Introduction to and Recent Progress in Lattice QCD

Z. Fodor

University of Wuppertal,
Forschungszentrum Juelich,
Eotvos University Budapest

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lattice field theory talk

examples to reach the physical limit (physical mass & continuum)

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Outline

Quantum Chromodynamics (QCD)

QCD: Currently the best known theory to describe the strong interaction.

SU(3) gauge theory with fermions in fundamental representation.

Fundamental degrees of freedom:

- gluons: A^a_μ, $a = 1, \ldots, 8$
- quarks: ψ, 3(color) $\times$ 4(spin) $\times$ 6(flavor) components

$$\mathcal{L}_{\mathrm{QCD}} = \underbrace{-\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu}}_{\text{pure gauge part}} + \underbrace{\overline{\psi}(\mathrm{i}D_\mu\gamma^\mu - m)\psi}_{\text{fermionic part}},$$

where

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu \qquad \text{field strength}$$

$$D_\mu = \partial_\mu + g A^a_\mu \frac{\lambda^a}{2\mathrm{i}} \qquad \text{covariant derivative} \quad \longrightarrow \quad \text{gives quark–gluon interaction}$$

SU(3) group

SU(3): group of 3×3 unitary matrices with unit determinant:

$U \in \mathrm{SU}(3) \iff$

1. $UU^\dagger = \mathbf{1}_{3\times 3}$, that is, $U^{-1} = U^\dagger$,
2. $\det U = 1$.

8 generators: Gell–Mann matrices λ^a $(a = 1, \ldots, 8)$

Lie algebra of $SU(3)$: Linear combinations $A = A^a \dfrac{\lambda^a}{2}$

1. Hermitean: $A^\dagger = A$,
2. traceless: $\mathrm{Tr}\, A = 0$.

$U = \exp(\mathrm{i}A) = \exp\left(\mathrm{i}A^a \dfrac{\lambda^a}{2}\right)$: elements of group SU(3).

$[A, B] = \mathrm{i} f^{abc} A^b B^c \dfrac{\lambda^a}{2}$, f^{abc}: structure coefficients.

Quantum Chromodynamics (2)

$\mathcal{L}_{\rm QCD}$ is invariant under local gauge transformations:

$$A'_\mu(x) = G(x)A_\mu(x)G(x)^\dagger - \frac{\mathrm{i}}{g}\left(\partial_\mu G(x)\right)G(x)^\dagger$$
$$\psi'(x) = G(x)\psi(x)$$
$$\overline{\psi}'(x) = \overline{\psi}(x)G^\dagger(x)$$

Only gauge invariant quantities are physical.

Properties of QCD:

- Asymptotic freedom:
  Coupling constant $g \to 0$ when energy scale $\mu \to \infty$.
  $\Longrightarrow$ Perturbation theory can be used at high energies.
- Confinement:
  Coupling constant is large at low energies.
  $\Longrightarrow$ Nonperturbative methods are required.

Quantum Chromodynamics (3)

Quantization using Feynman path integral:

$$\langle 0|\, T[\mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)]\,|0\rangle = \frac{\int [\mathrm{d}\psi]\,[\mathrm{d}\overline{\psi}]\,[\mathrm{d}A_\mu]\,\mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)\,e^{\mathrm{i}S[\psi,\overline{\psi},A_\mu]}}{\int [\mathrm{d}\psi]\,[\mathrm{d}\overline{\psi}]\,[\mathrm{d}A_\mu]\,e^{\mathrm{i}S[\psi,\overline{\psi},A_\mu]}}$$

$e^{\mathrm{i}S}$ oscillates $\longrightarrow$ hard to evaluate integrals.
Wick rotation: $t \rightarrow -\mathrm{i}t$ analytic continuation to Euclidean spacetime.
$\Longrightarrow \quad e^{\mathrm{i}S} \longrightarrow e^{-S_\mathrm{E}}$, where

$$S_\mathrm{E} = \int \mathrm{d}^4x\ \mathcal{L}_\mathrm{E} = \int \mathrm{d}^4x \left[\frac{1}{4}F^a_{\mu\nu}F^a_{\mu\nu} + \overline{\psi}(D_\mu\gamma^\mu + m)\psi\right]$$

positive definite Euclidean action.

