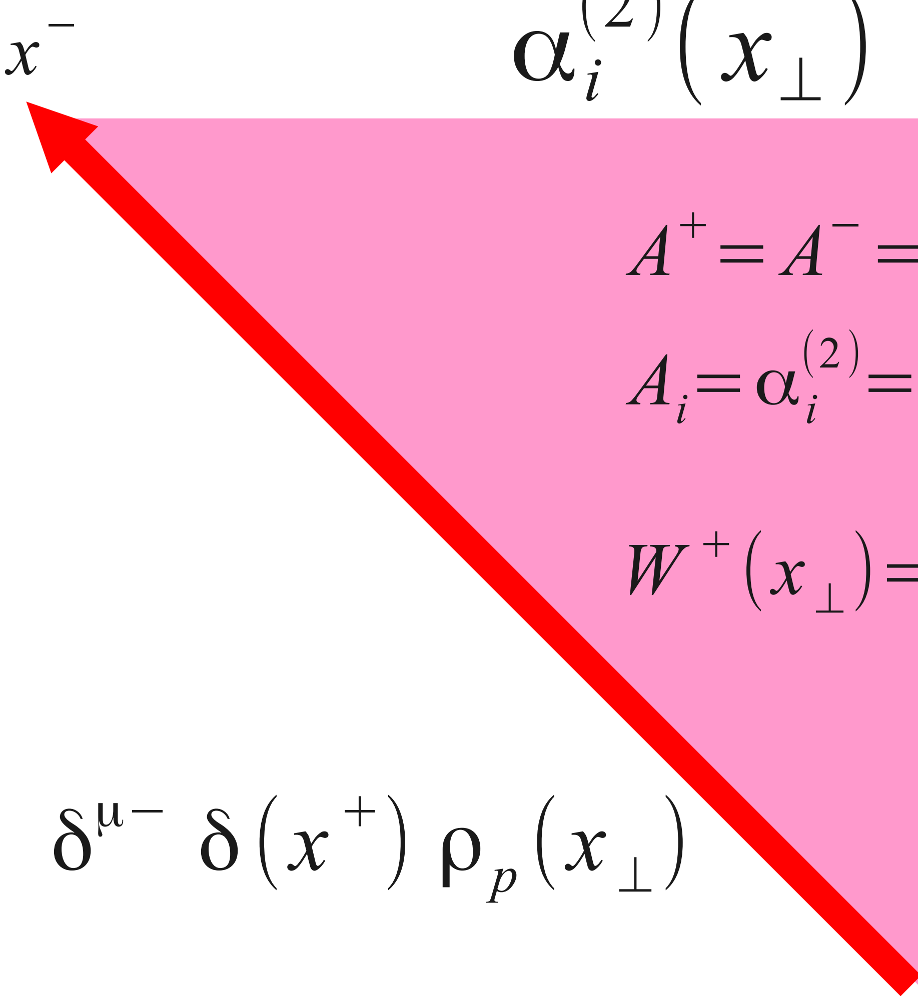
Proper Time and (space-time) Rapidity

$$\tau = \sqrt{t^2 - z^2}\,, \qquad \eta = \frac{1}{2}\ln\left[\frac{t+z}{t-z}\right]$$



Fields from the Other Source

Similar to the one source problem



$$A^+ = A^- = 0$$

$$A_i = \alpha_i^{(2)} = -\frac{1}{ig} W(x_\perp)\,\partial_i W^\dagger(x_\perp)$$

$$W^+(x_\perp) = P\exp\left[-ig\int dz^+ \frac{1}{\partial_\perp^2}\rho_p(x_\perp)\,\delta(z^+)\right]$$

Two Source Problem

Two-source problem is not solvable analytically

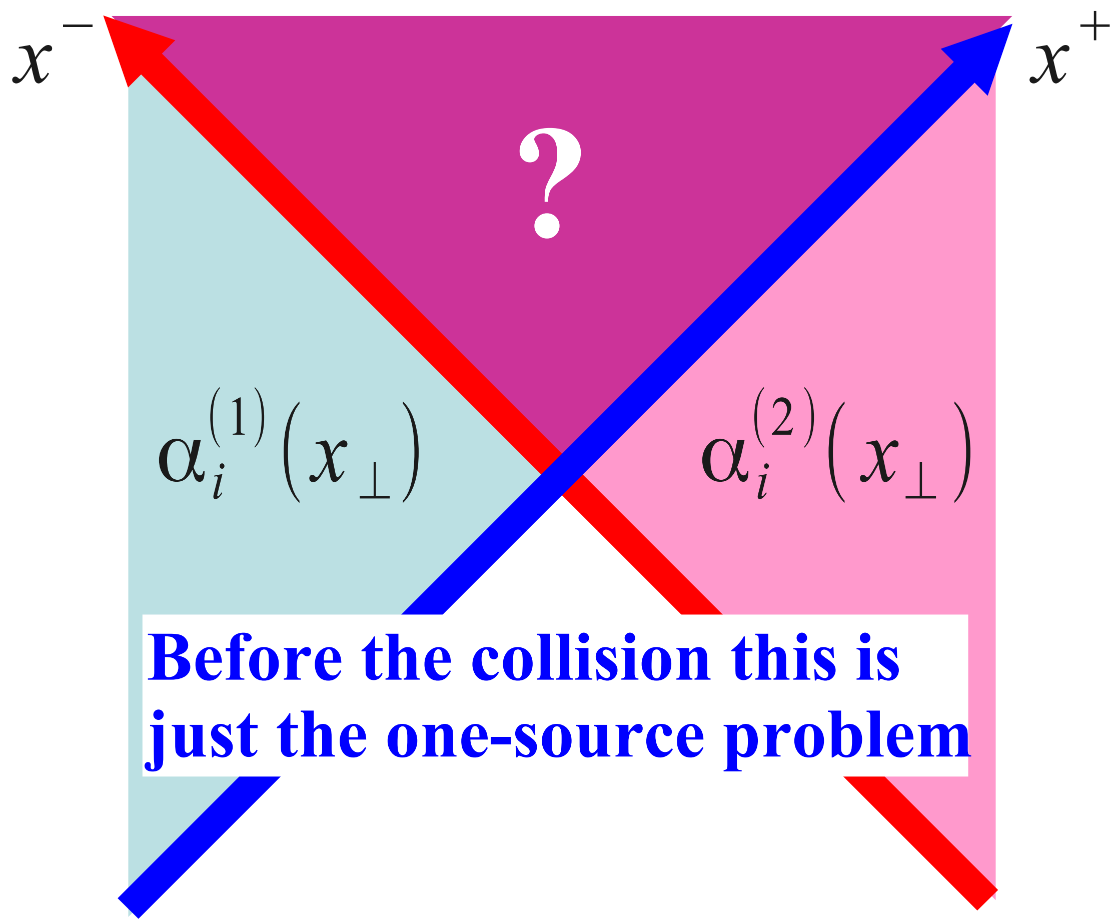
Initial condition is known on the light-cone

$$\mathcal{A}_i = \alpha_i^{(1)} + \alpha_i^{(2)}$$

$$\mathcal{A}_\eta = 0$$

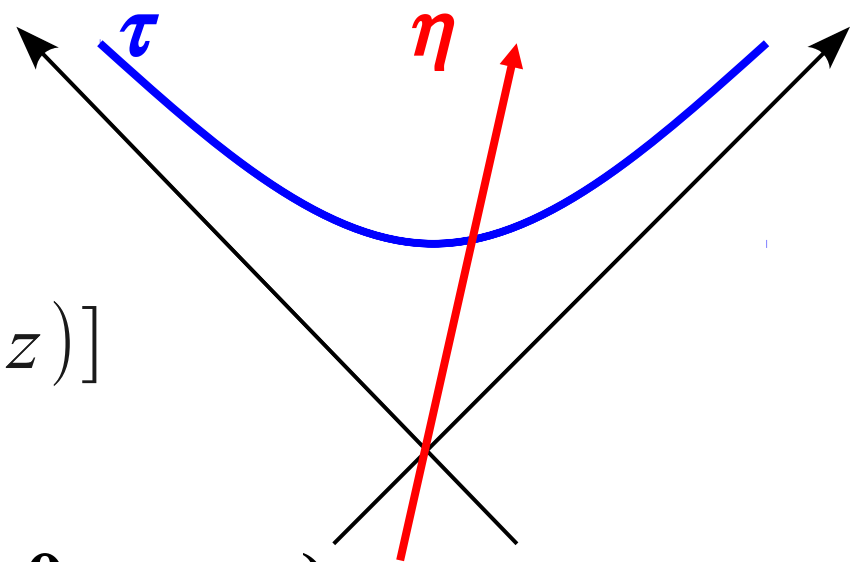
$$\mathcal{E}^i = 0$$

$$\mathcal{E}^\eta = ig\left[\alpha_i^{(1)}, \alpha_i^{(2)}\right]$$



Kovner-McLerran-Weigert (1995)

Equations of Motion to be Solved



Coordinates

proper time $\tau = \sqrt{t^2 - z^2}$

rapidity $\eta = \frac{1}{2} \ln [(t+z)/(t-z)]$

Equations to be solved (in A_τ=0 gauge)

$$E^i = \tau \partial_\tau A_i, \qquad E^\eta = \tau^{-1} \partial_\tau A_\eta$$

$$\partial_\tau E^i = \tau^{-1} D_\eta F_{\eta i} + \tau D_j F_{ji}$$

$$\partial_\tau E^\eta = \tau^{-1} D_j F_{j\eta}$$

Initial Condition

Chromo-Electric and Magnetic fields

$$E^i_{(0)} = 0\,,$$

$$E^\eta_{(0)} = ig\Big(\big[\alpha^{(1)}_1, \alpha^{(2)}_1\big] + \big[\alpha^{(1)}_2, \alpha^{(2)}_2\big]\Big)$$

$$B^i_{(0)} = 0\,, \qquad B^\eta_{(0)} = F_{12(0)}$$

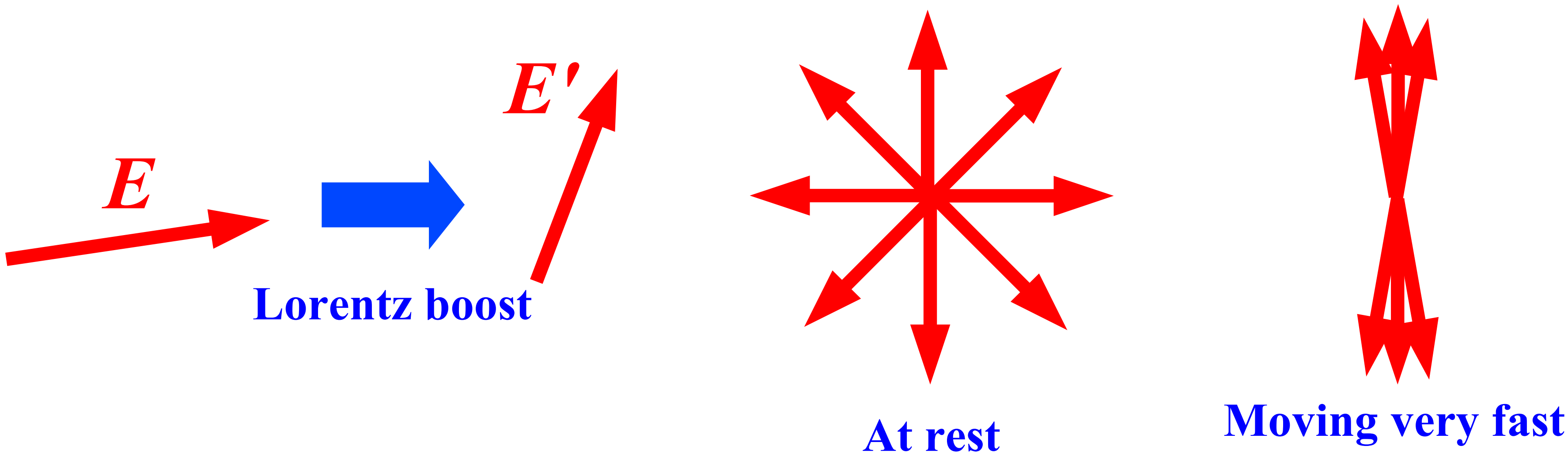
$$F_{ij(0)} = -ig\Big(\big[\alpha^{(1)}_i, \alpha^{(2)}_j\big] + \big[\alpha^{(2)}_i, \alpha^{(1)}_j\big]\Big)$$

After the collision only the longitudinal fields dominate.
This is "very" non-trivial initial condition...

Before Collision

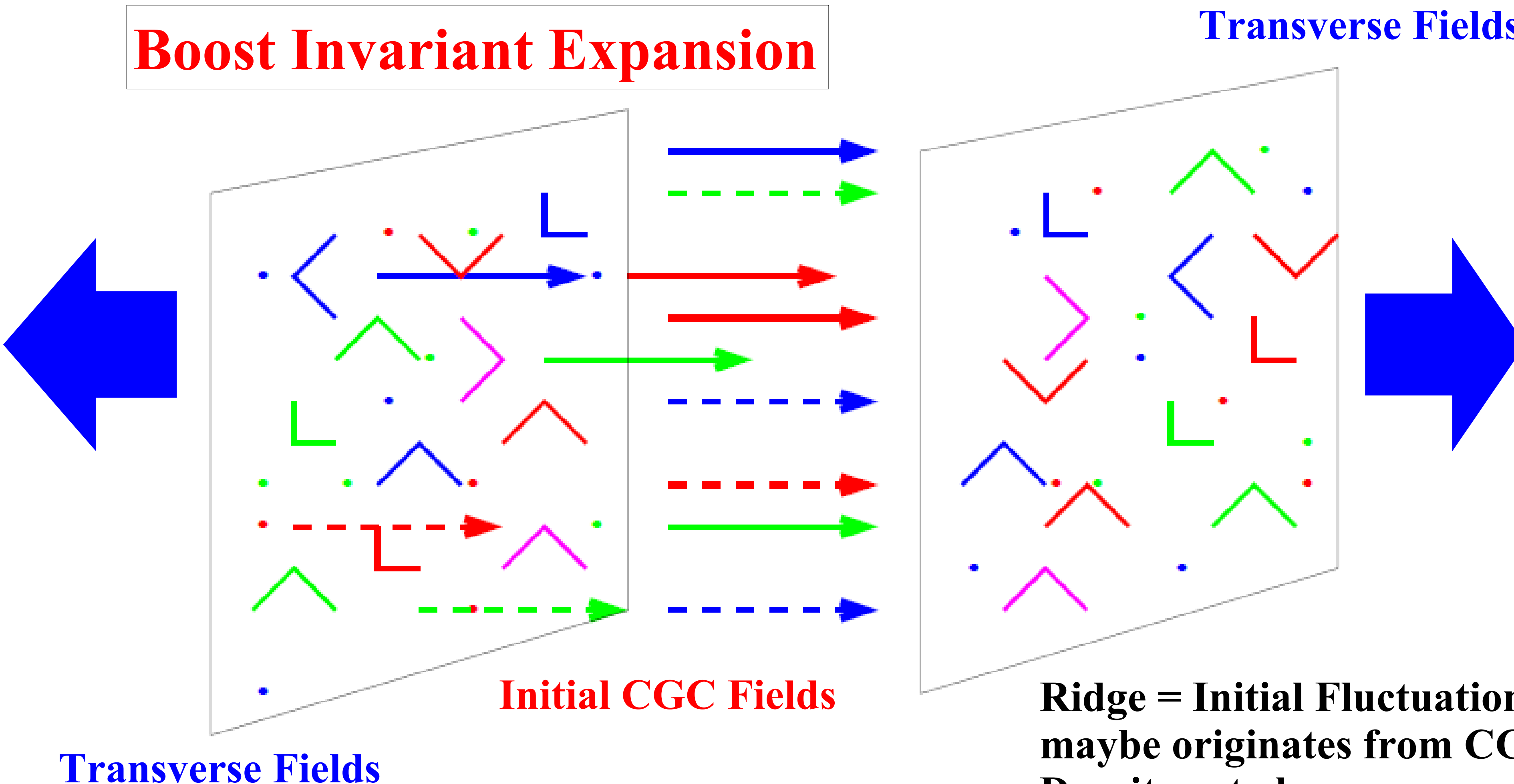
Two sources do not talk to each other
Just one-source problem

No longitudinal fields but only transverse fields attached on the nucleus sheet



After Collision

Longitudinal Fields between Nucleus Sheets



Ridge = Initial Fluctuation maybe originates from CGC Dumitru et al.

