# ON SINGULARITIES OF REDUCED GAUGE THEORIES

Daniele Dorigoni, Gabriele Veneziano, J W

## 1 How to calculate masses of particles ?

- Lattice
- Diagonalize Hamiltonian
- Light Cone Discretization
- QCD equations: coupled Bethe-Salpeter equations on the LC
- Simplifications: large N planar diagrams - single traces
- less dimensions
- even quantum mechanics (but at $N \to \infty$)
- supersymmetry

## 2 Planar gauge theory in 1+1 dimensions

- **The history**

**FT on the light cone – C. Thorn ('77)**
**Warm-up: D=1+1, $QCD_2$ – 't Hooft ('74)**
**fermions in funamental irrep $\stackrel{\text{LargeN}}{\longrightarrow}$ no multiparton states.**

**YM+with addjoint matter – Klebanov et al. ('93)**
**matter = fermions or *scalars* ( = reduced $YM_3$ )**
**$SYM_2$ – Matsumura et al. ('95)**
**D=4 Wilson and Glazek ('93)**
**Hiller et al. ('98)**
**$QCD_4$ on the light cone – Brodsky et al. (since '70)**

### 2.1 One way: Light Cone Discretization

$$
\begin{aligned}
P^+ &= \sum_{n=2}^{\infty} \sum_{i=1}^{n} p_i^+, \quad p_i^+ \geq 0 \\
K &= \sum_{n=2}^{\infty} \sum_{i=1}^{n} r_i, \quad K, r_i - \text{natural},
\end{aligned}
$$

**Cutoff $K \Longrightarrow$ partitions $\{r_1, r_2, \ldots\} \Longrightarrow$ states**

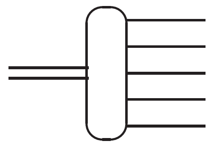
$$
|\{r\}\rangle = Tr[a^\dagger(r_1)a^\dagger(r_2)...a^\dagger(r_p)]|0\rangle \tag{1}
$$

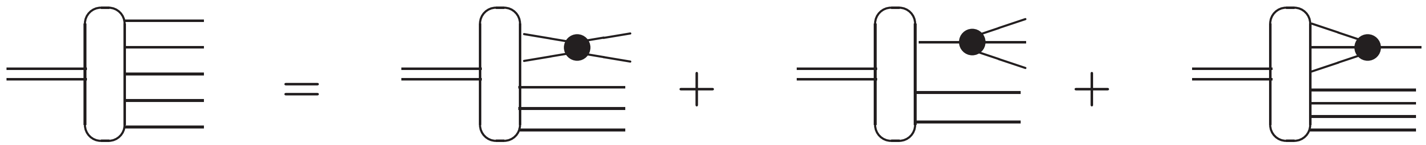
$|\{r\}\rangle \Longrightarrow \langle\{r\}|H|\{r'\}\rangle \Longrightarrow E_n$

## 2.2 Second way: integral equations in the continuum

- **Different cutoff – directly in the continuum**

$$H|\Phi\rangle = M^2|\Phi\rangle \tag{2}$$

$|\Phi\rangle \rightarrow \Phi_n(x_1, x_2, \ldots, x_n) \qquad \leftrightarrow$ 



$$M^2\Phi_n(x_1 \ldots x_n) = A \otimes \Phi_n + B \otimes \Phi_{n-2} + C \otimes \Phi_{n+2} \tag{3}$$

- **Interpretation: proton is invariant against elementary processes**
- **Fundamental: contain DGLAP and BFKL evolution eqns.**
- **Emission and absorption are present (parton recombination)**

**The cutoff:**

$$n \leq n_{max} \tag{4}$$

$n_{max} = 2$ **'t Hooft equation – exact for** $QCD_2$ **(with fundamental fermions)**