Quantum Chromodynamics (4)

Vector components: $\mu = 0, 1, 2, 3 \quad \longrightarrow \quad \mu = 1, 2, 3, 4$

Euclidean correlator

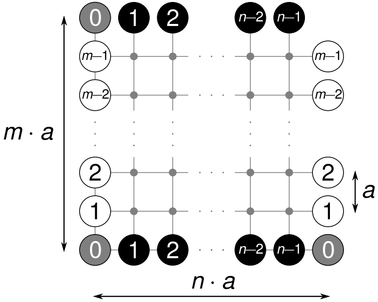
$$\langle 0 | \, \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \, | 0 \rangle_E = \frac{\int [\mathrm{d}\psi] \, [\mathrm{d}\overline{\psi}] \, [\mathrm{d}A_\mu] \, \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \, e^{-S_E[\psi, \overline{\psi}, A_\mu]}}{\int [\mathrm{d}\psi] \, [\mathrm{d}\overline{\psi}] \, [\mathrm{d}A_\mu] \, e^{-S_E[\psi, \overline{\psi}, A_\mu]}}$$

Expectation value of $\mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n)$
with respect to positive definit measure $[\mathrm{d}\psi] \, [\mathrm{d}\overline{\psi}] \, [\mathrm{d}A_\mu] \, e^{-S_E}$.

Lattice regularization

"Most sytematic" nonperturbative approach:
lattice QFT

Take a finite segment of spacetime,
put fields at vertices of hypercubic lattice with lattice spacing a:



Usual boundary conditions:
Bosons:

Periodic in all directions

Fermions:

Time direction: antiperiodic

Space directions: periodic

Lattice regularization (2)

We have to discretize the action:

intagral over spacetime $\int \mathrm{d}^4 x$	$\longrightarrow$	sum over sites	$a^4 \sum_x$
derivatives ∂_μ	$\longrightarrow$	finite differences	

Momentum $p \leq \frac{\pi}{a}$ $\Longrightarrow$ natural UV cutoff.

At finite "a" results differ from the continuum value.

$$R^{\text{latt.}} = R^{\text{cont.}} + O(a^\nu)$$

for some dimensionless quantity R.

To get physical results, need to perform:

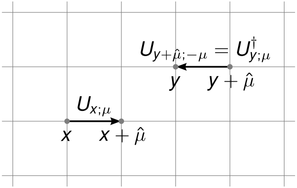
1. Infinite volume limit $(V \to \infty)$,
2. Continuum limit $(a \to 0)$.

Yang–Mills theories on the lattice

Regularization has to maintain lattice version of gauge invariance.

Gauge fields $\longrightarrow$ on links connecting neighboring sites.

- Continuum: A_μ, elements of Lie algebra of SU(3).
- Lattice: $U_\mu = e^{iagA_\mu}$, elements of group SU(3) itself.



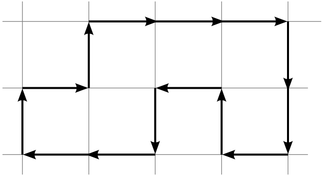
$$U_{x+\hat{\mu};-\mu} = U^{-1}_{x;\mu} = U^\dagger_{x;\mu}$$

Lattice gauge transformation:

$$U'_{x;\mu} = G_x U_{x;\mu} G^\dagger_{x+\hat{\mu}}$$
$$\psi'_x = G_x \psi_x$$
$$\overline{\psi}'_x = \overline{\psi}_x G^\dagger_x$$

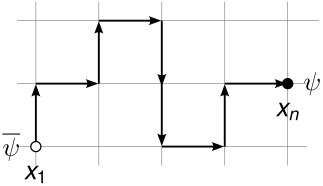
Gauge invariant quantities on the lattice

- Gluon loops



$$\mathrm{Tr}\left[U_{x_1;\mu}\, U_{x_1+\hat{\mu};\nu} \cdots U_{x_1-\hat{\epsilon};\epsilon}\right]$$

- Gluon lines connecting q and $\overline{q}$



$$\overline{\psi}_{x_1} U_{x_1;\mu}\, U_{x_1+\hat{\mu};\nu} \cdots U_{x_n-\hat{\epsilon};\epsilon} \psi_{x_n}$$

