lattice field theory talk
examples to reach the physical limit (physical mass & continuum)
Outline

1. Quantum Chromodynamics
2. Lattice Regularization
3. Yang–Mills theories on the lattice
4. Fermions on the lattice
5. Algorithms
6. Setting the scale
Quantum Chromodynamics (QCD)

QCD: Currently the best known theory to describe the strong interaction.

SU(3) gauge theory with fermions in fundamental representation.

Fundamental degrees of freedom:

- **gluons:** $A^a_\mu$, $a = 1, \ldots, 8$
- **quarks:** $\psi$, 3(color) $\times$ 4(spin) $\times$ 6(flavor) components

\[
\mathcal{L}_{\text{QCD}} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + \bar{\psi} (iD_\mu \gamma^\mu - m) \psi,
\]

where

\[
F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu \quad \text{field strength}
\]

\[
D_\mu = \partial_\mu + g A^a_\mu \frac{\lambda^a}{2i} \quad \text{covariant derivative} \quad \rightarrow \quad \text{gives quark–gluon interaction}
\]
SU(3) group

SU(3): group of $3 \times 3$ unitary matrices with unit determinant:

$$U \in \text{SU}(3) \iff UU^\dagger = 1_{3 \times 3}, \quad \text{that is,} \quad U^{-1} = U^\dagger,$$

$$\det U = 1.$$ 

8 generators: Gell–Mann matrices $\lambda^a \quad (a = 1, \ldots, 8)$

Lie algebra of $SU(3)$: Linear combinations $A = A^a \frac{\lambda^a}{2}$

1. Hermitean: $A^\dagger = A,$
2. Traceless: $\text{Tr} A = 0.$

$$U = \exp(iA) = \exp \left( iA^a \frac{\lambda^a}{2} \right): \quad \text{elements of group SU(3)}.$$ 

$$[A, B] = i f^{abc} A^b B^c \frac{\lambda^a}{2}, \quad f^{abc}: \quad \text{structure coefficients.}$$
Quantum Chromodynamics (2)

$L_{\text{QCD}}$ is invariant under local gauge transformations:

\[ A'_\mu(x) = G(x) A_\mu(x) G(x)^\dagger - \frac{i}{g} (\partial_\mu G(x)) G(x)^\dagger \]

\[ \psi'(x) = G(x) \psi(x) \]

\[ \overline{\psi}'(x) = \overline{\psi}(x) G^\dagger(x) \]

Only gauge invariant quantities are physical.

Properties of QCD:

- **Asymptotic freedom:**
  
  Coupling constant $g \to 0$ when energy scale $\mu \to \infty$.

  \[ \implies \text{Perturbation theory can be used at high energies.} \]

- **Confinement:**
  
  Coupling constant is large at low energies.

  \[ \implies \text{Nonperturbative methods are required.} \]
Quantum Chromodynamics (3)

Quantization using Feynman path integral:

\[
\langle 0 | T[O_1(x_1) \cdots O_n(x_n)] | 0 \rangle = \frac{\int [d\psi] [d\bar{\psi}] [dA_\mu] O_1(x_1) \cdots O_n(x_n) e^{iS[\psi, \bar{\psi}, A_\mu]} \int [d\psi] [d\bar{\psi}] [dA_\mu] e^{iS[\psi, \bar{\psi}, A_\mu]} }{\int [d\psi] [d\bar{\psi}] [dA_\mu] e^{iS[\psi, \bar{\psi}, A_\mu]}}
\]

\( e^{iS} \) oscillates \( \rightarrow \) hard to evaluate integrals.

Wick rotation: \( t \rightarrow -it \) analytic continuation to Euclidean spacetime.

\[ e^{iS} \rightarrow e^{-S_E}, \text{ where} \]

\[
S_E = \int d^4x \ L_E = \int d^4x \left[ \frac{1}{4} F_{\mu\nu}^a F_{\mu\nu}^a + \bar{\psi} (D_\mu \gamma^\mu + m) \psi \right]
\]

positive definite Euclidean action.
Quantum Chromodynamics (4)

Vector components: \( \mu = 0, 1, 2, 3 \rightarrow \mu = 1, 2, 3, 4 \)