## 2.3 A sample of results

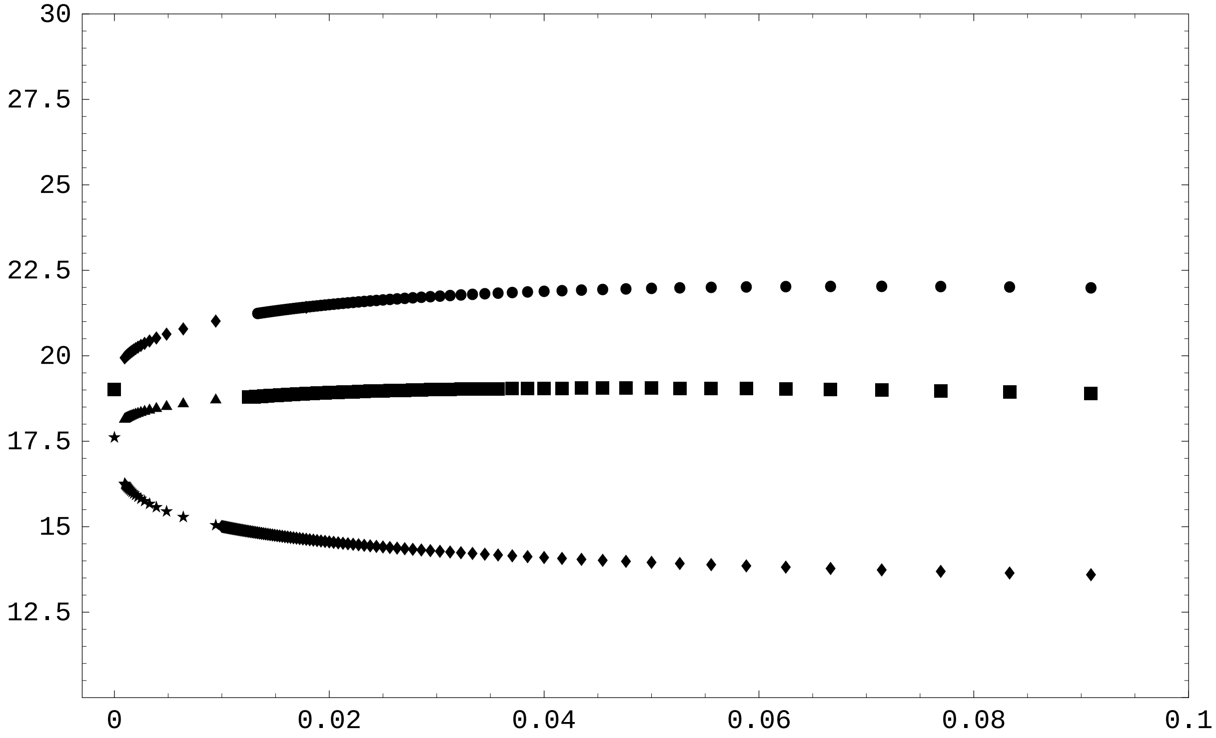



Figure 1: $M^2$ **vs. 1/K for the lowest state, two partons only**

## • EQUATIONS

$$|\Phi\rangle = \sum_{n=2}^{\infty} \int [dx]\delta(1 - x_1 - x_2 - \ldots x_n)\Phi_n(x_1, x_2, \ldots x_n)Tr[a^\dagger(x_1)a^\dagger(x_2)\ldots a^\dagger(x_n)]|0\rangle$$

**EXAMPLE 1:** $QCD_2$ **( fundamental fermions )**

$$M^2 f(x) = m^2 \left(\frac{1}{x} + \frac{1}{1-x}\right) f(x) + \frac{2\lambda}{\pi} \int_0^1 \frac{dy}{(y-x)^2} [f(x) - f(y)]$$

$$f(x) = \Phi_2(x, 1-x)$$

**EXAMPLE 2: $SYM_2$ restricted to the two-parton sector**

**There are two coupled equations in the bosonic sector**

$$M^2\phi_{bb}(x) = m_b^2\left(\frac{1}{x}+\frac{1}{1-x}\right)\phi_{bb}(x) + \frac{\lambda}{2}\frac{\phi_{bb}(x)}{\sqrt{x(1-x)}}$$

$$-\frac{2\lambda}{\pi}\int_0^1 \frac{(x+y)(2-x-y)}{4\sqrt{x(1-x)y(1-y)}}\frac{[\phi_{bb}(y)-\phi_{bb}(x)]}{(y-x)^2}dy + \frac{\lambda}{2\pi}\int_0^1 \frac{1}{(y-x)}\frac{\phi_{ff}(y)}{\sqrt{x(1-x)}}dy$$

$$M^2\phi_{ff}(x) = m_f^2\left(\frac{1}{x}+\frac{1}{1-x}\right)\phi_{ff}(x)$$

$$-\frac{2\lambda}{\pi}\int_0^1 \frac{[\phi_{ff}(y)-\phi_{ff}(x)]}{(y-x)^2}dy + \frac{\lambda}{2\pi}\int_0^1 \frac{1}{(x-y)}\frac{\phi_{bb}(y)}{\sqrt{y(1-y)}}dy$$