Gauge action

Continuum gauge action:

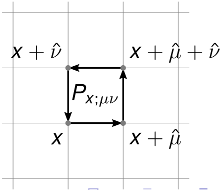
$$S_{\mathrm{g}}^{\text{cont.}} = \int \mathrm{d}^4x \, \frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu}$$

Simplest gauge invariant lattice action: Wilson action

$$S_{\mathrm{g}}^{\text{Wilson}} = \beta \sum_{\substack{x \\ \nu<\mu}} \left(1 - \frac{1}{3} \mathrm{Re}\left[P_{x;\mu\nu} \right] \right) , \quad \beta = \frac{6}{g^2}, \quad S_g^{latt.} = S_g^{cont} + O(a^2),$$

where $P_{x;\mu\nu}$ is the plaquette:

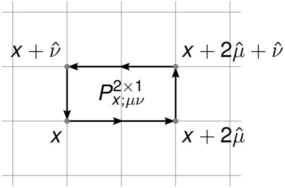
$$P_{x;\mu\nu} = \mathrm{Tr}\left[U_{x;\mu} \, U_{x+\hat{\mu};\nu} \, U^\dagger_{x+\hat{\nu};\mu} \, U^\dagger_{x;\nu} \right]$$



Gauge action – Symanzik improvement

Add 2×1 gluon loops to Wilson action:

$$S_g^{\text{Symanzik}} = \beta \sum_{\substack{x \\ \nu<\mu}} \left\{ 1 - \frac{1}{3} \left(c_0 \, \mathrm{Re}[P_{x;\mu\nu}] + c_1 \, \mathrm{Re}\left[P_{x;\mu\nu}^{2\times1}\right] + c_1 \, \mathrm{Re}\left[P_{x;\nu\mu}^{2\times1}\right] \right) \right\}$$



Consistency condition: $c_0 + 8c_1 = 1$.

$c_1 = -\frac{1}{12}$ gives tree level improvement $\Longrightarrow S_g^{latt.} = S_g^{cont.} + O(a^4)$

Fermion doubling

Continuum fermion action

$$S_\mathrm{f} = \int d^4x\, \overline{\psi}(\gamma^\mu \partial_\mu + m)\psi.$$

Naively discretized:

$$S_\mathrm{f}^\mathrm{naive} = a^4 \sum_x \left[\overline{\psi}_x \sum_{\mu=1}^{4} \gamma_\mu \frac{\psi_{x+\hat{\mu}} - \psi_{x-\hat{\mu}}}{2a} + m\overline{\psi}_x \psi_x \right]$$

Inverse propagator:

$$G_\mathrm{naive}^{-1}(p) = \mathrm{i}\gamma_\mu \frac{\sin p_\mu a}{a} + m.$$

Extra zeros at $p_\mu = 0, \pm\frac{\pi}{a}$ $\Longrightarrow$ 16 zeros in 1$^\mathrm{st}$ Brillouin zone.

In d dimensions 2^d fermions instead of 1 $\Longrightarrow$ fermion doubling.

Wilson fermions

$$S_f^W = S_f^{\text{naive}} \underbrace{- a \cdot \frac{r}{2} a^4 \sum_x \overline{\psi}_x \Box \psi_x}_{\text{Wilson term}},$$

where

$$\Box \psi_x = \sum_{\mu=1}^{4} \frac{\psi_{x+\hat{\mu}} - 2\psi_x + \psi_{x-\hat{\mu}}}{a^2}.$$

$0 < r \leq 1$ Wilson parameter, usually $r = 1$.

$$G_W^{-1}(p) = G_{\text{naive}}^{-1}(p) + \frac{2r}{a} \sum_{\mu=1}^{4} \sin^2(p_\mu a/2)$$

$m_{\text{doublers}} = O(a^{-1}) \implies$ doublers disappear in continuum limit.

Wilson fermions (2)

Work with dimensionless quantities: $a^{3/2}\psi \to \psi$

$$S_f^W = \sum_x \left\{ \overline{\psi}_x \sum_\mu [(\gamma_\mu - r)\,\psi_{x+\hat\mu} - (\gamma_\mu + r)\,\psi_{x-\hat\mu}] + (ma + 4r)\,\overline{\psi}_x\psi_x \right\}$$

Rescale ψ by $\sqrt{2\kappa}$, $\kappa = \dfrac{1}{2ma + 8r}$ hopping parameter.