Euclidean correlator

\[
\langle 0 | \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) | 0 \rangle_E = \frac{\int [d\psi] [d\overline{\psi}] [dA_\mu] \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) e^{-S_E[\psi, \overline{\psi}, A_\mu]}}{\int [d\psi] [d\overline{\psi}] [dA_\mu] e^{-S_E[\psi, \overline{\psi}, A_\mu]}}
\]

Expectation value of \( \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \) with respect to positive definite measure \([d\psi] [d\overline{\psi}] [dA_\mu] e^{-S_E} \).
"Most systematic" nonperturbative approach: lattice QFT

Take a finite segment of spacetime, put fields at vertices of hypercubic lattice with lattice spacing $a$:

Usual boundary conditions:
- **Bosons:** Periodic in all directions

- **Fermions:**
  - Time direction: antiperiodic
  - Space directions: periodic
Lattice regularization (2)

We have to discretize the action:

\[\int d^4 x \rightarrow \text{sum over sites} \quad a^4 \sum_x\]

\[\partial_\mu \rightarrow \text{finite differences}\]

Integral over spacetime: \(\int d^4 x\)

Derivatives: \(\partial_\mu\)

Momentum \(p \leq \frac{\pi}{a}\) \(\Rightarrow\) natural UV cutoff.

At finite "a" results differ from the continuum value.

\[R^{\text{latt.}} = R^{\text{cont.}} + O(a^\nu)\]

for some dimensionless quantity \(R\).

To get physical results, need to perform:

1. Infinite volume limit \((V \rightarrow \infty)\),
2. Continuum limit \((a \rightarrow 0)\).
Yang–Mills theories on the lattice

Regularization has to maintain lattice version of gauge invariance.

Gauge fields $\rightarrow$ on links connecting neighboring sites.
- Continuum: $A_\mu$, elements of Lie algebra of SU(3).
- Lattice: $U_\mu = e^{iagA_\mu}$, elements of group SU(3) itself.

Lattice gauge transformation:

$$U_{x+\hat{\mu};-\mu} = U_{x;\mu}^{-1} = U_{x;\mu}^\dagger$$

$$U'_{x;\mu} = G_x U_{x;\mu} G_{x+\hat{\mu}}^\dagger$$

$$\psi'_x = G_x \psi_x$$

$$\bar{\psi}'_x = \bar{\psi}_x G_x^\dagger$$
Gauge invariant quantities on the lattice

- **Gluon loops**

\[
\text{Tr} \left[ U_{x_1;\mu} U_{x_1+\hat{\mu};\nu} \cdots U_{x_1-\hat{\epsilon};\epsilon} \right]
\]

- **Gluon lines connecting} \, q \text{ and } \overline{q}$$

\[
\overline{\psi}_{x_1;\mu} U_{x_1+\hat{\mu};\nu} \cdots U_{x_n-\hat{\epsilon};\epsilon} \psi_{x_n}
\]
Gauge action

Continuum gauge action:

\[ S_{g}^{\text{cont.}} = \int d^4 x \frac{1}{4} F_{\mu \nu}^a F_{\mu \nu}^a \]

Simplest gauge invariant lattice action: Wilson action

\[ S_{g}^{\text{Wilson}} = \beta \sum_{x} \left( 1 - \frac{1}{3} \text{Re} \left[ P_{x; \mu \nu} \right] \right), \quad \beta = \frac{6}{g^2}, \quad S_{g}^{\text{latt.}} = S_{g}^{\text{cont}} + O(a^2), \]

where \( P_{x; \mu \nu} \) is the plaquette:

\[ P_{x; \mu \nu} = \text{Tr} \left[ U_{x; \mu} U_{x+\hat{\mu}; \nu} U_{x+\hat{\nu}; \mu} U_{x; \nu} \right] \]
Gauge action – Symanzik improvement

Add $2 \times 1$ gluon loops to Wilson action:

$$S_g^{\text{Symanzik}} = \beta \sum_{x} \left\{ 1 - \frac{1}{3} \left( c_0 \ \text{Re}[P_{x;\mu\nu}] + c_1 \ \text{Re}[P_{x;\mu\nu}^{2\times1}] + c_1 \ \text{Re}[P_{x;\nu\mu}^{2\times1}] \right) \right\}$$

Consistency condition: $c_0 + 8c_1 = 1$.