Numerical Method

Lattice Discretization

$$A_\mu(x) \;\rightarrow\; U = e^{-iga\,A_\mu(x)} \quad \text{(Link Variable)}$$

Why link variables than naïve discretization?

Formal Answer:
Gauss law is not compatible with the time evolution.
It becomes more and more violated at later time.

Practical Answer:
Keeping numerical stability is very important.
It is very sensitive to the order of the discretization.
The correct ordering is guaranteed in the lattice formulation.

EoM on the Lattice

Canonical Momenta

Leap-frog Method

$$\tau' = \tau + \Delta\tau$$
$$\tau'' = \tau + 2\Delta\tau$$

$$U_i(\tau'') = \exp\left[-2\Delta\tau \cdot \mathrm{i}gE^i(\tau')/\tau'\right]U_i(\tau) \ ,$$
$$U_\eta(\tau'') = \exp\left[-2\Delta\tau \cdot \mathrm{i}ga_\eta\tau' E^\eta(\tau')\right]U_\eta(\tau)$$

Krasnitz, Venugopalan
Nara, Lappi, Romatschke

Equations of Motion

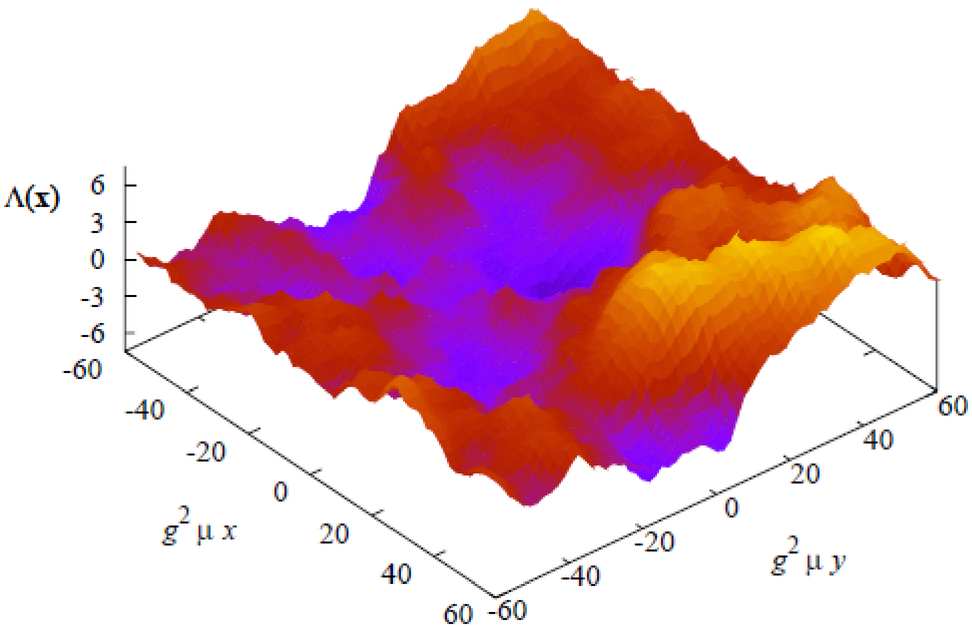
$$E^i(\tau') = E^i(\tau - \Delta\tau) + 2\Delta\tau\frac{\mathrm{i}}{2ga_\eta^2\tau}\Big[U_{\eta i}(x) + U_{-\eta i}(x) - (\mathrm{h.c.})\Big]_\tau$$
$$+ 2\Delta\tau\frac{\mathrm{i}\tau}{2g}\sum_{j\neq i}\Big[U_{ji}(x) + U_{-ji}(x) - (\mathrm{h.c.})\Big]_\tau \ ,$$

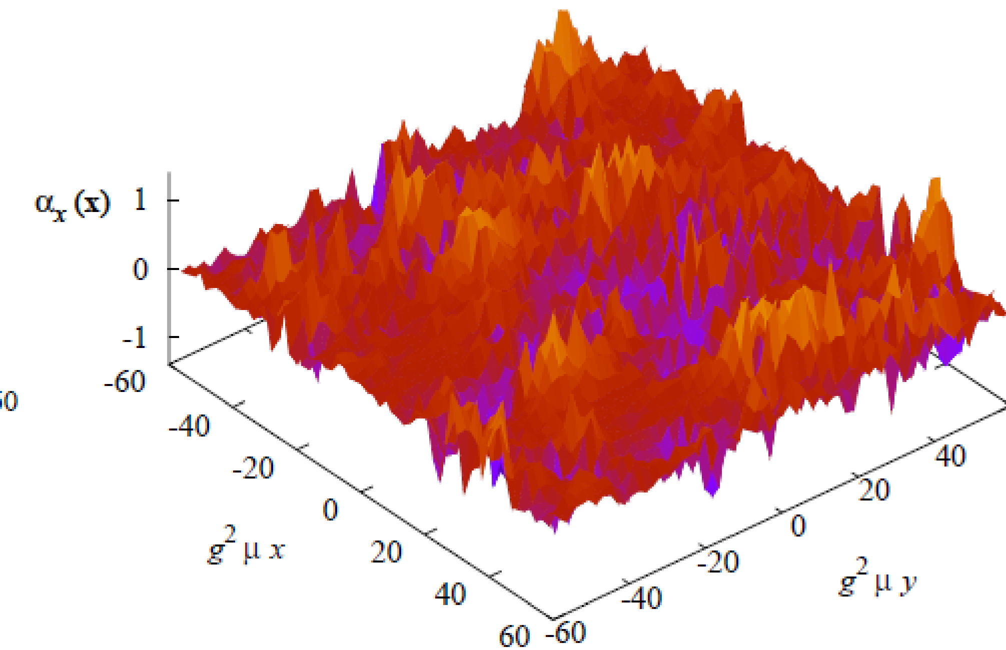
$$E^\eta(\tau') = E^\eta(\tau - \Delta\tau) + 2\Delta\tau\frac{\mathrm{i}}{2ga_\eta\tau}\sum_{j=x,y}\Big[U_{j\eta}(x) + U_{-j\eta}(x) - (\mathrm{h.c.})\Big]_\tau$$

Initial Configurations

One Configuration

Flux-tube missing
MV should be improved by JIMWLK





Spatial distribution of the solution of the Poisson eq.

$$\partial_\perp^2 \Lambda^{(m)}(\boldsymbol{x}_\perp) = -\rho^{(m)}(\boldsymbol{x}_\perp)$$

Spatial distribution of the gauge field

$$e^{-ig\Lambda(\boldsymbol{x}_\perp)} e^{ig\Lambda(\boldsymbol{x}_\perp + \hat{i})} = \exp[-\mathrm{i}g\alpha_i(\boldsymbol{x}_\perp)]$$

Parameter Fixing

Model Parameters

- **Saturation Scale Parameter**

$\boxed{g^2\mu L = 120} \longleftarrow g^2\mu R_A = 67.7$ with $\pi R_A^2 = L^2$

$\nwarrow$ $g^2\mu \approx 2\ \mathrm{GeV}$ with $R_A \approx 7\ \mathrm{fm}$

$g = 2$

- **Transverse Size**

  Physics should not depend on the transverse size N

  Check the robustness with various N's

- **Longitudinal Size**

  Physics should not depend on the longitudinal size N_η

  When the boost inv. is not broken, the continnum limit is easily taken ($a_\eta \to 0$)

Merit and Demerit of HIC

Merit

The largest merit is that Q_s is multiplied by $A^{1/6}$

In the Au-Au case $A^{1/6} \sim 2.4$

c.f. RHIC (200GeV) $\rightarrow$ LHC (5.5TeV)
Energy is 27 times bigger, but Q_s only 2.6 times.

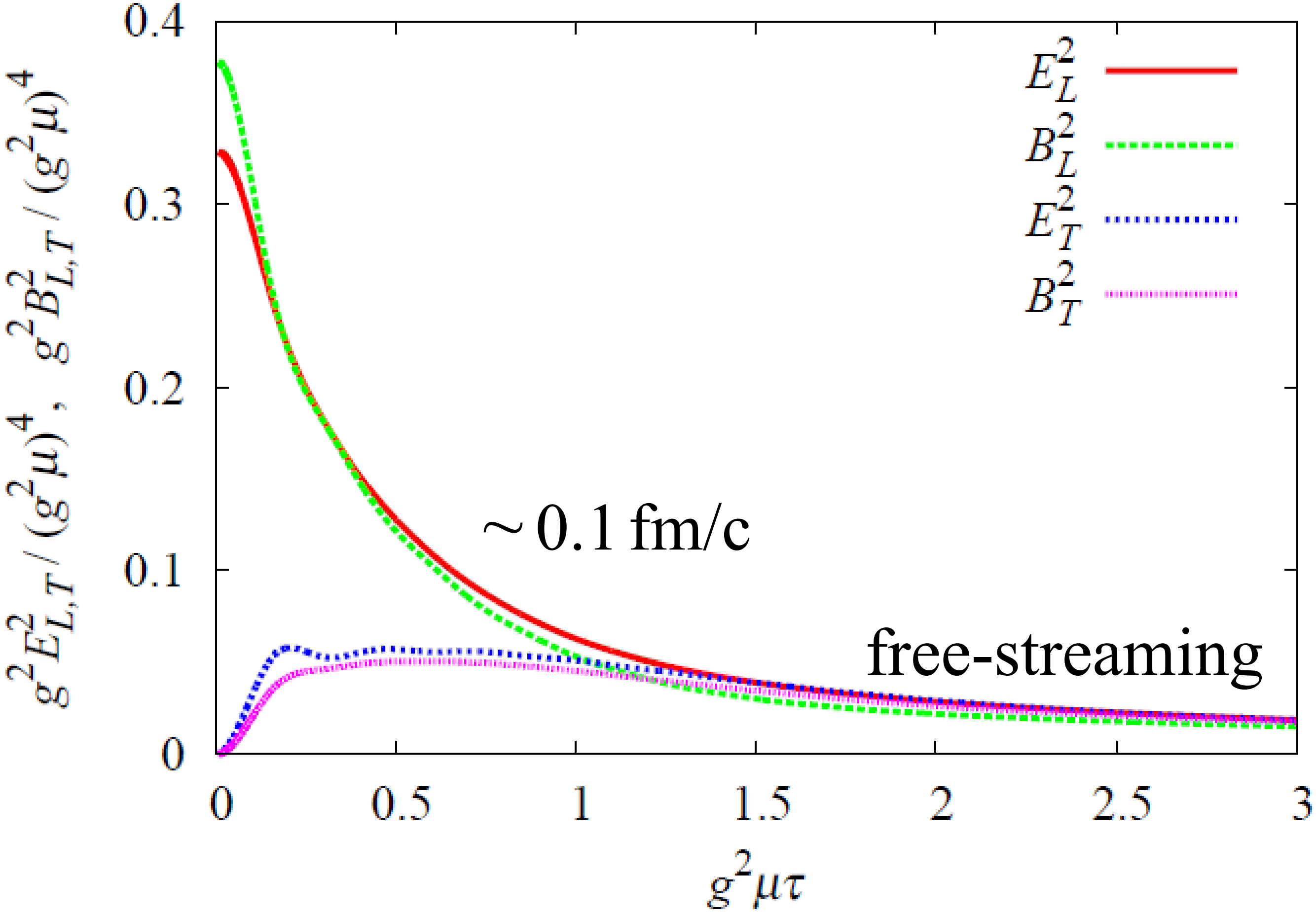
Demerit

Bjorken x is not fixed uniquely

Dirty environment unlike *ep* or *eH*

Chromo-Electric and Magnetic Fields

Longitudinal and Transverse Fields

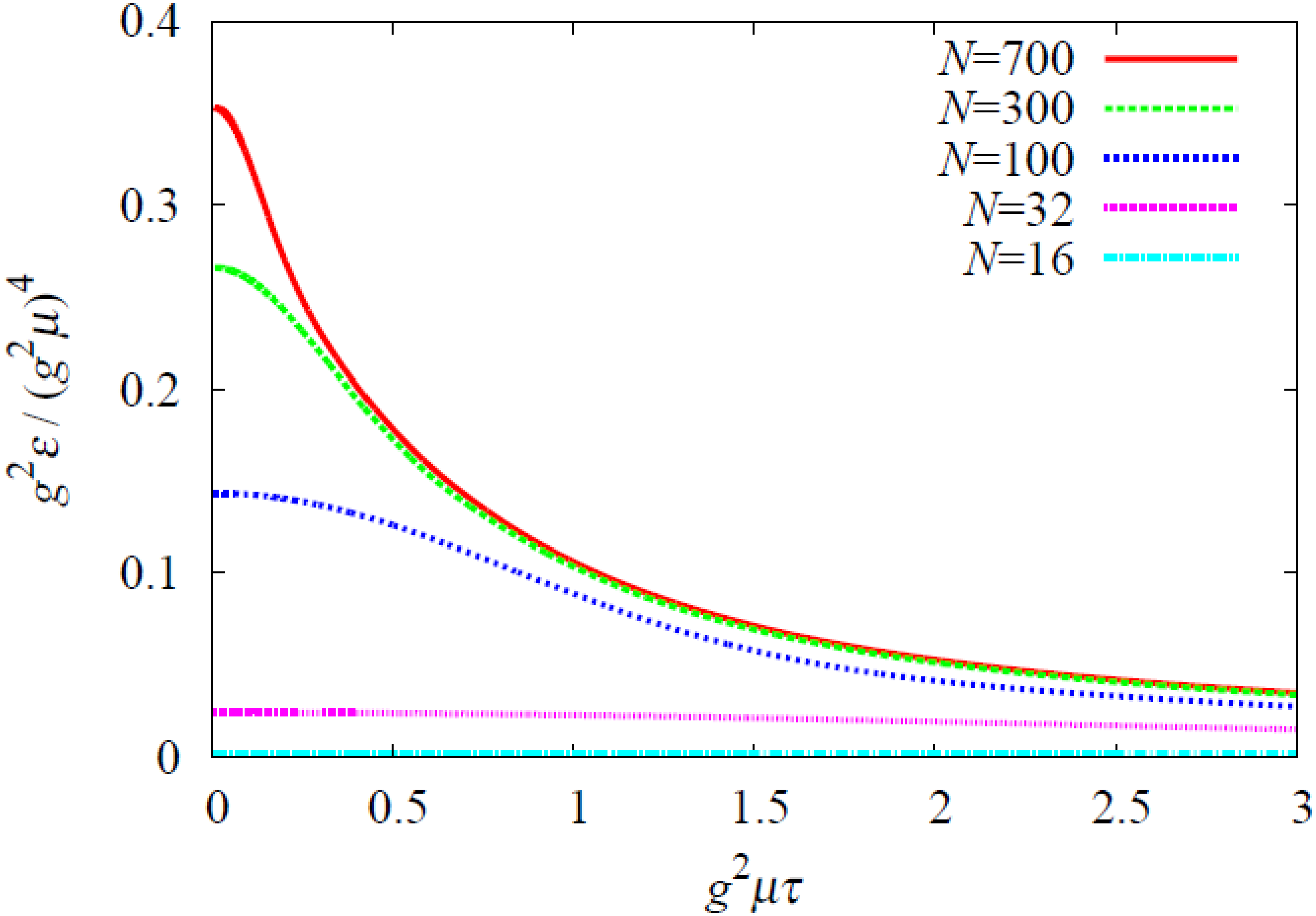


Lappi-McLerran (2006)
Fukushima-Gelis (2011)