Action including gauge fields:

$$S_f^W = \sum_x \left\{ \kappa \left[\sum_\mu \overline{\psi}_x\,(\gamma_\mu - r)\,U_{x;\mu}\psi_{x+\hat\mu} - \overline{\psi}_{x+\hat\mu}\,(\gamma_\mu + r)\,U^\dagger_{x;\mu}\psi_x \right] + \overline{\psi}_x\psi_x \right\}$$

Wilson fermions (3)

- Advantages
  1. Kills all doublers.

- Disadvantages
  1. No chiral symmetry at $a \neq 0$.

     $\Longrightarrow$ Massless pions at $\kappa_c \neq \frac{1}{8r}$.

     Additive quark mass renormalization.

  2. Large discretization errors:

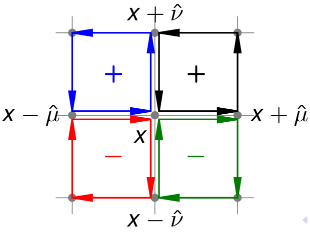
     $S_f^W = S_f^{\text{cont.}} + O(a)$

Wilson fermions – Clover improvement

$$S_f^{\text{clover}} = S_f^{W} - \underbrace{\frac{iac\kappa r}{4} \sum_x \overline{\psi}_x \sigma_{\mu\nu} \mathcal{F}_{x;\mu\nu} \psi_x}_{\text{clover term}} = S_f^{\text{cont.}} + O(a^2), \qquad \sigma_{\mu\nu} = \frac{i}{4}[\gamma_\mu, \gamma_\nu]$$

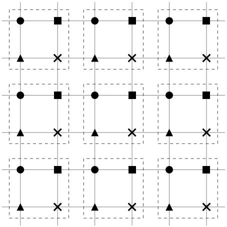
$$\mathcal{F}_{x;\mu\nu} = \frac{1}{4}\Big(U_{x;\mu} U_{x+\hat\mu;\nu} U^\dagger_{x+\hat\nu;\mu} U^\dagger_{x;\nu} - U^\dagger_{x-\hat\nu;\nu} U^\dagger_{x-\hat\mu-\hat\nu;\mu} U_{x-\hat\mu-\hat\nu;\nu} U_{x-\hat\nu;\nu} + + U_{x;\nu} U^\dagger_{x-\hat\mu+\hat\nu;\mu} U^\dagger_{x-\hat\mu;\nu} U_{x-\hat\mu;\mu} - U_{x;\mu} U^\dagger_{x+\hat\mu-\hat\nu;\nu} U^\dagger_{x-\hat\nu;\mu} U_{x-\hat\nu;\nu} \Big)$$

discretized version of field strength $F_{\mu\nu}$.



Kogut–Susskind (staggered) fermions

Fermion degrees of freedom $\longrightarrow$ corners of hypercube.



In d dimensions:

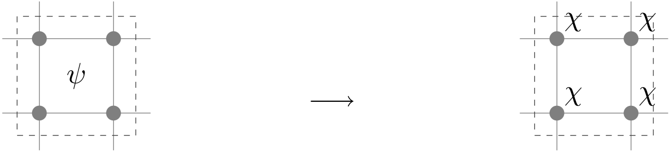
$2^{d/2}$ spinor components of Dirac spinors

2^d corners of hypercube

$\Longrightarrow$ describes $2^d/2^{d/2} = 2^{d/2}$ flavors (tastes).

If $d = 4 \Longrightarrow$ 4 flavors (tastes) $\Longrightarrow$ 4^{th} rooting required.

Kogut–Susskind (staggered) fermions (2)



3(color) × 4(spin) components 3(color) × 1(spin) components

$$S_{\mathrm{f}}^{\mathrm{S}} = \sum_x \overline{\chi}_x \left\{ \frac{1}{2} \sum_\mu \eta_{x,\mu} \left(U_{x;\mu} \chi_{x+\hat{\mu}} - U^\dagger_{x-\hat{\mu};\mu} \chi_{x-\hat{\mu}} \right) + ma \chi_x \right\},$$

where

$$\eta_{x,\mu} = (-1)^{\sum_{\nu=1}^{\mu-1} x_\nu}$$ staggered phase.