$c_1 = -\frac{1}{12}$ gives tree level improvement $\implies S_g^{\text{cont.}} = S_g^{\text{latt.}} + O(a^4)$
Fermion doubling

Continuum fermion action

\[ S_f = \int d^4x \bar{\psi}(\gamma^\mu \partial_\mu + m)\psi. \]

Naively discretized:

\[ S_{f, \text{naive}} = a^4 \sum_x \sum_{\mu=1}^{4} \left[ \bar{\psi}_x \frac{\gamma_\mu \psi_{x+\hat{\mu}} - \psi_{x-\hat{\mu}}}{2a} + m \bar{\psi}_x \psi_x \right] \]

Inverse propagator:

\[ G_{f, \text{naive}}^{-1}(p) = i \gamma_\mu \frac{\sin p_\mu a}{a} + m. \]

Extra zeros at \( p_\mu = 0, \pm \frac{\pi}{a} \) \( \Rightarrow \) 16 zeros in 1\textsuperscript{st} Brillouin zone.

In \( d \) dimensions \( 2^d \) fermions instead of 1 \( \Rightarrow \) fermion doubling.
Wilson fermions

\[
S_f^W = S_f^{\text{naive}} - a \cdot \frac{r}{2} a^4 \sum_x \bar{\psi}_x \Box \psi_x,
\]

where

\[
\Box \psi_x = \sum_{\mu=1}^4 \frac{\psi_{x+\hat{\mu}} - 2\psi_x + \psi_{x-\hat{\mu}}}{a^2}.
\]

0 < r ≤ 1 Wilson parameter, usually r = 1.

\[
G_W^{-1}(p) = G_{\text{naive}}^{-1}(p) + \frac{2r}{a} \sum_{\mu=1}^4 \sin^2 \left( p_{\mu} a / 2 \right)
\]

\[
m_{\text{doublers}} = O(a^{-1}) \quad \Rightarrow \quad \text{doublers disappear in continuum limit.}
\]
Wilson fermions (2)

Work with dimensionless quantities: \( a^{3/2} \psi \rightarrow \psi \)

\[
S^W_f = \sum_x \left\{ \bar{\psi}_x \sum_\mu \left[ (\gamma_\mu - r) \psi_{x+\hat{\mu}} - (\gamma_\mu + r) \psi_{x-\hat{\mu}} \right] + (ma + 4r) \bar{\psi}_x \psi_x \right\}
\]

Rescale \( \psi \) by \( \sqrt{2\kappa} \),

\( \kappa = \frac{1}{2ma + 8r} \)

hopping parameter.

Action including gauge fields:

\[
S^W_f = \sum_x \left\{ \kappa \left[ \sum_\mu \bar{\psi}_x (\gamma_\mu - r) U_{x;\mu} \psi_{x+\hat{\mu}} - \bar{\psi}_{x+\hat{\mu}} (\gamma_\mu + r) U_{x;\mu}^\dagger \psi_x \right] + \bar{\psi}_x \psi_x \right\}
\]
Wilson fermions (3)

**Advantages**

1. Kills all doublers.

**Disadvantages**

1. No chiral symmetry at $a \neq 0$.

   $\Rightarrow$ Massless pions at $\kappa_c \neq \frac{1}{8r}$.

   Additive quark mass renormalization.

2. Large discretization errors:

   $S_f^W = S_f^{\text{cont.}} + O(a)$
Wilson fermions – Clover improvement

\[ S_f^{\text{clover}} = S_f^W - \frac{i a c_k r}{4} \sum_x \psi_x \sigma_{\mu\nu} F_{x;\mu\nu} \psi_x = S_f^{\text{cont.}} + O(a^2), \]

\[ \sigma_{\mu\nu} = \frac{i}{4} [\gamma_\mu, \gamma_\nu] \]

clover term

\[ F_{x;\mu\nu} = \frac{1}{4} \left( U_{x;\mu} U_{x+\hat{\mu};\nu} U_{x+\hat{\nu};\mu} U_{x;\nu} - U_{x-\hat{\nu};\nu} U_{x-\hat{\mu}-\hat{\nu};\mu} U_{x-\hat{\mu};\nu} U_{x-\hat{\nu};\nu} + U_{x;\nu} U_{x-\hat{\mu}+\hat{\nu};\mu} U_{x-\hat{\mu};\nu} U_{x-\hat{\mu};\nu} - U_{x;\mu} U_{x+\hat{\mu}-\hat{\nu};\nu} U_{x-\hat{\nu};\mu} U_{x-\hat{\nu};\nu} \right) \]

discretized version of field strength \( F_{\mu\nu} \).
Kogut–Susskind (staggered) fermions

Fermion degrees of freedom $\rightarrow$ corners of hypercube.