Time evolution of fields after averaging over 30 configurations

Ultra-violet Stability

Numerical results for different site-number *N*



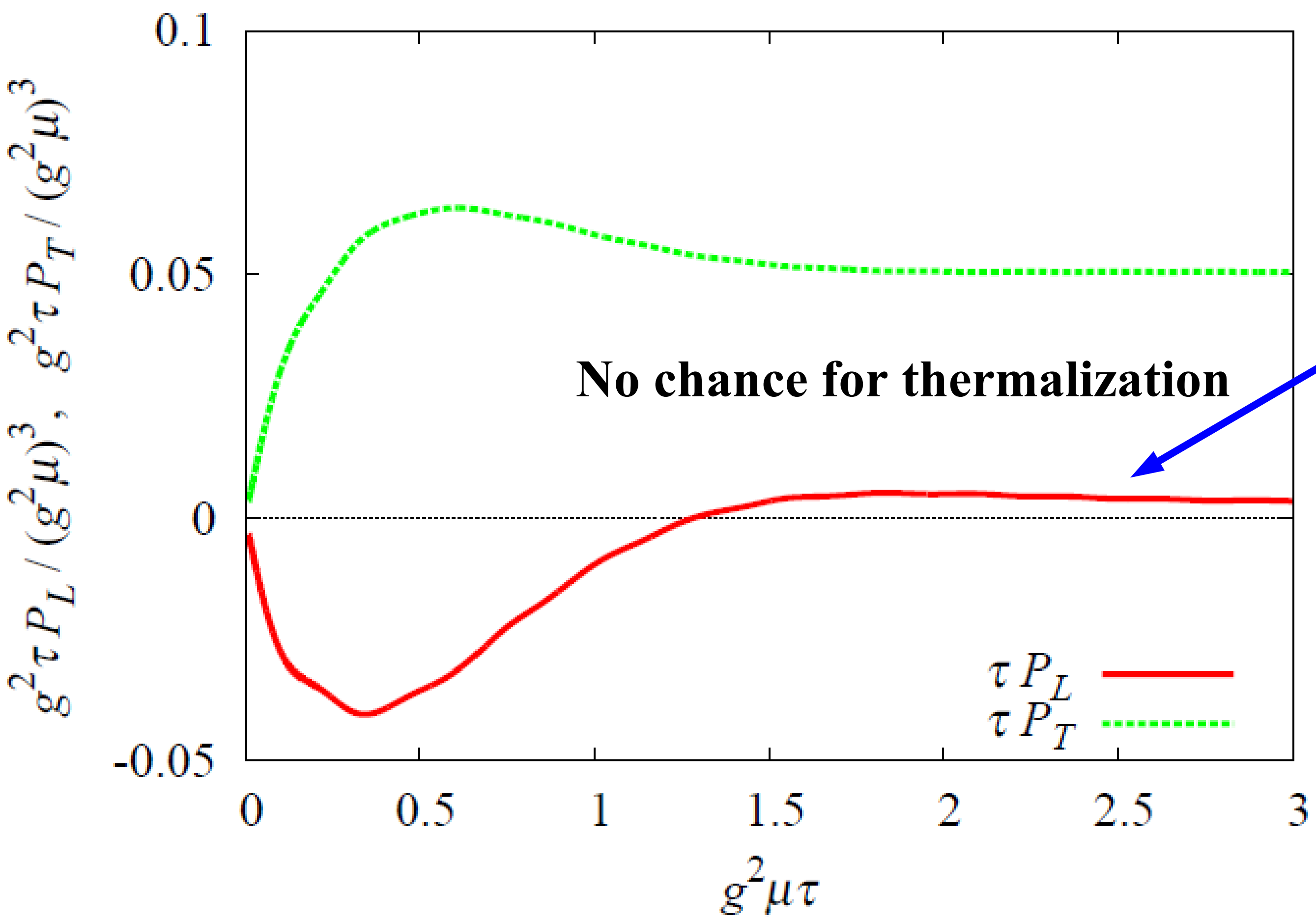
Results stabilized quickly at finite τ

First discussed by Lappi (2007)

Longitudinal and Transverse Pressure

$$P_T = \frac{1}{2}\left\langle T^{xx} + T^{yy} \right\rangle = \left\langle \mathrm{tr}\left[E_L^2 + B_L^2\right] \right\rangle ,$$

$$P_L = \left\langle \tau^2 T^{\eta\eta} \right\rangle = \left\langle \mathrm{tr}\left[E_T^2 + B_T^2 - E_L^2 - B_L^2\right] \right\rangle$$



(Almost) free-streaming

If thermalized

$P_T = P_L$

Fukushima-Gelis (2011)

Summary of Part I

The idea and the formalism of the Color Glass Condensate (CGC) was introduced.

Gaussian approximation for the CGC weight is the McLerran-Venugopalan model.

The initial condition right after the heavy-ion collision is given by the CGC which describes boost-invariant longitudinal fields.

How to reach thermalization??? → Lecture II
Answer is not known yet, but some progresses

Part II

Introduction to the Glasma and its instability and the resulting spectrum

Glasma

Glasma = (Color) **Glass** (Condensate) + **Plasma**

Initial State ~0.1fm/c

Color Glass Condensate

Coherent (pure) state far from thermal equilibrium

Transient State – CGC decaying to Plasma

Microscopic mechanism for thermalization still lacks a clear understanding

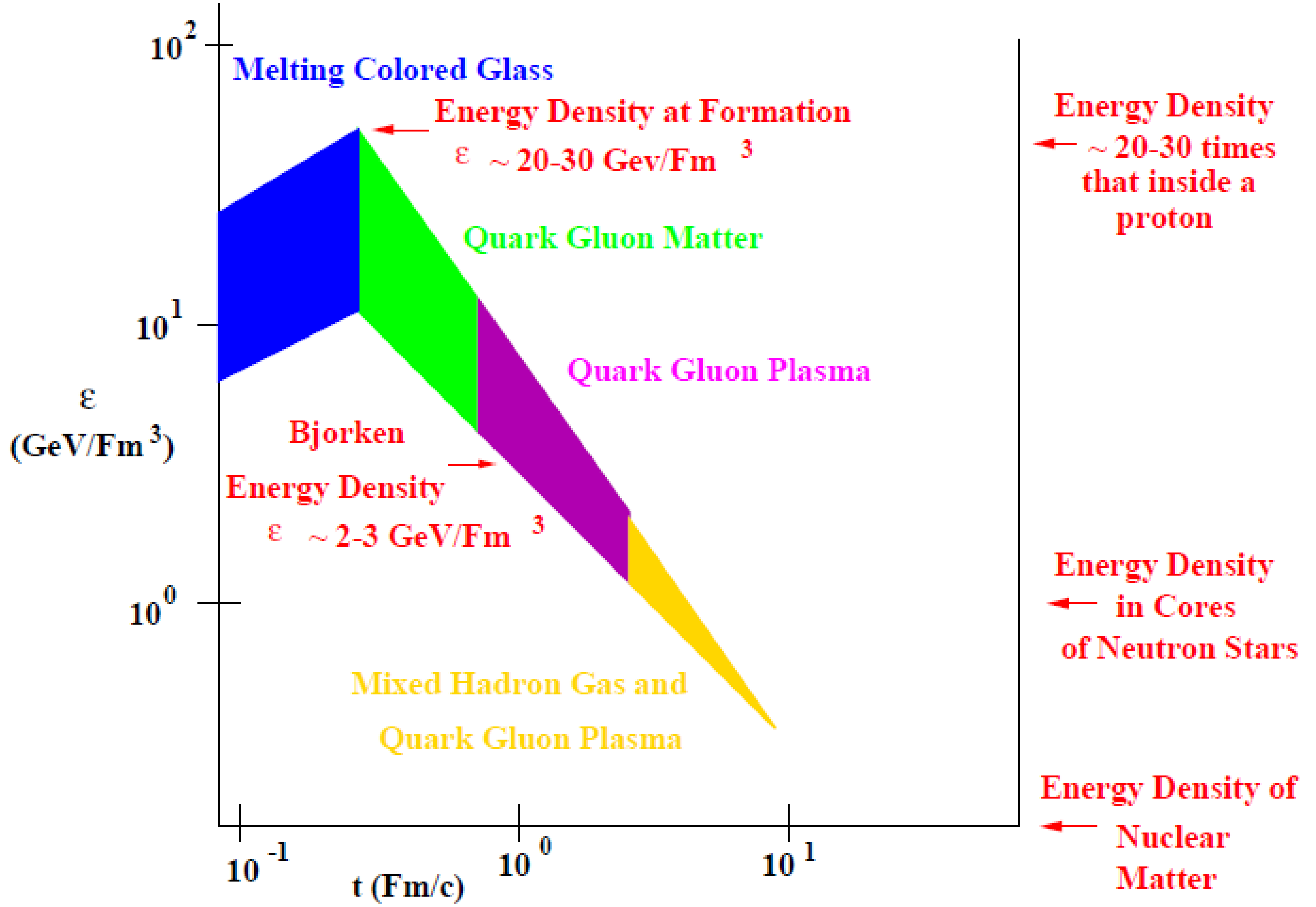
Final State ~0.6fm/c

Plasma

Local thermal equilibrium (mixed state) realized

Larry's Picture

Melting Colored Glass = Glasma



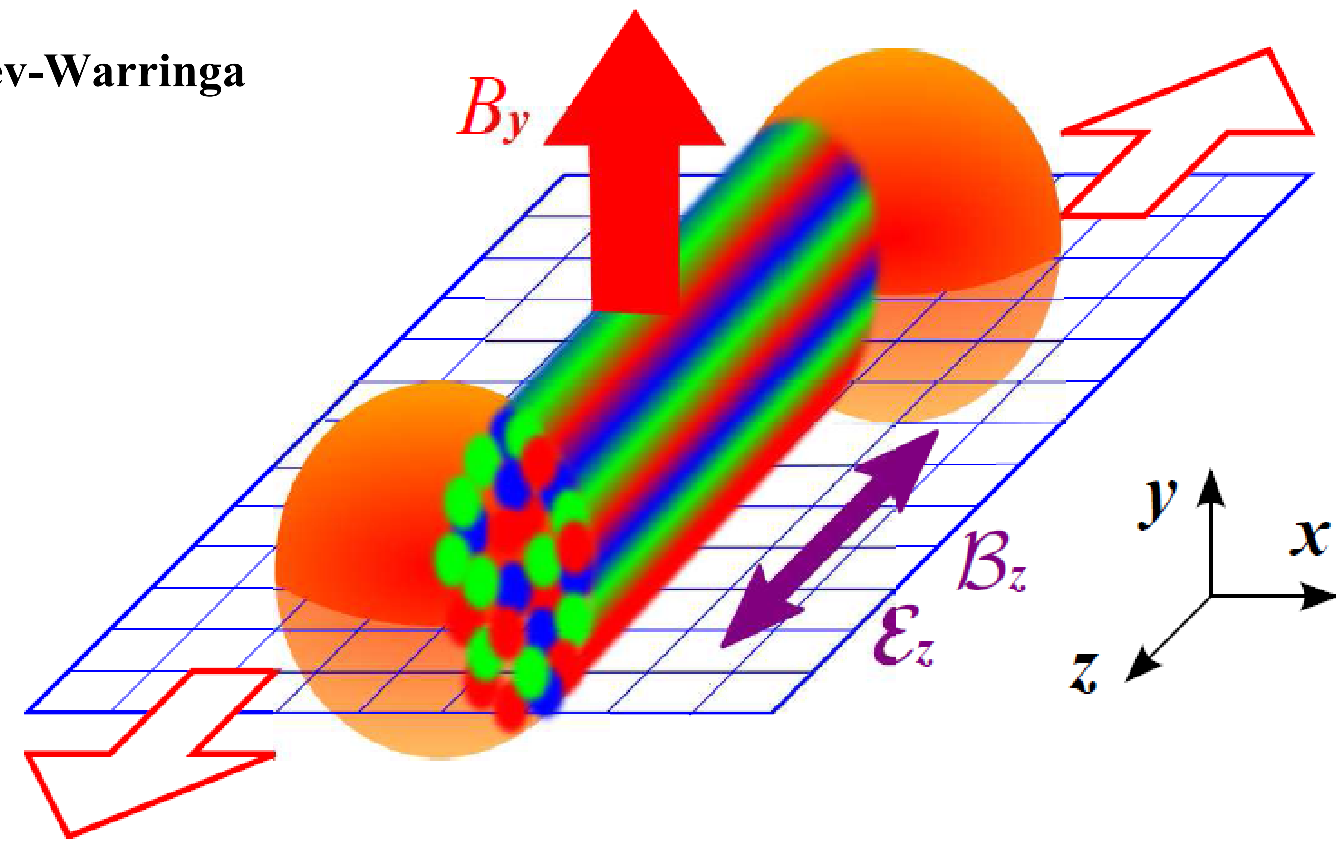
Glasma Characteristics

Very Strong Longitudinal Fields

Negative longitudinal pressure

Topological charge density → Chiral magnetic effect

KF-Kharzeev-Warringa



Chiral Magnetic Effect (CME)

Topological Charge Density

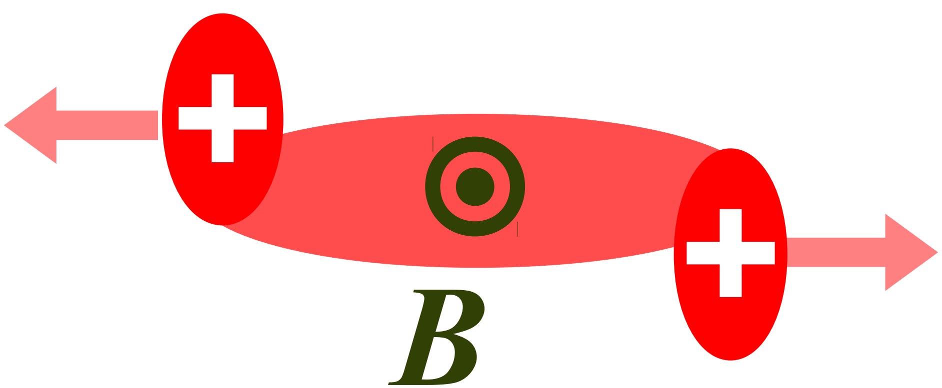
$$\langle\theta|\theta\rangle \to \quad S_{\mathrm{QCD}} = -\frac{1}{2g^2}\mathrm{tr}\,F_{\mu\nu}F^{\mu\nu} + \theta\frac{1}{16\pi^2}\mathrm{tr}\,F_{\mu\nu}\tilde{F}^{\mu\nu}$$