**and the single one in the fermionic sector**

$$M^2\phi_{bf}(x) = \left(\frac{m_b^2}{x}+\frac{m_f^2}{1-x}\right)\phi_{bf}(x) + \frac{2\lambda}{\pi}\frac{\phi_{bf}(x)}{\sqrt{x}+x}$$

$$-\frac{2\lambda}{\pi}\int_0^1 \frac{(x+y)}{2\sqrt{xy}}\frac{[\phi_{bf}(y)-\phi_{bf}(x)]}{(y-x)^2}dy - \frac{\lambda}{2\pi}\int_0^1 \frac{1}{(1-y-x)}\frac{\phi_{bf}(y)}{\sqrt{xy}}dy \qquad (5)$$

**Example 3: $YM_2$ with addjoint fermionc matter - all parton-number sectors**

$$
\begin{aligned}
M^2\phi_n(x_1\ldots x_n) &= \frac{m^2}{x_1}\phi_n(x_1\ldots x_n) \\
&+ \frac{\lambda}{\pi}\frac{1}{(x_1+x_2)^2}\int_0^{x_1+x_2} dy\phi_n(y,x_1+x_2-y,x_3\ldots x_n) \\
&+ \frac{\lambda}{\pi}\int_0^{x_1+x_2}\frac{dy}{(x_1-y)^2}\{\phi_n(x_1,x_2,x_3\ldots x_n) \\
&\quad -\phi_n(y,x_1+x_2-y,x_3\ldots x_n)\} \\
&+ \frac{\lambda}{\pi}\int_0^{x_1} dy\int_0^{x_1-y} dz\phi_{n+2}(y,z,x_1-y-z,x_2\ldots x_n)\left[\frac{1}{(y+z)^2}-\frac{1}{(x_1-y)^2}\right] \\
&+ \frac{\lambda}{\pi}\phi_{n-2}(x_1+x_2+x_3,x_4\ldots x_n)\left[\frac{1}{(x_1+x_2)^2}-\frac{1}{(x_1-x_3)^2}\right] \\
&\pm\ cyclic\ \ permutations\ \ of\ \ (x_1\ldots x_n)
\end{aligned}
$$

## 3 IR divergencies (scalar matter only)

$$\left(m_0^2 + \frac{\lambda}{\pi}\int_0^1 \frac{dy}{y} - \frac{2\lambda}{\pi}\right)\left(\frac{1}{x} + \frac{1}{1-x}\right)\phi_{bb}(x) + \frac{\lambda}{2}\frac{\phi_{bb}(x)}{\sqrt{x(1-x)}}$$
$$-\frac{\lambda}{2\pi}\int_0^1 \frac{(x+y)(2-x-y)}{\sqrt{x(1-x)y(1-y)}}\frac{\phi_{bb}(y)-\phi_{bb}(x)}{(y-x)^2}dy = M^2\phi_{bb}(x) \tag{6}$$

- **Mass renormalization (Klebanov, Pinsky) (IR ??)**
- **Mass counterterms $\leftrightarrow$ SUSY ??**
- **LCD: cutoff dependence ?**
- **Integral equations: multiplicity *and* momentum cutoffs ?**