In $d$ dimensions:

- $2^{d/2}$ spinor components of Dirac spinors
- $2^d$ corners of hypercube

$\Rightarrow$ describes $2^d/2^{d/2} = 2^{d/2}$ flavors (tastes).

If $d = 4 \Rightarrow 4$ flavors (tastes) $\Rightarrow 4^{\text{th}}$ rooting required.
Kogut–Susskind (staggered) fermions (2)

3(color) × 4(spin) components $\rightarrow$ 3(color) × 1(spin) components

$$S_f^S = \sum_x \bar{\chi}_x \left\{ \frac{1}{2} \sum_{\mu} \eta_{x,\mu} \left( U_{x;\mu} \chi_x + \hat{\mu} - \bar{U}_{x-\hat{\mu};\mu} \chi_x - \hat{\mu} \right) + ma\chi_x \right\},$$

where

$$\eta_{x,\mu} = (-1)^{\sum_{\nu=1}^{\mu-1} x_{\nu}}$$

staggered phase.
Kogut–Susskind (staggered) fermions (3)

Advantages

1. Remnant chiral symmetry at \( a \neq 0 \)
   \[ \Rightarrow \] no additive quark mass renormalization.

2. \( O(a^2) \) discretization errors.

3. Fast.

Disadvantages

1. 4 tastes (flavors) instead of 1
   \[ \Rightarrow \] rooting trick required.

2. Taste symmetry breaking.
Integral over fermions

Full lattice QCD action

\[ S(U, \psi, \bar{\psi}) = S_g(U) - \bar{\psi} M(U) \psi \]

Fermions are described by Grassmann variables \( \longrightarrow \) have to integrate out analytically.

\[ \int [dU] [d\bar{\psi}] [d\psi] e^{-S_g(U) + \bar{\psi} M(U) \psi} = \int [dU] e^{-S_g(U)} \det M(U) \]

\( \longrightarrow \) Effective action for gluons

\[ S_{\text{eff.}}(U) = S_g(U) - \ln (\det M(U)) . \]

Staggered fermion matrix describes 4 tastes.

Rooting trick: for \( n_f \) flavors, take power \( \frac{n_f}{4} \) of determinant:

\[ S_{\text{eff.}}^S(U) = S_g(U) - \ln \left( \det M(U)^{n_f/4} \right) = S_g(U) - \frac{n_f}{4} \ln (\det M(U)) \]
Expectation values of fermionic quantities

\[ \mathcal{O}(x, y) = \left( \overline{\psi}_u \psi^d \right)_y \left( \overline{\psi}^d \psi^u \right)_x \quad \text{fermionic operator} \]

\[
\langle 0 | \mathcal{O}(x, y) | 0 \rangle = \frac{\int \left[ dU \right] \left[ d\overline{\psi} \right] \left[ d\psi \right] \overline{\psi}^u, a \psi^d, a \overline{\psi}^d, b \psi^u, b e^{-S_g(U) + \overline{\psi} M(U) \psi} \chi}{\int \left[ dU \right] \left[ d\overline{\psi} \right] \left[ d\psi \right] e^{-S_g(U) + \overline{\psi} M(U) \psi}} = \\
= \frac{\int \left[ dU \right] \left[ M^{-1}_{x,y} U \right]^{ab} \left[ M_{y,x}^{-1} U \right]^{ba} \det M(U) e^{-S_g(U)}}{\int \left[ dU \right] \det M(U) e^{-S_g(U)}} = \\
= \int \left[ dU \right] \text{Tr}_{\text{color,spin}} \left[ \left( M^{-1}_{x,y} U \right) \left( M_{y,x}^{-1} U \right) \right] e^{-S_{\text{eff.}}(U)}
\]

\[ = \int \left[ dU \right] e^{-S_{\text{eff.}}(U)} \]
Expectation values of fermionic quantities (2)

Expectation value of

\[ \mathcal{O} = \left( \overline{\psi}^u \psi^d \right)_y \left( \overline{\psi}^d \psi^u \right)_x \]

with respect to action

\[ S(U, \psi, \overline{\psi}) = S_g(U) - \overline{\psi} M(U) \psi. \]

\[ \Rightarrow \]