Kogut–Susskind (staggered) fermions (3)

- Advantages
  1. Remnant chiral symmetry at $a \neq 0$

     $\Longrightarrow$ no additive quark mass renormalization.
  2. $O(a^2)$ discretization errors.
  3. Fast.
- Disadvantages
  1. 4 tastes (flavors) instead of 1

     $\Longrightarrow$ rooting trick required.
  2. Taste symmetry breaking.

Integral over fermions

Full lattice QCD action

$$S(U,\psi,\overline{\psi}) = \underbrace{S_{\rm g}(U)}_{\text{gluonic part}} - \underbrace{\overline{\psi}\, M(U)\,\psi}_{\text{fermionic part}}$$

Fermions are described by Grassmann variables $\longrightarrow$ have to integrate out analytically.

$$\int [\mathrm{d}U]\,[\mathrm{d}\overline{\psi}]\,[\mathrm{d}\psi]\, e^{-S_{\rm g}(U)+\overline{\psi}\,M(U)\,\psi} = \int [\mathrm{d}U]\, e^{-S_{\rm g}(U)} \det M(U)$$

$\Longrightarrow$ Effective action for gluons

$$S_{\rm eff.}(U) = S_{\rm g}(U) - \ln\left(\det M(U)\right).$$

Staggered fermion matrix describes 4 tastes.
Rooting trick: for n_f flavors, take power $\frac{n_f}{4}$ of determinant:

$$S^{\rm S}_{\rm eff.}(U) = S_{\rm g}(U) - \ln\left(\det M(U)^{n_f/4}\right) = S_{\rm g}(U) - \frac{n_f}{4}\ln\left(\det M(U)\right)$$

Expectation values of fermionic quantities

$$\mathcal{O}(x,y) = \left(\overline{\psi}^u \psi^d\right)_y \left(\overline{\psi}^d \psi^u\right)_x$$ fermionic operator

$$\langle 0|\,\mathcal{O}(x,y)\,|0\rangle = \frac{\int [\mathrm{d}U]\,[\mathrm{d}\overline{\psi}]\,[\mathrm{d}\psi]\,\overline{\psi}_y^{u,a}\,\psi_y^{d,a}\,\overline{\psi}_x^{d,b}\,\psi_x^{u,b}\,e^{-S_g(U)+\overline{\psi}\,M(U)\,\psi}}{\int [\mathrm{d}U]\,[\mathrm{d}\overline{\psi}]\,[\mathrm{d}\psi]\,e^{-S_g(U)+\overline{\psi}\,M(U)\,\psi}} =$$

$$= \frac{\int [\mathrm{d}U]\,\left[M_{x,y}^{-1,u}(U)\right]^{ab}\left[M_{y,x}^{-1,d}(U)\right]^{ba}\,\det M(U)\;e^{-S_g(U)}}{\int [\mathrm{d}U]\,\det M(U)\;e^{-S_g(U)}} =$$

$$= \frac{\int [\mathrm{d}U]\,\mathrm{Tr}_{\text{color,spin}}\left[\left(M_{x,y}^{-1,u}\right)\left(M_{y,x}^{-1,d}\right)\right]\,e^{-S_{\text{eff.}}(U)}}{\int [\mathrm{d}U]\,e^{-S_{\text{eff.}}(U)}}.$$

Expectation values of fermionic quantities (2)

Expectation value of $\mathcal{O} = \left(\overline{\psi}^u \psi^d\right)_y \left(\overline{\psi}^d \psi^u\right)_x$

with respect to action $S(U,\psi,\overline{\psi}) = S_g(U) - \overline{\psi}\, M(U)\, \psi$.

$$\downarrow$$

Expectation value of $\mathcal{O}' = \mathrm{Tr}_{\text{color,spin}}\left[\left(M^{-1,u}_{x,y}\right)\left(M^{-1,d}_{y,x}\right)\right]$

with respect to action $S_{\text{eff.}}(U) = S_g(U) - \ln\left(\det M(U)\right)$.