$$F_{\mu\nu}\tilde{F}^{\mu\nu} = 2\boldsymbol{E}\cdot\boldsymbol{B}$$

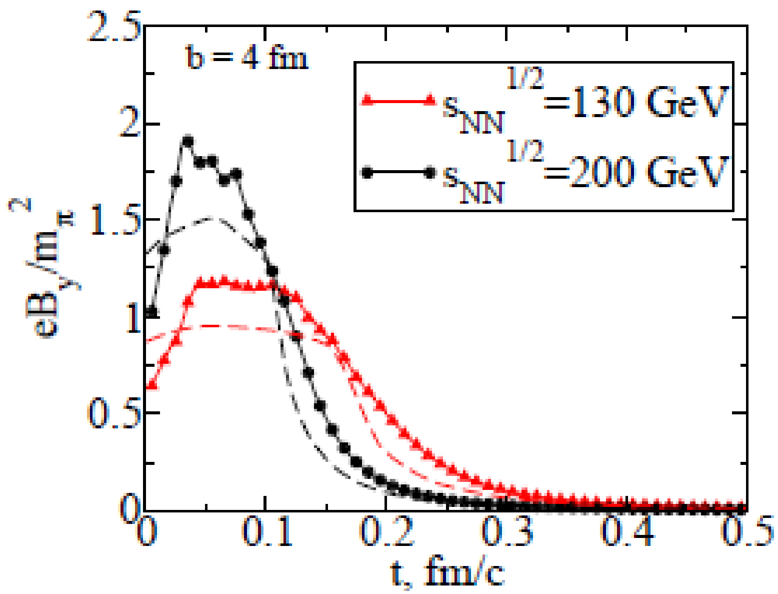
$\mathcal{P}$ and $\mathcal{CP}$ odd

This θ term effectively arises in the Glasma

(QED) Magnetic Field



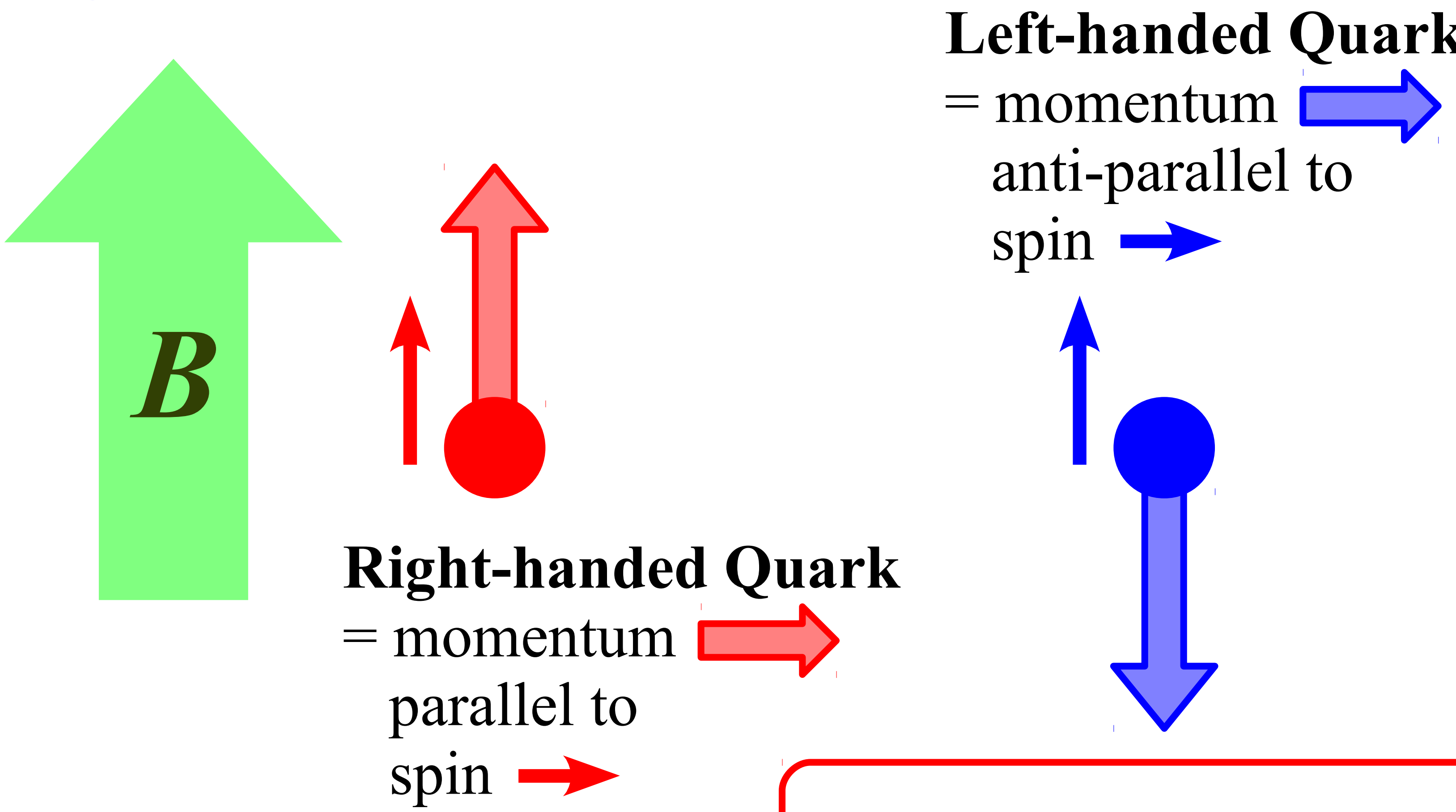
$$eB \sim m_\pi^2 \quad (B \sim 10^{18}\,\mathrm{G})$$



(Skokov-Illarionev-Toneev)

CME-induced Current

Classical Picture



$$\boldsymbol{J} \neq 0 \quad \text{if} \quad N_5 = N_R - N_L \neq 0$$

Kharzeev-McLerran-Warringa (2007)
Fukushima-Kharzeev-Warringa (2008)

Anomaly Relations

Induced N_5 by Topological Effects

$$\frac{dN_5}{dt} = -\frac{g^2 N_f}{8\pi^2}\int d^3x \;\mathrm{tr}\, F_{\mu\nu}\tilde{F}^{\mu\nu}$$

QCD Anomaly Relation

Introduce μ_5 to describe induced N_5

Induced J by the presence of N_5 and B

$$\boldsymbol{j} = \frac{e^2 \mu_5}{2\pi^2}\boldsymbol{B}$$

QED Anomaly Relation

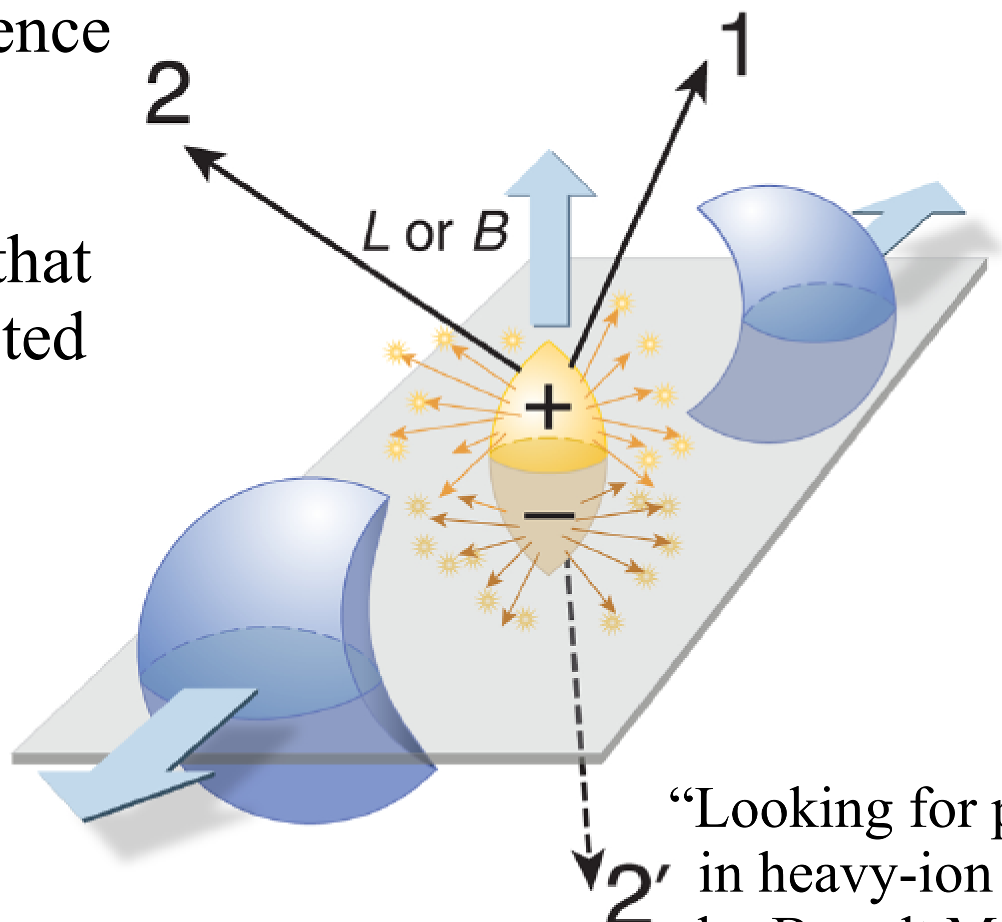
$$\left(\boldsymbol{j} = \sum_{i=\mathrm{flavor}} \frac{q_i^2 \mu_5}{2\pi^2}\boldsymbol{B} \quad \text{in QCD} \right)$$

Metlitski-Zhitnitsky (2005)
Fukushima-Kharzeev-Warringa (2008)

Charge Separation

No experimental evidence has been found so far.

(Once it was claimed that the STAR data suggested the CME, but now...)

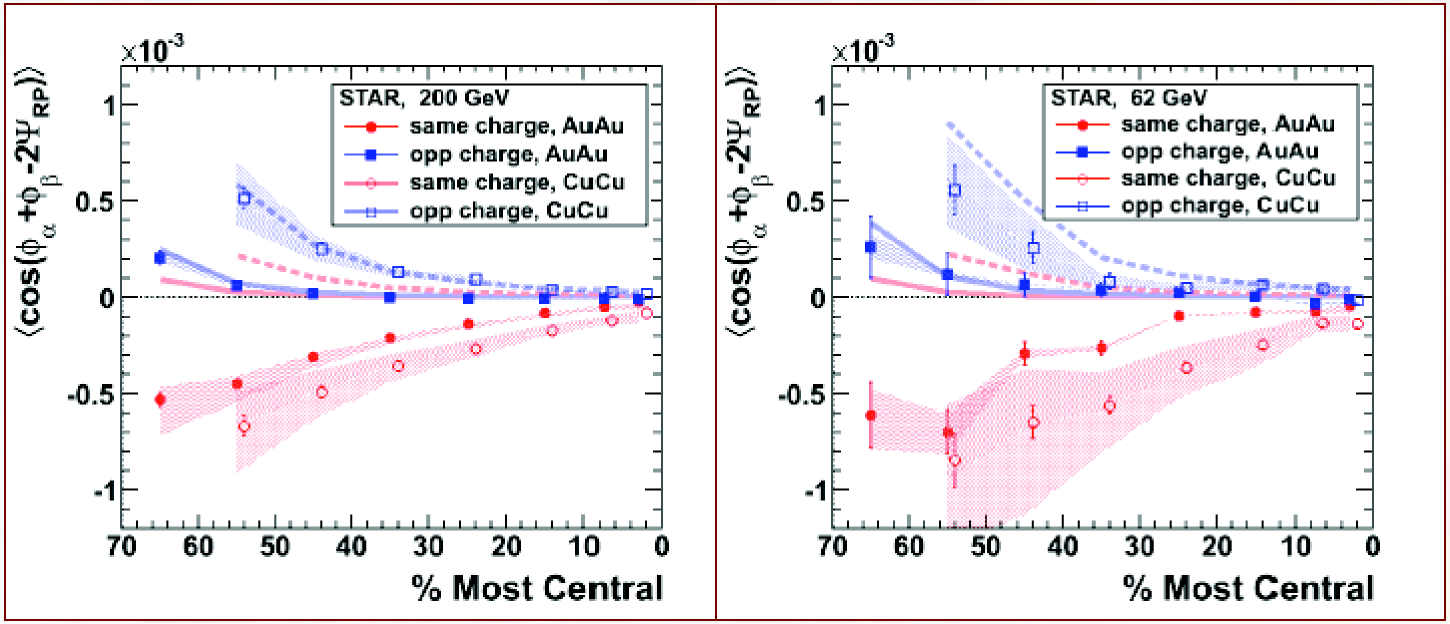


"Looking for parity violation in heavy-ion collisions"
by Berndt Müller
Physics 2, 104 (2009)

"Former" Evidence

No longer a clear evidence...

$$\langle\langle \cos(\Delta\phi_\alpha + \Delta\phi_\beta)\rangle\rangle \equiv \left\langle\left\langle \frac{1}{N_\alpha N_\beta}\sum_{i=1}^{N_\alpha}\sum_{j=1}^{N_\beta}\cos(\Delta\phi_{\alpha,i}+\Delta\phi_{\beta,j}) \right\rangle\right\rangle$$
$$= \langle\langle \cos\Delta\phi_\alpha \cos\Delta\phi_\beta\rangle\rangle - \langle\langle \sin\Delta\phi_\alpha \sin\Delta\phi_\beta\rangle\rangle$$
$$= \left(\langle\langle v_{1,\alpha}v_{1,\beta}\rangle\rangle + B^{\rm in}_{\alpha\beta}\right) - \left(\langle\langle a_\alpha a_\beta\rangle\rangle + B^{\rm out}_{\alpha\beta}\right).$$



STAR
also
ALICE

Glasma – Stable or Unstable?

Boost invariant solution is stable

Something missing

(Anisotropic) Plasma is unstable

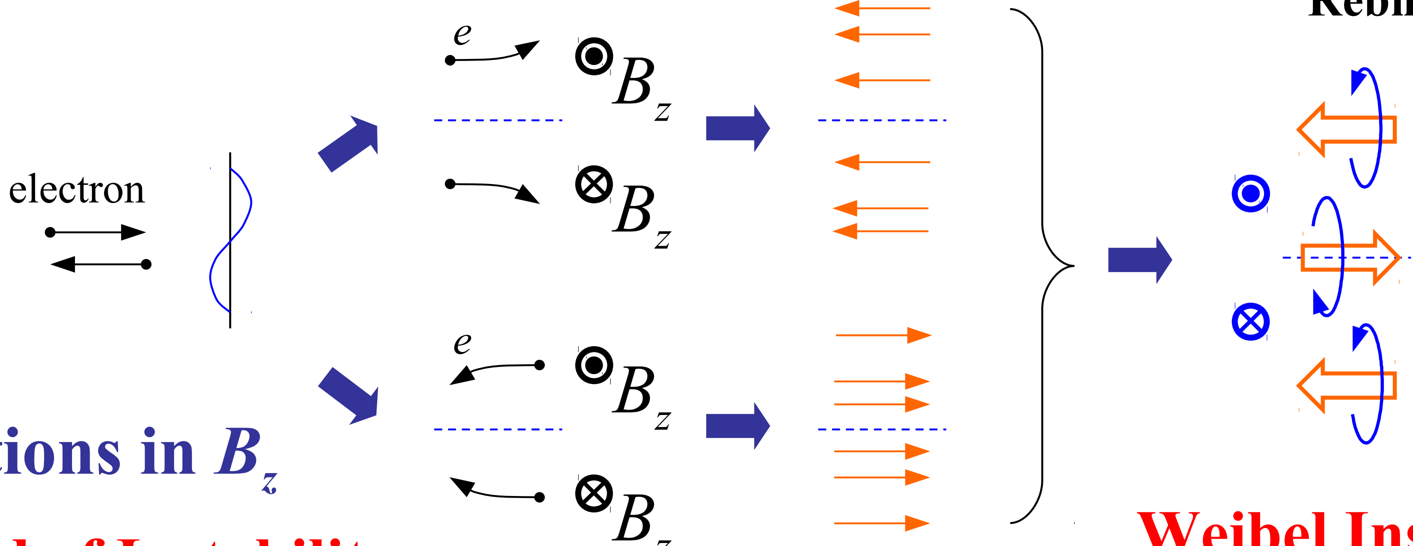
c.f. Classical YM eq.
is unstable (chaotic)