Expectation value of

\[ \mathcal{O}' = \text{Tr}_{\text{color,spin}} \left[ \left( M^{-1, u}_{x, y} \right) \left( M^{-1, d}_{y, x} \right) \right] \]

with respect to action

\[ S_{\text{eff.}}(U) = S_g(U) - \ln \det(M(U)). \]

\[ \langle 0 | \mathcal{O} | 0 \rangle = \frac{\int [dU] [d\overline{\psi}] [d\psi] \mathcal{O} e^{-S(U, \psi, \overline{\psi})}}{\int [dU] [d\overline{\psi}] [d\psi] e^{-S(U, \psi, \overline{\psi})}} = \frac{\int [dU] \mathcal{O}' e^{-S_{\text{eff.}}(U)}}{\int [dU] e^{-S_{\text{eff.}}(U)}} \]
Importance sampling

Monte Carlo simulation: calculate \( \langle 0 | \mathcal{O} | 0 \rangle \) stochastically.

Naive way: take random gauge configurations \( U_\alpha \) according to the uniform distribution and calculate the weighed average:

\[
\langle 0 | \mathcal{O} | 0 \rangle = \frac{\sum_\alpha \mathcal{O}_\alpha e^{-S_\alpha}}{\sum_\alpha e^{-S_\alpha}}
\]

\( S_\alpha \): value of \( S_{\text{eff.}} \) at \( U_\alpha \),
\( \mathcal{O}_\alpha \): value of \( \mathcal{O} \) at \( U_\alpha \).

\( S_\alpha \) large for most configurations \( \rightarrow \) small portion of configurations give significant contribution.

Importance sampling: generate configurations with probability based on their importance \( \rightarrow \) probability of \( U_\alpha \) is proportional to \( e^{-S_\alpha} \).

Then

\[
\langle 0 | \mathcal{O} | 0 \rangle = \frac{1}{N} \sum_{\alpha=1}^{N} \mathcal{O}_\alpha \quad \text{with relative error} \quad \frac{1}{\sqrt{N}}.
\]
Importance sampling (2)

Simplest method: Metropolis algorithm.
Choose an initial configuration $U_0$.

1. Generate $U_{k+1}$ from $U_k$ with a small random change.
2. Measure the change $\Delta S$ in the action.
3. If $\Delta S \leq 0$, keep $U_{k+1}$.
4. If $\Delta S > 0$, keep $U_{k+1}$ with a probability of $e^{-\Delta S}$.

- $U_0$ is far from the region where $e^{-S}$ is significant.
  $\Rightarrow$ Many steps required to reach equilibrium distribution: Thermalization time.
- $U_k \rightarrow U_{k+1}$ by small change.
  $\Rightarrow$ Subsequent configurations are not independent.
  Number of steps required to reach next independent configuration: Autocorrelation time.
Setting the scale

All quantities in the calculation are in lattice units
\( \rightarrow \) lattice spacing \( a \) has to be determined.

Process of obtaining \( a \):

1. Choose physical quantity \( A \) such that
   - experimental value \( A_{\text{exp.}} \) is well known,
   - easily measurable on the lattice,
   - not sensitive to discretization errors,
   - \([A] = (\text{GeV})^\nu\), \( \nu \neq 0 \).

2. Measure dimensionless \( A'_{\text{latt.}} = A_{\text{latt.}} \cdot a^\nu \) on the lattice.

3. Setting \( A_{\text{latt.}} = A_{\text{exp.}} \) yields
   \[ a = \left( \frac{A'_{\text{latt.}}}{A_{\text{exp.}}} \right)^{1/\nu}. \]
A = \sigma \text{ string tension}

\sigma = \lim_{R \to \infty} \frac{dV(R)}{dR}

Experimental value: \sqrt{\sigma} = 465 \text{ MeV}

V(R) = -\lim_{T \to \infty} \frac{1}{T} \ln[W(R, T)], \quad W(R, T) = \ldots

Static q–\bar{q} potential
Setting the scale (3)

2 \( A = r_0 \) Sommer parameter,

\[
R^2 \cdot \left. \frac{dV(R)}{dR} \right|_{R=r_0} = 1.65
\]

Experimental value: \( r_0 = 0.469(7) \text{ fm} \)

3 \( A = F_K \) leptonic decay constant of Kaon

Experimental value: \( f_K = 159.8 \text{ MeV} \)