$$\langle 0|\, \mathcal{O}\, |0\rangle = \frac{\int [\mathrm{d}U]\,[\mathrm{d}\overline{\psi}]\,[\mathrm{d}\psi]\,\mathcal{O}\, e^{-S(U,\psi,\overline{\psi})}}{\int [\mathrm{d}U]\,[\mathrm{d}\overline{\psi}]\,[\mathrm{d}\psi]\, e^{-S(U,\psi,\overline{\psi})}} = \frac{\int [\mathrm{d}U]\,\mathcal{O}'\, e^{-S_{\text{eff.}}(U)}}{\int [\mathrm{d}U]\, e^{-S_{\text{eff.}}(U)}}$$

Importance sampling

Monte Carlo simulation: calculate $\langle 0|\,\mathcal{O}\,|0\rangle$ stochastically.

Naive way: take random gauge configurations U_α according to the uniform distribution and calculate the weighed average:

$$\langle 0|\,\mathcal{O}\,|0\rangle = \frac{\sum_\alpha \mathcal{O}_\alpha\, e^{-S_\alpha}}{\sum_\alpha e^{-S_\alpha}}$$

S_α: value of $S_{\text{eff.}}$ at U_α,
$\mathcal{O}_\alpha$: value of $\mathcal{O}$ at U_α.

S_α large for most configurations $\longrightarrow$ small portion of configurations give significant contribution.

Importance sampling: generate configurations with probability based on their importance $\longrightarrow$ probability of U_α is proportional to e^{-S_α}.

Then $\langle 0|\,\mathcal{O}\,|0\rangle = \frac{1}{N}\sum_{\alpha=1}^{N} \mathcal{O}_\alpha$ with relative error $\frac{1}{\sqrt{N}}$.

Importance sampling (2)

Simplest method: Metropolis algorithm.
Choose an initial configuration U_0.

1. Generate U_{k+1} from U_k with a small random change.
2. Measure the change ΔS in the action.
3. If $\Delta S \leq 0$, keep U_{k+1}.
4. If $\Delta S > 0$, keep U_{k+1} with a probability of $e^{-\Delta S}$.

- U_0 is far from the region where e^{-S} is significant.
  $\Longrightarrow$ Many steps required to reach equilibrium distribution: Thermalization time.
- $U_k \longrightarrow U_{k+1}$ by small change.
  $\Longrightarrow$ Subsequent configurations are not independent. Number of steps required to reach next independent configuration: Autocorrelation time.

Setting the scale

All quantities in the calculation are in lattice units
$\longrightarrow$ lattice spacing a has to be determined.

Process of obtaining a:

1. Choose physical quantity A such that
   - experimental value $A_{\text{exp.}}$ is well known,
   - easily measurable on the lattice,
   - not sensitive to discretization errors,
   - $[A] = (\text{GeV})^\nu$, $\nu \neq 0$.
2. Measure dimensionless $A'_{\text{latt.}} = A_{\text{latt.}} \cdot a^\nu$ on the lattice.
3. Setting $A_{\text{latt.}} = A_{\text{exp.}}$ yields $a = \left(\frac{A'_{\text{latt.}}}{A_{\text{exp.}}}\right)^{1/\nu}$.

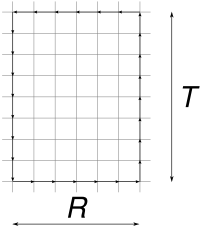
Setting the scale (2)

1. $A = \sigma$ string tension

$$\sigma = \lim_{R\to\infty} \frac{\mathrm{d}V(R)}{\mathrm{d}R}$$

Experimental value: $\sqrt{\sigma} = 465\,\mathrm{MeV}$

$$V(R) = -\lim_{T\to\infty} \frac{1}{T} \ln[W(R,T)]\,, \qquad W(R,T) \quad =$$



Static q–$\bar{q}$ potential

Setting the scale (3)

2. $A = r_0$ Sommer parameter,

$$R^2 \cdot \left.\frac{\mathrm{d}V(R)}{\mathrm{d}R}\right|_{R=r_0} = 1.65$$

   Experimental value: $r_0 = 0.469(7)\,\mathrm{fm}$

3. $A = F_K$ leptonic decay constant of Kaon
   Experimental value: $f_K = 159.8\,\mathrm{MeV}$