Bolte-Muller-Schafer (1999)
Kunihiro-Muller-et al. (2010)

electron motion

current density

Mrowczynski
Rebhan, Berges, etc



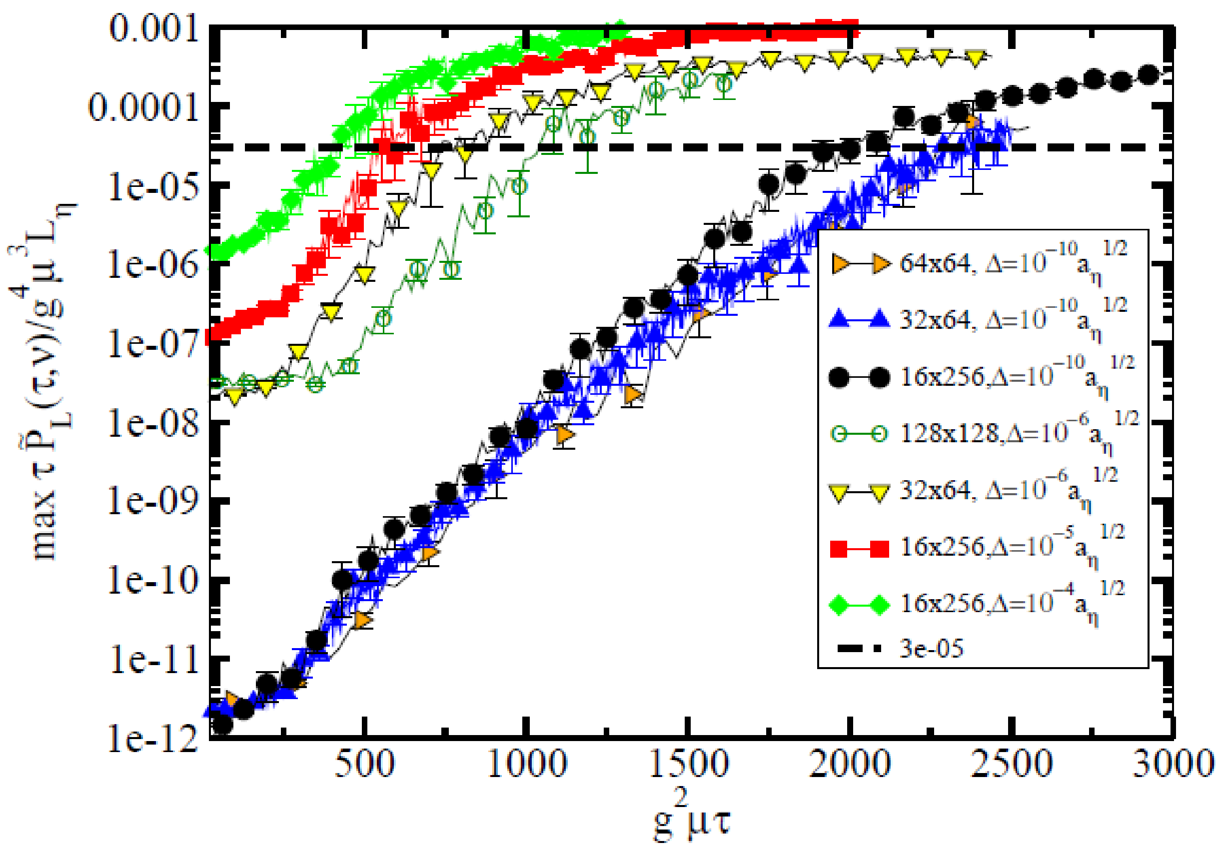
Fluctuations in B_z

Seed of Instability

Weibel Instability

Unstable Glasma

Important Hint – Glasma Instability



Add η-dependent fluct.
η-dependent modes grow exponentially.

$\sim \exp[C g^2 \mu \tau]$ (non-expanding)

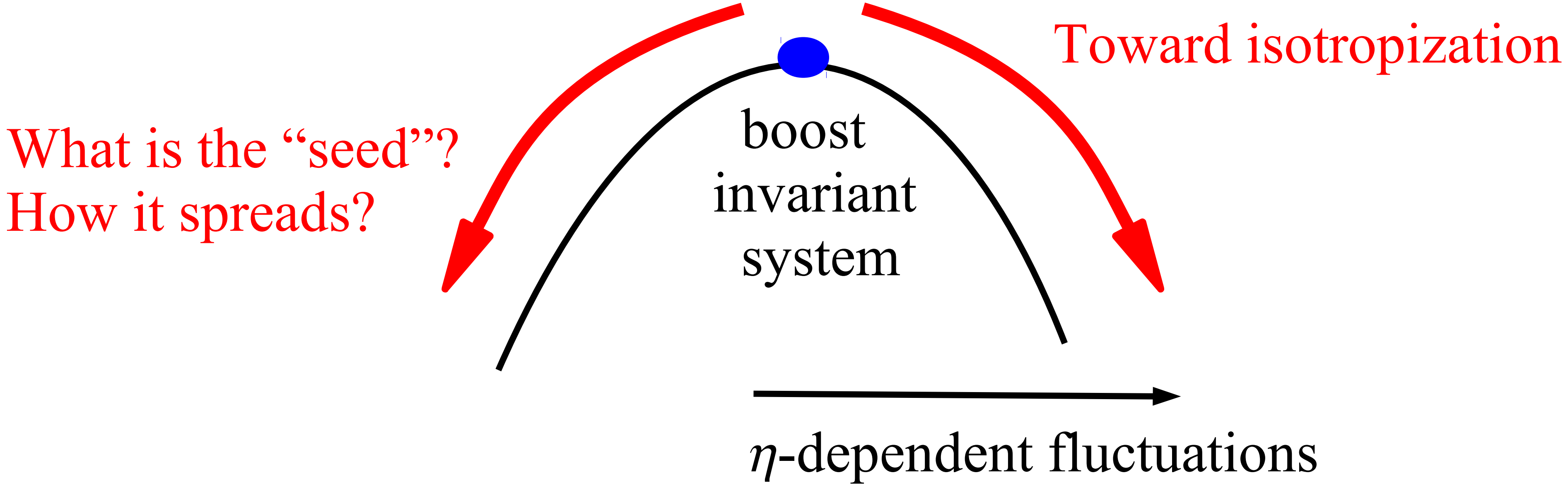
$\sim \exp[C \sqrt{g^2 \mu \tau}]$ (expanding)

Instability time-scale is too slow
System size dep. not under control

Romatschke-Venugopalan (2005)

Boost Inv. Violation

Boost-invariant Glasma sits on the top of the potential maximum (seemingly stable without any perturbation)



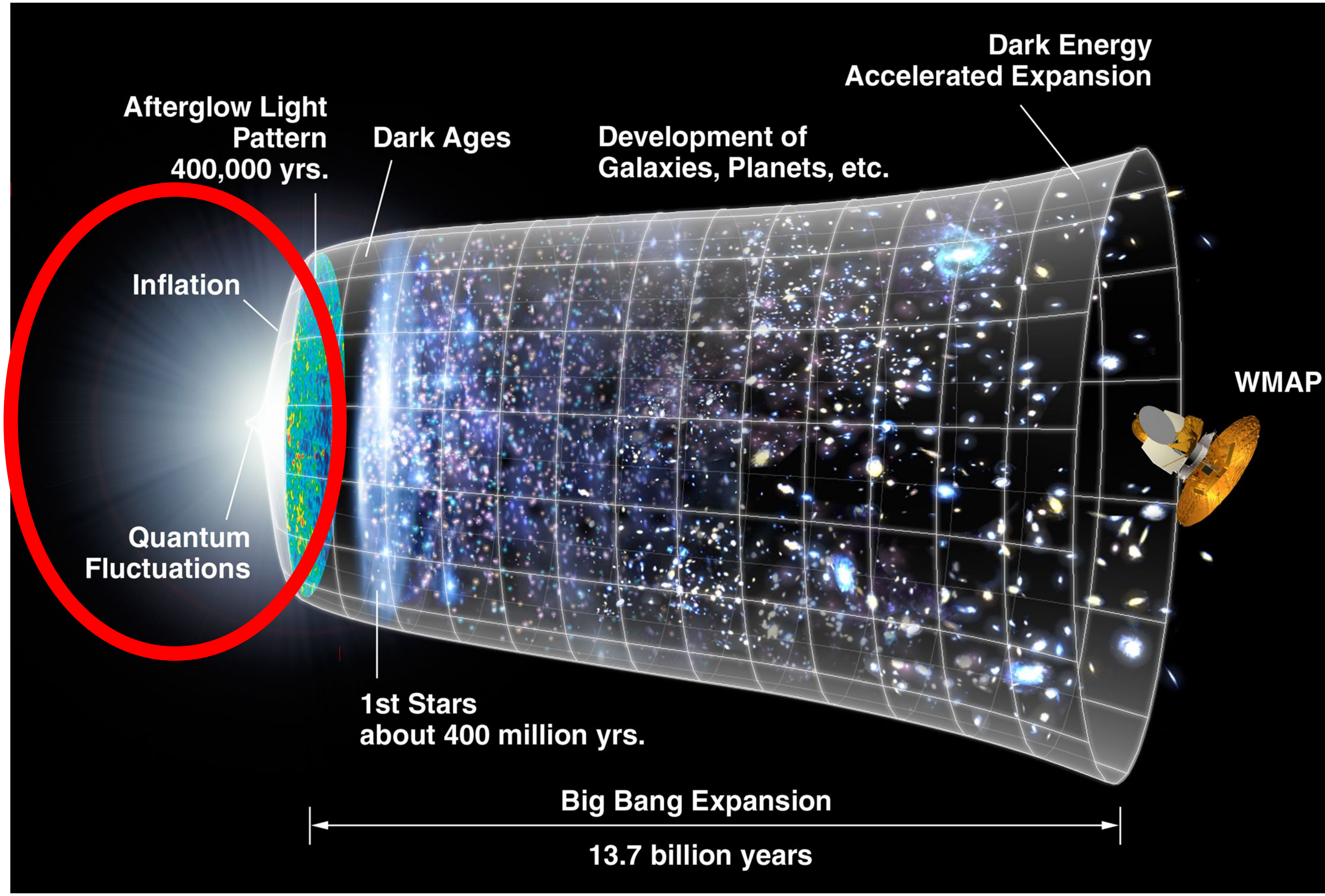
Isotropization does not necessarily mean thermalization.
If thermalized, the system must be isotropic.

Cosmology Analogue

Quantum Fluctuations

Inflation
(Instability)

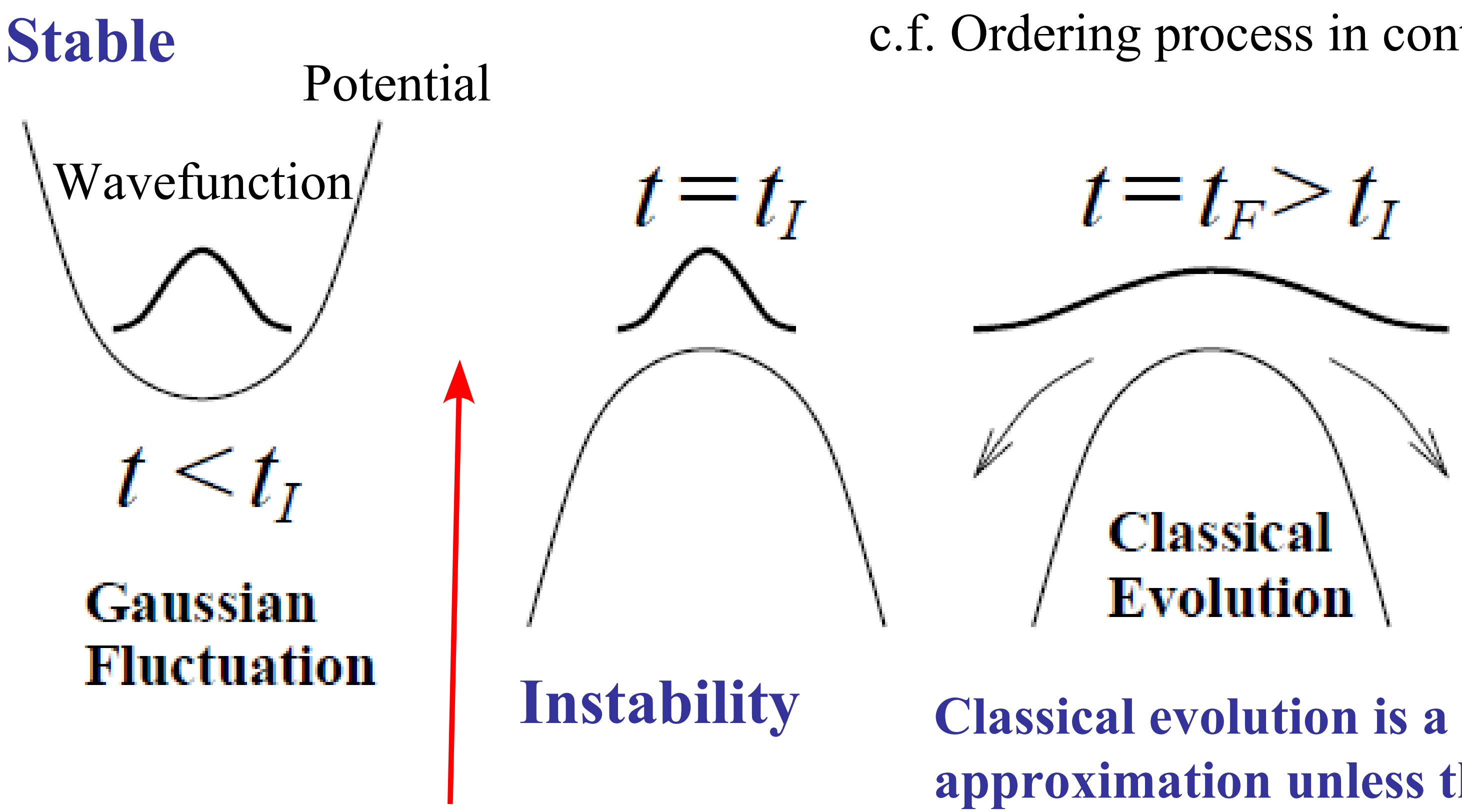
Reheating
(Thermalization)



Fluctuations and Instability

Time Evolution of Fluctuations under Instability

c.f. Ordering process in continuous transition



Classical evolution is a good approximation unless the instability is weak (i.e. potential is flat).

Singularity
depending on the problem

Formulation

Computed Physical Observables

$$\langle \mathcal{O} \rangle_\tau = \int [\mathrm{d}a_i\, \mathrm{d}a_\eta\, \mathrm{d}e^i\, \mathrm{d}e^\eta]\, \underline{W[a,e]}\, \mathcal{O}\big[\mathcal{A}[\mathcal{A}_i + a_i, a_\eta, e^i, \mathcal{E}^\eta + e^\eta; \tau]\big]$$

Fluctuation Spectrum

Boost-inv. CGC Backgrounds

Microscopically derived by Gelis-Lappi-Venugopalan

Time Evolution

$$\mathcal{E}^i = \tau \partial_\tau \mathcal{A}_i \qquad \mathcal{E}^\eta = \tau^{-1} \partial_\tau \mathcal{A}_\eta$$

$$\partial_\tau \mathcal{E}^i = -\frac{\delta(\tau H)}{\delta \mathcal{A}_i} = \frac{1}{\tau} \mathcal{D}_\eta \mathcal{F}_{\eta i} + \tau \mathcal{D}_j \mathcal{F}_{ji} \ ,$$

$$\partial_\tau \mathcal{E}^\eta = -\frac{\delta(\tau H)}{\delta \mathcal{A}_\eta} = -\frac{1}{\tau} \mathcal{D}_j \mathcal{F}_{\eta j} - \frac{\tau}{2} \underline{\left[\frac{1}{x^+} \rho_1\, \delta(x^-) - \frac{1}{x^-} \rho_2\, \delta(x^+) \right]}$$

Collision Singularity

KF-Gelis-McLerran (2006)

Quantum Fluctuations before Singularity

Zero-Point Oscillation of Empty Steady State from Infinite Past

Linearized Schroedinger Equation

$$\int d\eta\, d^2x_\perp\, \mathrm{tr}\Big\{ -\frac{1}{\tau}\frac{\delta^2}{\delta a_i^2} - \tau\frac{\delta^2}{\delta a_\eta^2} + \frac{1}{\tau}(\partial_\eta a_i - \mathcal{D}_i a_\eta)^2 + \frac{\tau}{2}\Big[(\mathcal{D}_i a_j - \mathcal{D}_j a_i)^2 - 2ig\mathcal{F}_{ij}[a_i, a_j]\Big]\Big\}\Psi_-[A] = E\Psi_-[A]$$

Ground-state Wavefunction (without background A_i=0)

$$\Psi_-[A] = N\exp\Big\{-\int d\eta\, d^2x_\perp\, \mathrm{tr}\Big[a_i\,\tau\sqrt{-\Big(\frac{\partial_\eta}{\tau}\Big)^2 - \partial_\perp^2}\,\Big(\delta_{ij} - \frac{\partial_i\partial_j}{(\frac{\partial_\eta}{\tau})^2 + \partial_\perp^2}\Big)a_j\Big]\Big\}$$

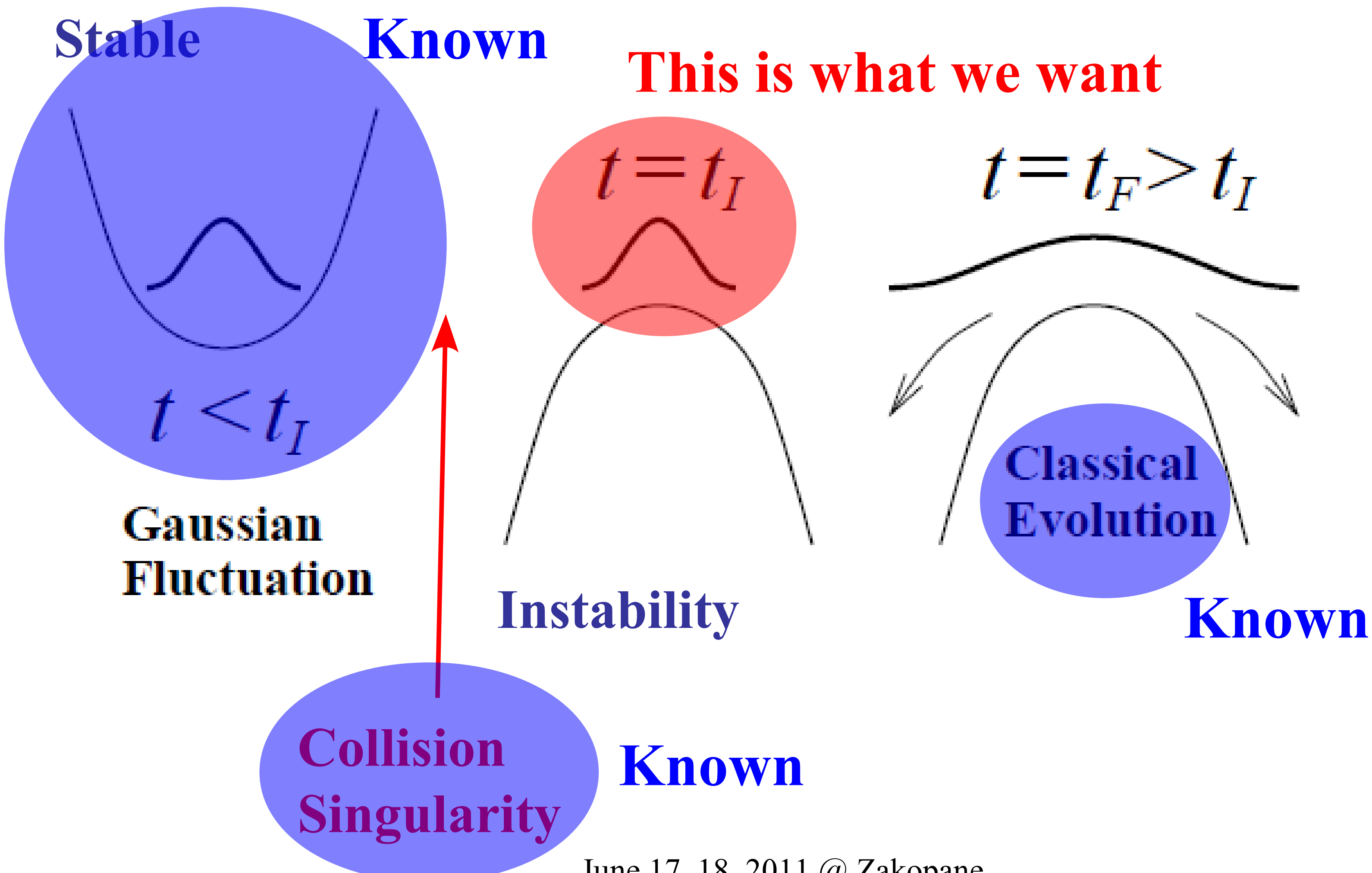
Gauss Law $\quad E^\eta = -\frac{1}{\partial_\eta}D_i E^i$

Zero-Point Oscillation Spectrum

KF-Gelis-McLerran (2006)

Fluctuations and Instability

Time Evolution of Fluctuations under Instability



Quantum Fluctuations after Singularity

Collision singularity simply shifts the fields by the CGC backgrounds

$$
\begin{aligned}
&W[a,e] \\
&= N\int [\mathrm{d}\delta a]\;\Psi_+^*[a-\tfrac{1}{2}\delta a]\,\Psi_+[a+\tfrac{1}{2}\delta a]\,\mathrm{e}^{-\mathrm{i}\int \mathrm{d}\eta\,\mathrm{d}^2\boldsymbol{x}_\perp\,2\mathrm{tr}[e^i\delta a_i]} \\
&= N\exp\Big\{-\int \mathrm{d}\eta\,\mathrm{d}^2\boldsymbol{x}_\perp\,2\mathrm{tr}\Big[a_i\,\tau\sqrt{-(\partial_\eta/\tau)^2-\boldsymbol{\partial}_\perp^2}\,\Big(\delta_{ij}-\frac{\partial_i\partial_j}{(\partial_\eta/\tau)^2+\boldsymbol{\partial}_\perp^2}\Big)\,a_j \\
&\qquad + e^i\frac{1}{\tau\sqrt{-(\partial_\eta/\tau)^2-\boldsymbol{\partial}_\perp^2}}\Big(\delta_{ij}+\frac{\partial_i\partial_j}{(\partial_\eta/\tau)^2}\Big)\,e^j\Big]\Big\} \\
&\qquad \times\delta[a_\eta]\,\delta\Big[e^\eta-e_0^\eta+\int_{\eta_0}^{\eta}\mathrm{d}\eta\,\mathcal{D}_ie^i\Big]\,,
\end{aligned}
$$

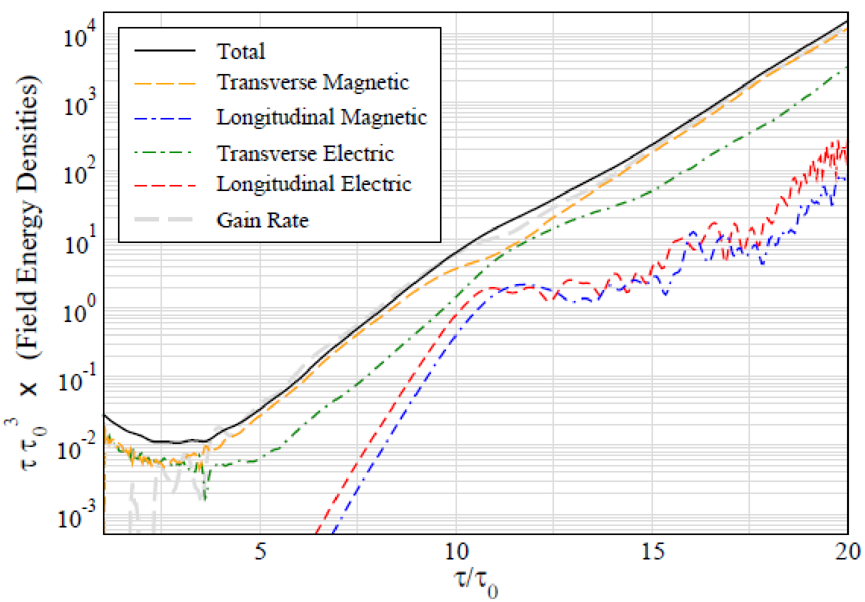
Uncertainty Principle

Gauss Law

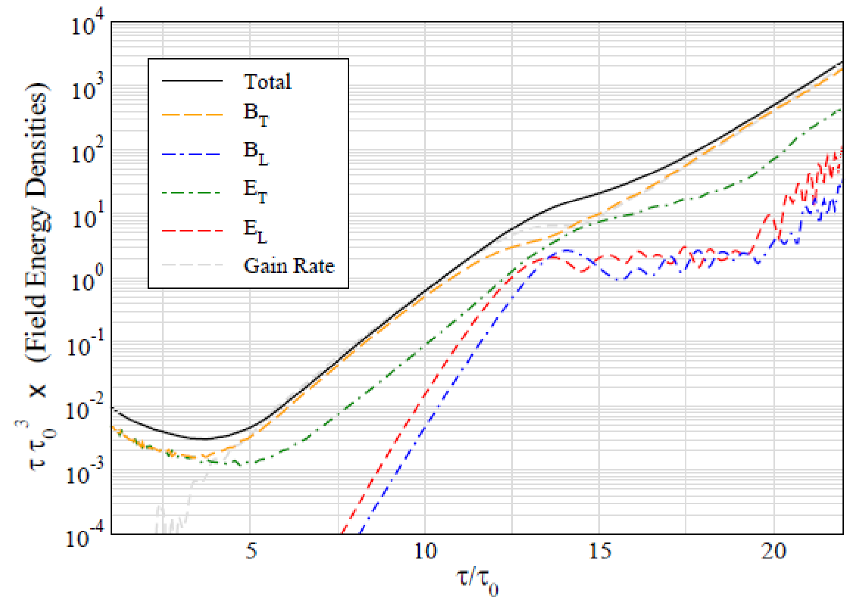
Continuous Spectrum → Ultraviolet Divergence **Rebhan et al...**

Fluctuation Spectrum Dependence

HLE Results by Rebhan-Strickland-Attems



Random Noise



FGM Spectrum

Glasma Simulation

Rapidity-dependent fluctuation should fulfill the Gauss law constraint:

Procedure **(Romatschke-Venugopalan)**

Generate random distributions on the transverse plane

$$\langle \xi^i(\boldsymbol{x}_\perp)\xi^j(\boldsymbol{x}'_\perp)\rangle = \delta^{ij}\delta^{(2)}(\boldsymbol{x}_\perp - \boldsymbol{x}'_\perp)$$

Generate fluctuations of chromo-electric fields

$$\delta E^i(x) = a_\eta^{-1}\Big[f(\eta - a_\eta) - f(\eta)\Big]\,\xi^i(\boldsymbol{x}_\perp)\ ,$$

$$\delta E^\eta(x) = -f(\eta)\sum_{i=x,y}\Big[U_i^\dagger(x-\hat{i})\xi^i(\boldsymbol{x}_\perp - \hat{i})U_i(x-\hat{i}) - \xi^i(\boldsymbol{x}_\perp)\Big]$$

Automatically satisfies the Gauss law

$$D_i E^i + \partial_\eta \delta E^\eta = 0$$

Why not $\boldsymbol{\delta U}$?

Choice of the η-Fluctuations

Zero-Point Spectrum **KF-Gelis-McLerran**

Suffering from UV divergences

White Noise **Romatschke-Venugopalan**

Adopted in many calculations – but why white noise?

Single Mode **KF-Gelis**

$$f(\eta) = \Delta \cos\left(\frac{2\pi\nu_0}{L_\eta}\eta\right)$$

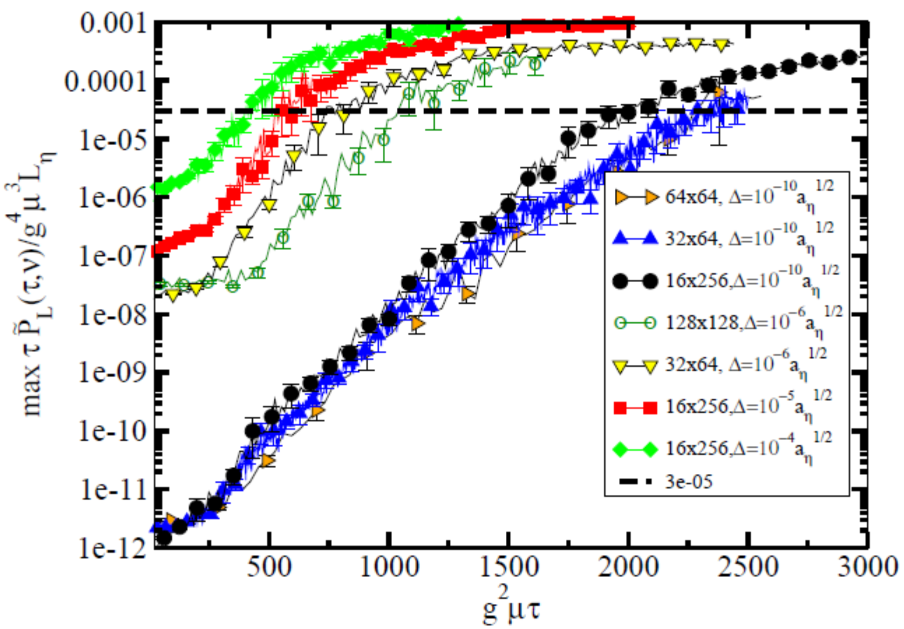
Clean environment to see the instability
As long as in the linear-regime (small fluctuations)
any spectrum can be described by a superposition
of single-mode results with different ν_0 's

Single-mode Analysis

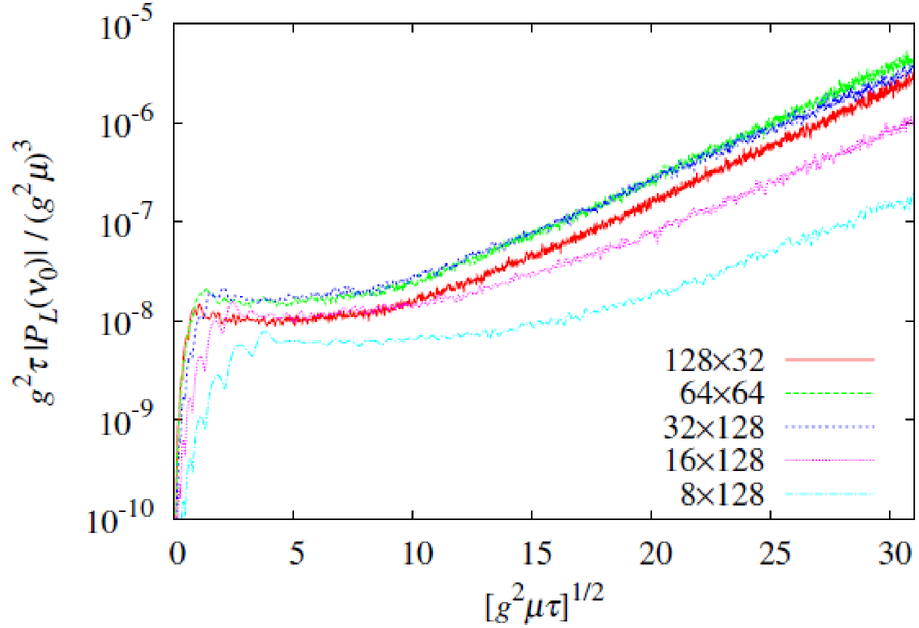
η-dependent Longitudinal Pressure

$$P_L(\nu) \equiv \frac{1}{L^2}\int d^2\boldsymbol{x}_\perp \frac{1}{L_\eta}\int_0^{L_\eta} d\eta\, P_L(\eta,\boldsymbol{x}_\perp)\, e^{i(2\pi\nu/L_\eta)\eta}$$

This quantity does not correspond to any "physical" observable but the same as adopted in the analysis by Romatschke-Venugopalan



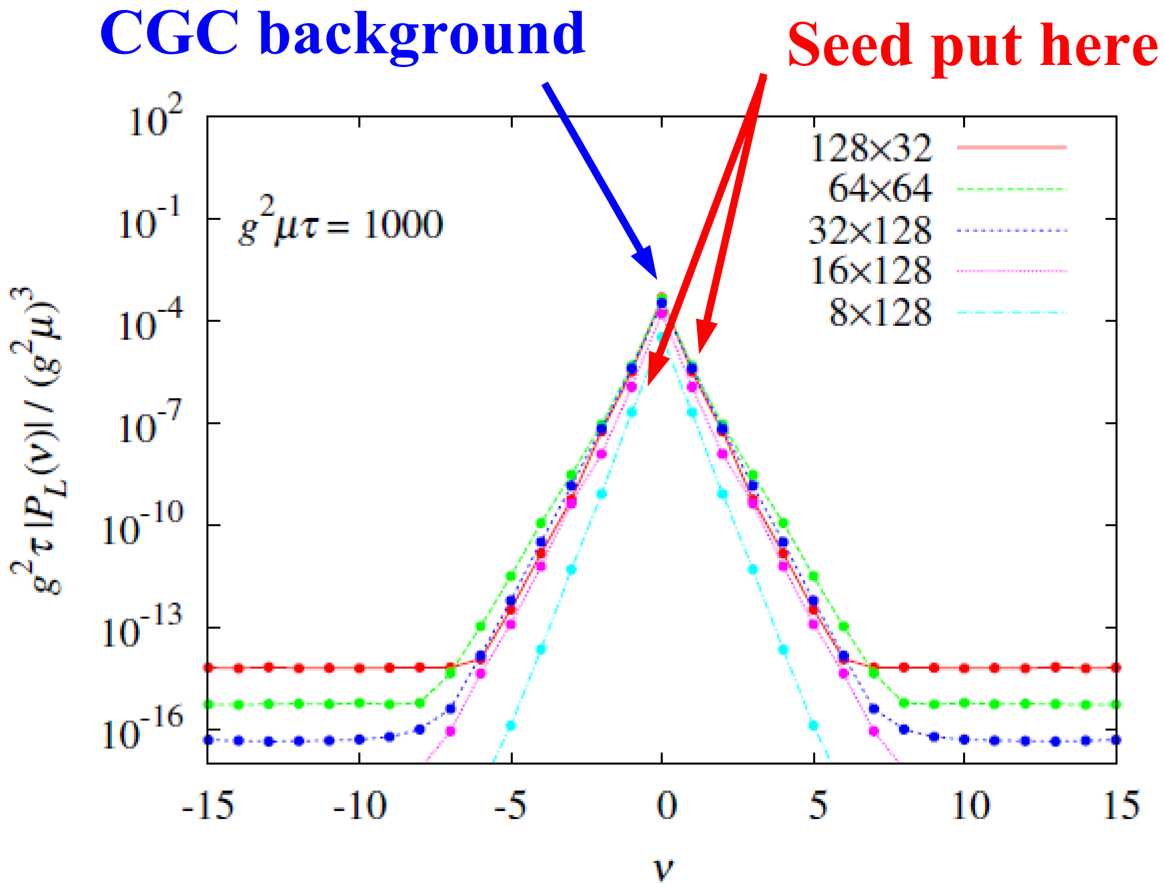
Romatschke-Venugopalan



Fukushima-Gelis

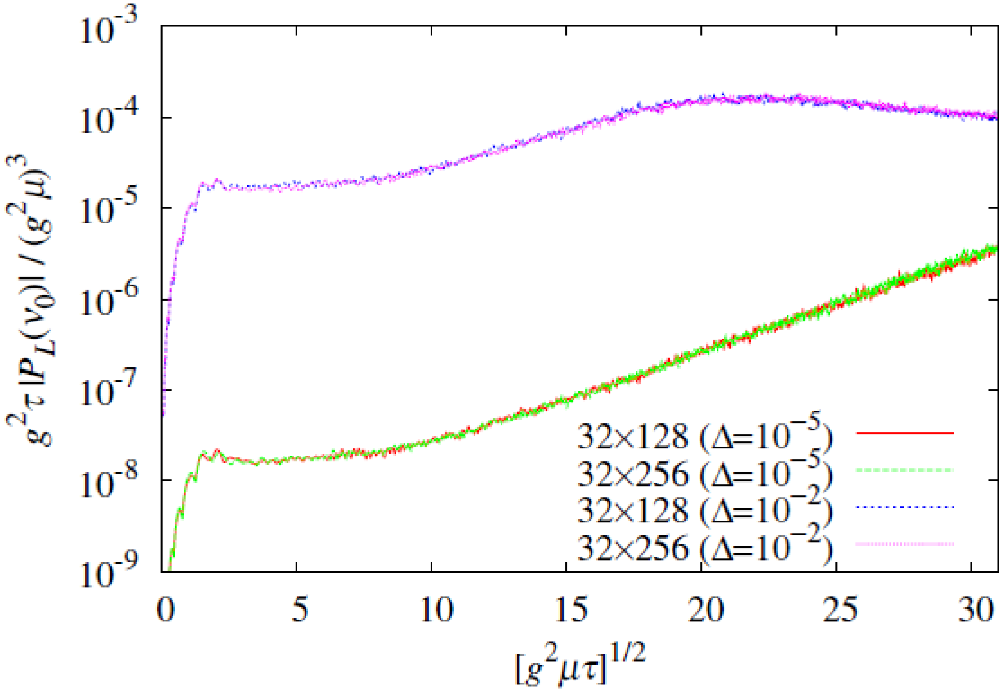
Mode Amplitudes

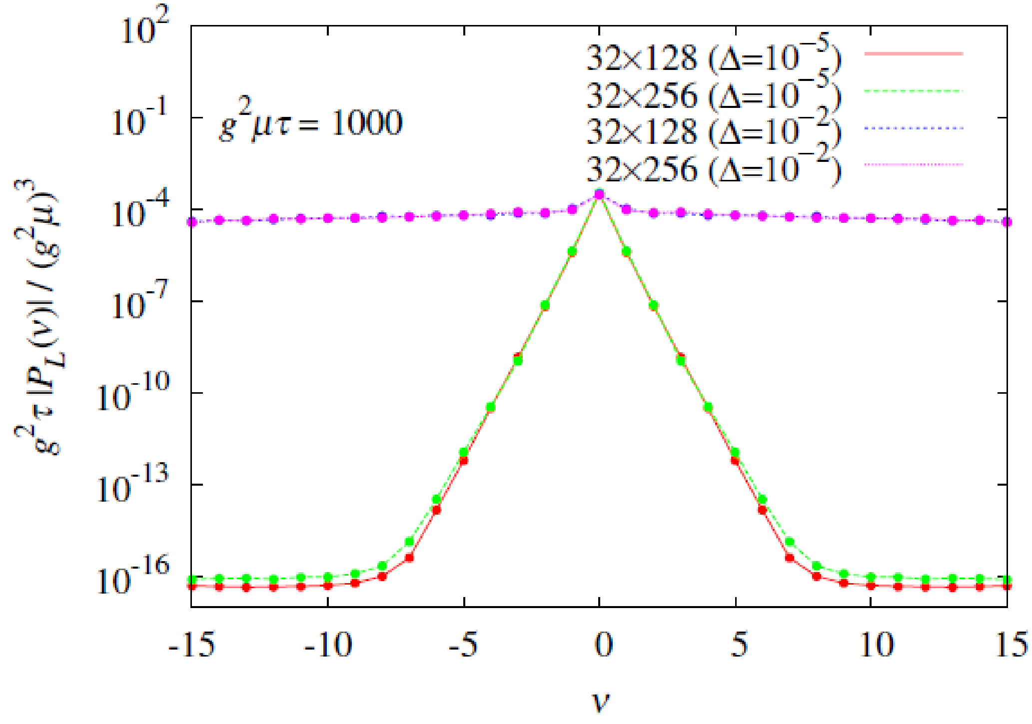
Instability spreads from lower to higher wave-number modes (canonical to rapidity)



Longitudinal Size Dependence

Completely free from artifacts

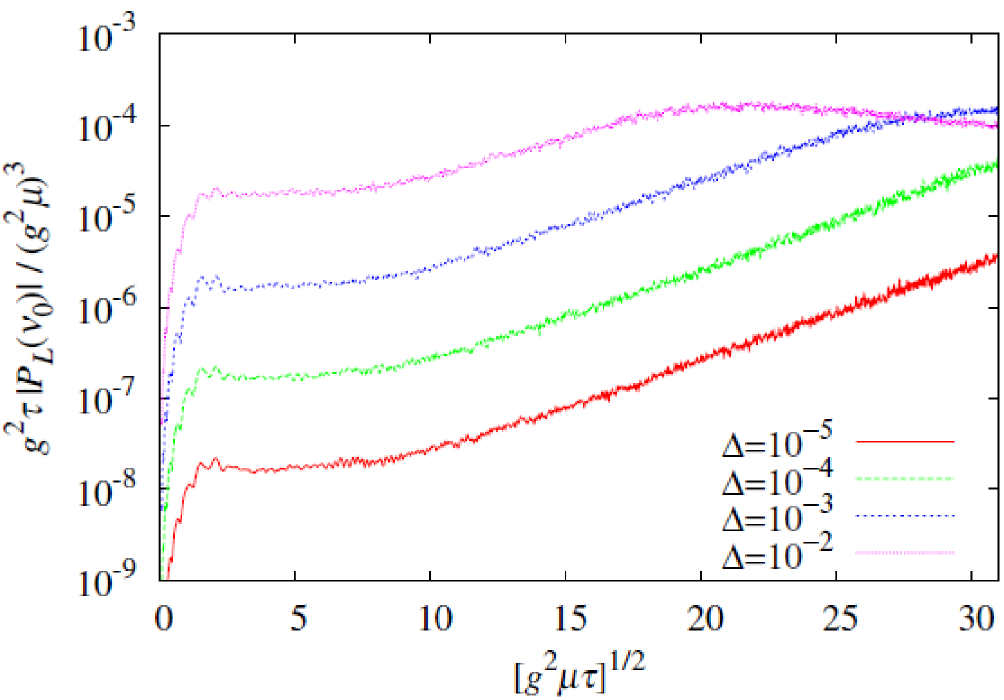


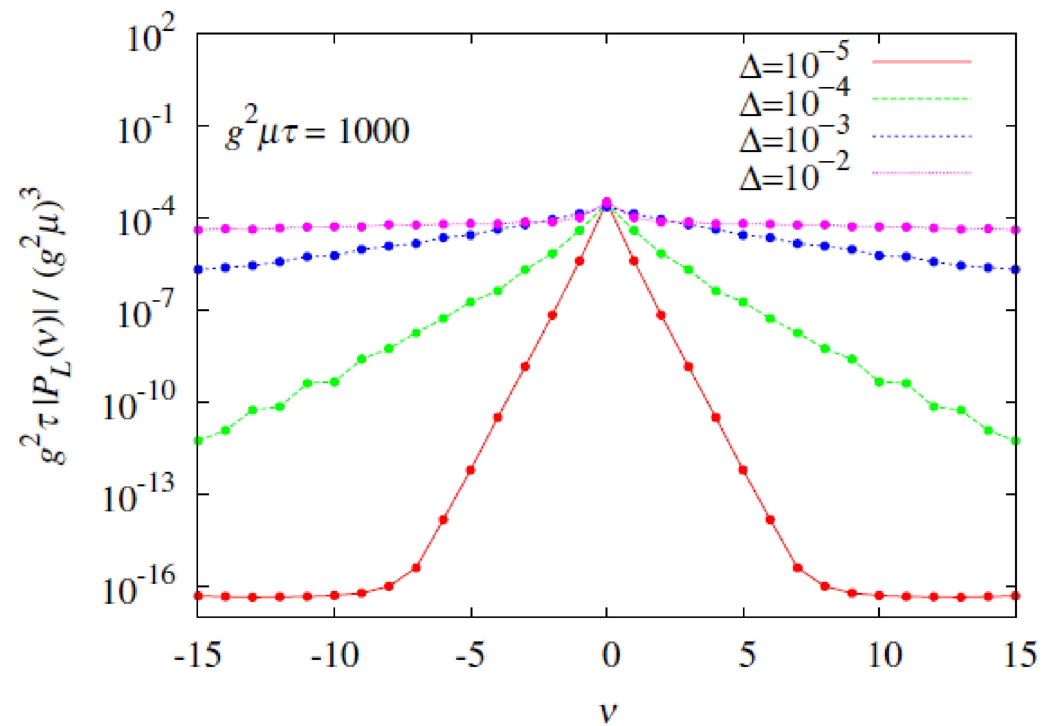


No dependence on the longitudinal size as it should not

Seed Magnitude Dependence

η-fluctuations are in the linear regime



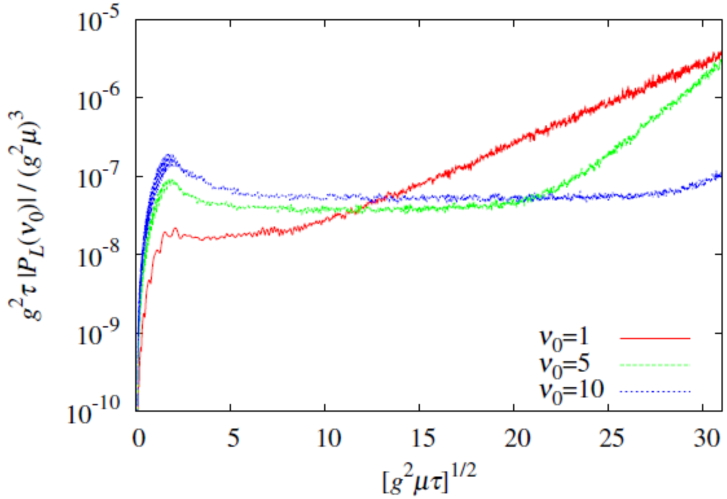


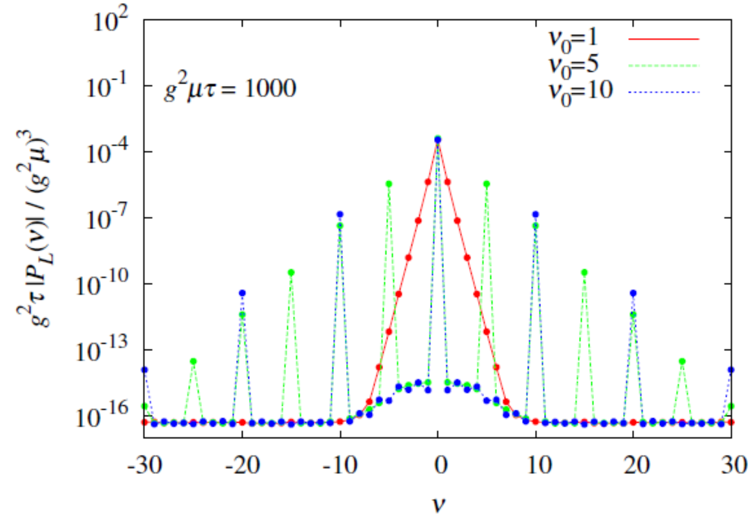
Simply proportional to the seed magnitude

How to fix it in principle? Ideally fixed by the thermalization time...

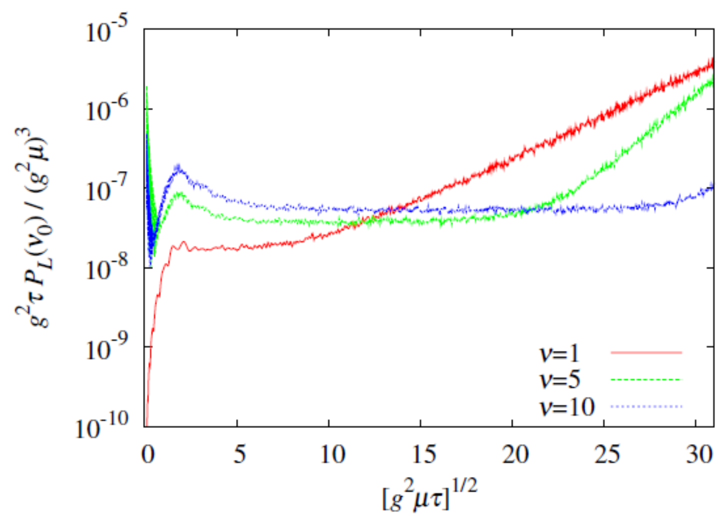
Fluctuations with Multiple Wave-Numbers

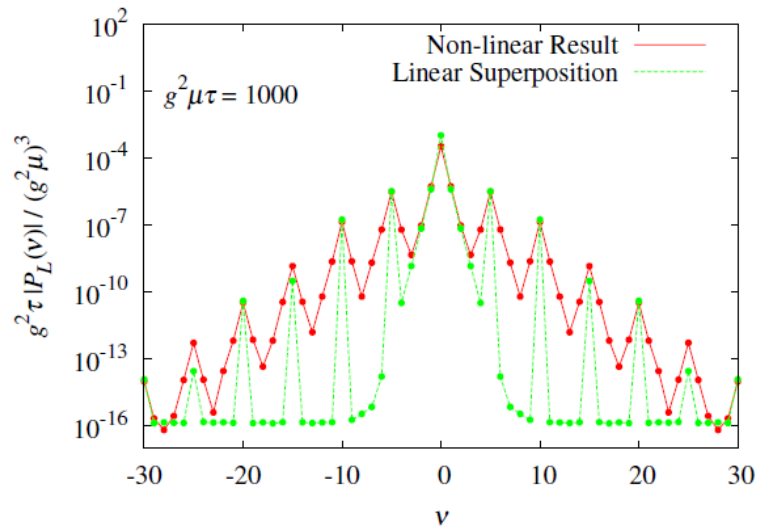
Individual simulations





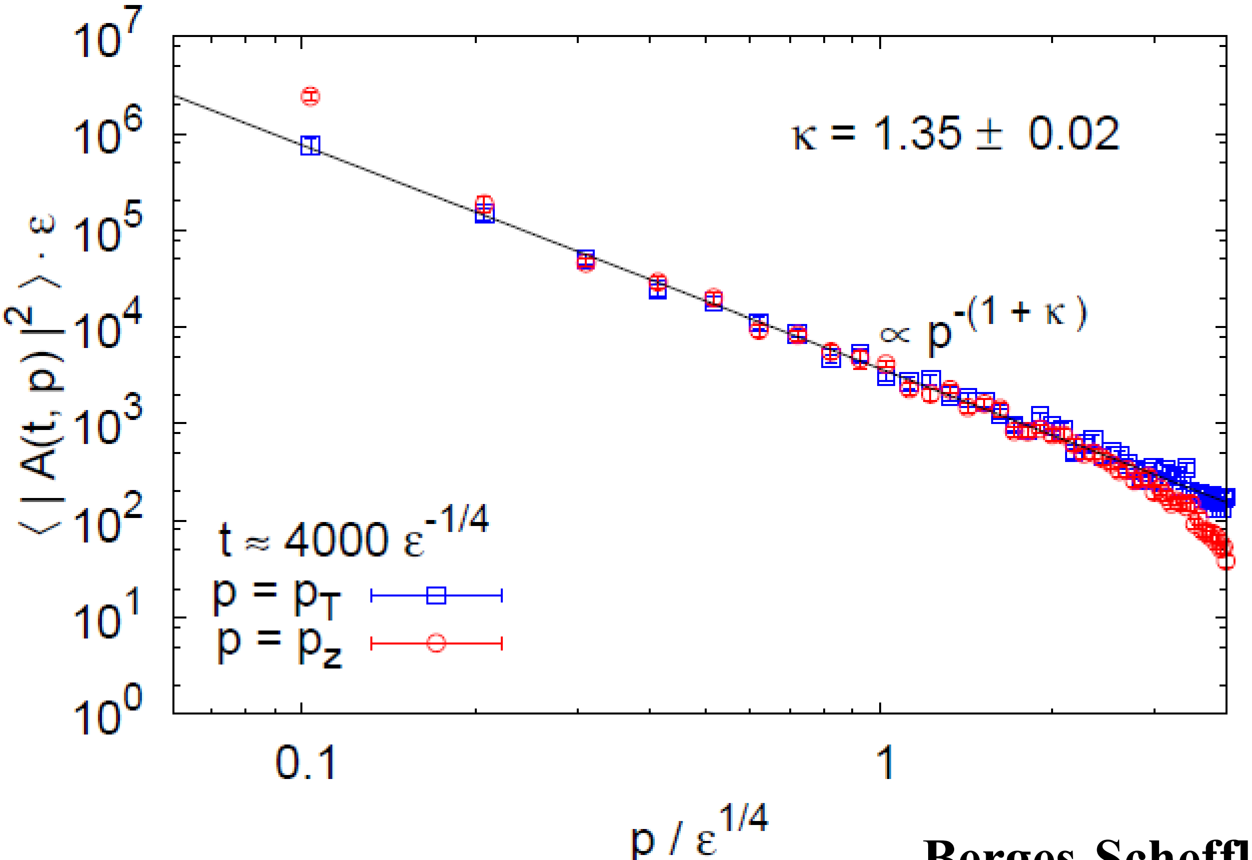
Simultaneous simulations





Kolmogorov Cascade Spectrum

Seen in the non-Abelian systems



Berges-Scheffler-Sexty (2008)

"Classical" Explanation

Dimensional Analysis

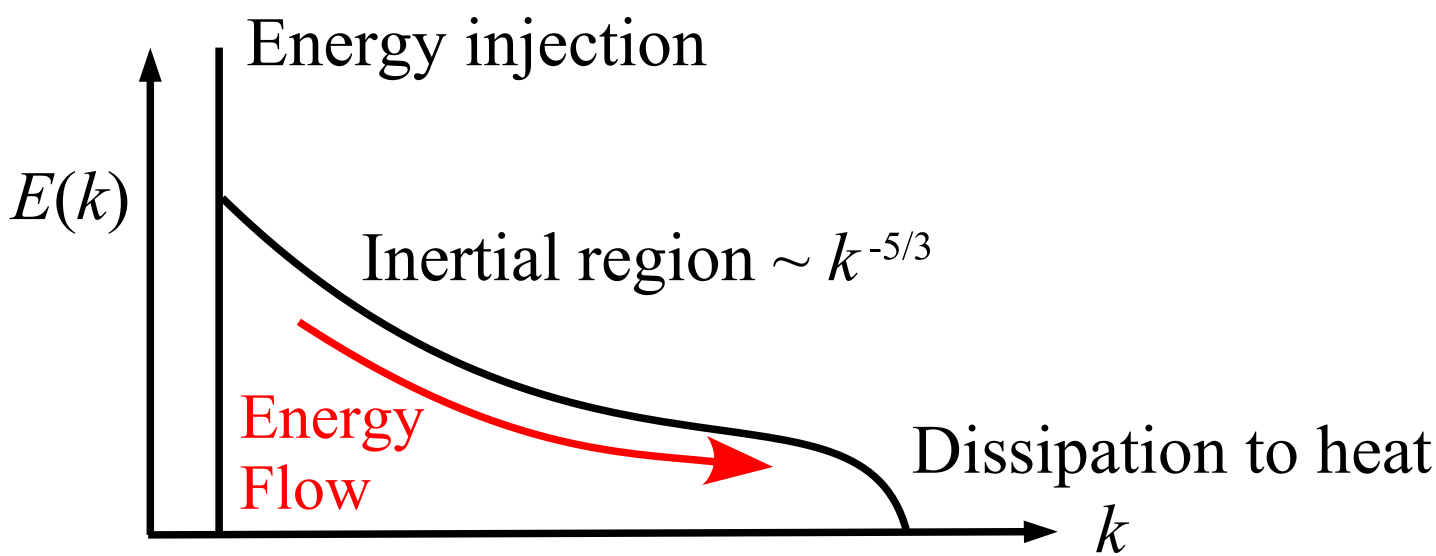
c.f. $[E]=l^2 t^{-2}$

wave-number: $[k]=l^{-1}$

Fourier component of the energy: $[E(k)]=l^3 t^{-2}$

energy flow rate: $[\psi]=l^2 t^{-3}$

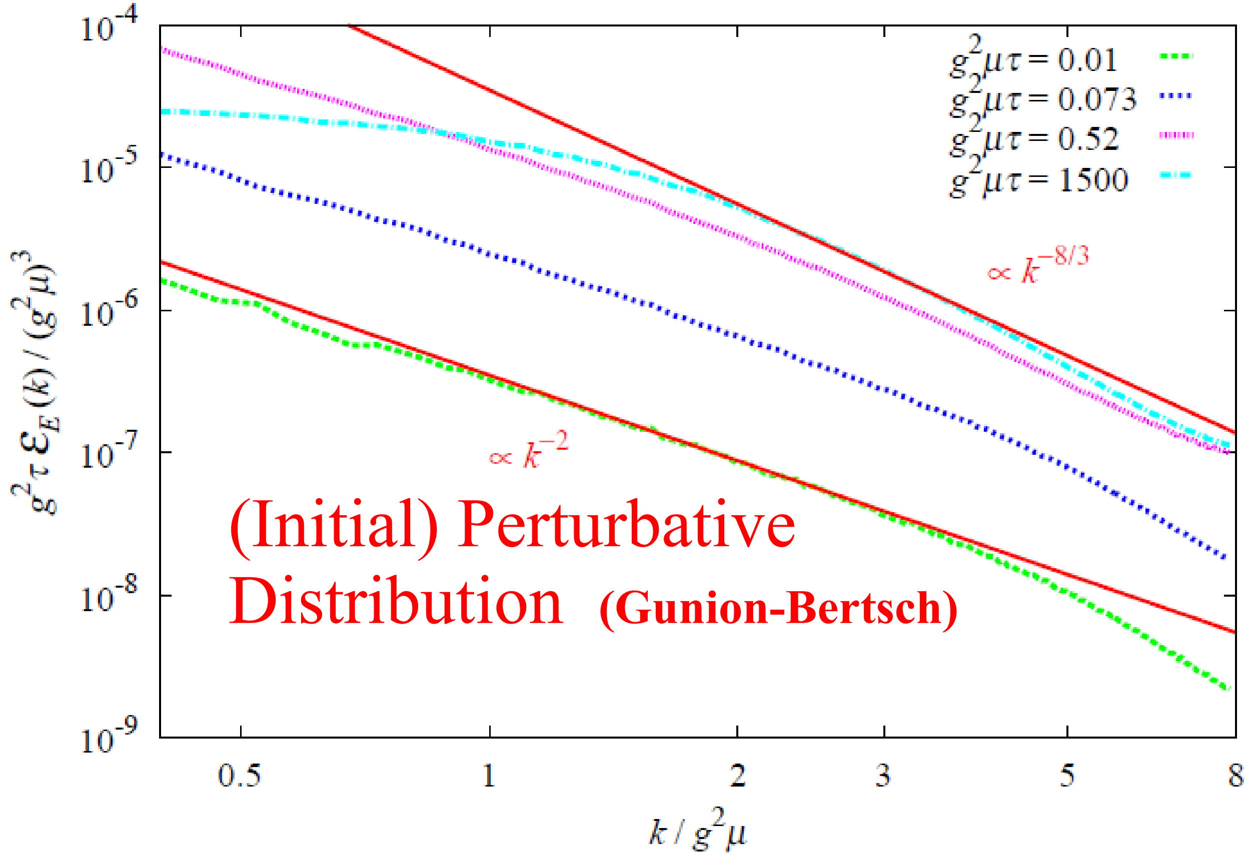
$$E(k)\propto k^{\alpha}\psi^{\beta} \quad \Rightarrow \quad \alpha=-5/3,\ \beta=2/3$$



c.f. bottom-up thermalization

Transverse Energy Spectrum

Kolmogorov's scaling is not manifest

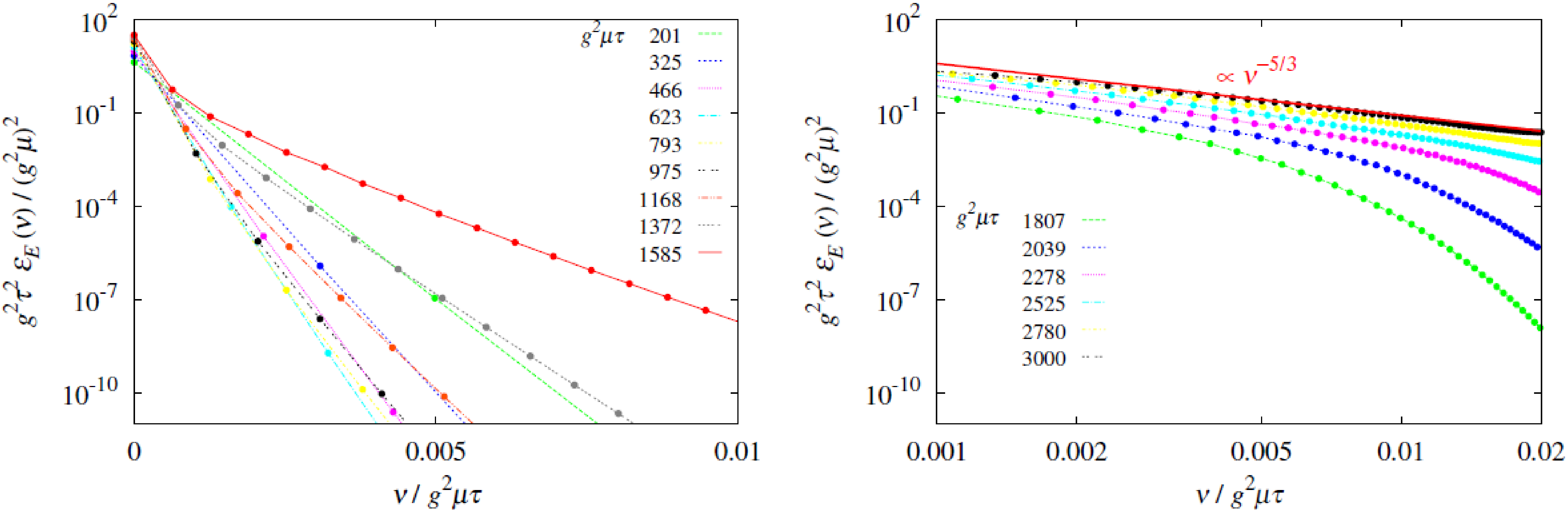


Small turbulence in the transverse flow

Energy source at small *k* does not remain dominant

Longitudinal Energy Spectrum

Kolmogorov's scaling is clearly seen



As far as we know, this is the first indication for Kolmogorov's scaling in the expanding system

Dimensional Analysis $[\nu/c\tau] = l^{-1}, \quad [V_\perp (c\tau)^2 \varepsilon_E(\nu)] = l^3 t^{-2}, \quad [\psi] = l^2 t^{-3}$

Summary of Part II

CGC simulation is supposed to give a right description of the early-time dynamics in the heavy ion collision.

So far it is not very successful... Glasma instability too slow, no isotropization, no thermalization...

At least the Kolmogorov cascade spectrum is seen, which is a promising indication.

Something missing – quantum fluctuations on top of CGC backgrounds and JIMWLK-type evolution