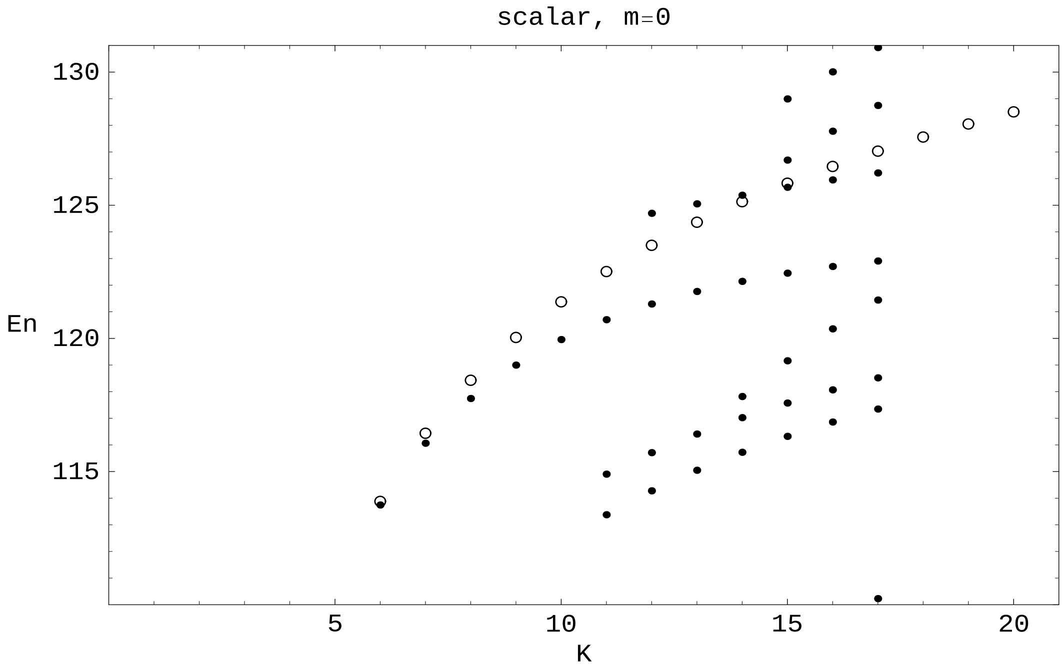



Figure 2: **LCD: a mass of the lowest state with (open circles) and without the multiplicity cutoff,** $n \leq 2$

## 4 Bloch-Nordsieck pattern of divergencies

1) **This is for the bound states.**

2) **Hamiltonian itself is divergent.**

**Nevertheless we suspect that**

- **IR divergencies cancel dynamically between *different* multiplicity sectors.**
- **The power of the dimensional reduction**

  **pattern of IR cancelations is similar to D=3+1, why ?**
  $YM_2$ **with "adjoined scalar matter" is not an exotic theory in two dimensions.**
  **It is the 3+1 YM theory reduced in the transverse directions.**
  **Adjoined scalars are just the transverse components of the 3+1 gauge field.**
- $\mapsto$ **Should study dimensional reduction of QCD singularities**

- **An abelian toy model (Bloch-Nordsieck inspired) in two dimensions**

$$H = \int_0^P dk \left[ {a^\dagger}_k a_k - j_k({a^\dagger}_k + a_k) + j_k^2 \right] = \sum_k H_k, \quad j_k = \frac{g}{\sqrt{k}}$$

$$H_k = A_k^\dagger A_k, \quad A_k = e^{-iP_k j_k} a_k e^{iP_k j_k}, \quad P_k = \frac{1}{i\sqrt{2}}(a_k - {a^\dagger}_k). \tag{7}$$

**The eigenstates are the BN coherent states**

$$|n_k\rangle_{new} = \frac{{A^\dagger}_k^n}{\sqrt{n!}} |0\rangle_{new} = e^{-iP_k j_k} |n_k\rangle_{old} \tag{8}$$

**and the eigenvalues (masses) are integer**

$$M^2 = \sum_k n_k \tag{9}$$

**The integral equations**

$$M^2 f_n(x_1, \ldots, x_n) = \left( n + \int_0^1 j(x)^2 dx \right) f_n(x_1, \ldots, x_n)$$

$$- \sum_{i=1,n} j(x_i) f_{n-1}(x_1, .., x_{j-1}, x_{j+1}, .., x_n) - \int_0^1 j(x) f_{n+1}(x_1, ..., x_n, x) dx \tag{10}$$

**Look divergent, but in fact they must ( and they do )give the finite spectrum!**

### 4.1 Solutions

**The wave functions of the lowest ($\lambda = 0$) state**

$$f_n^{(0)}(x_1, x_2, \dots x_n) = Z \frac{g^n}{\sqrt{x_1 \dots x_n}}, \quad x_1 \leq x_2 \leq \dots \leq x_n \tag{11}$$

**with**

$$Z = \exp\left(-\frac{g^2}{2}\int_0^1 \frac{dx}{x}\right) \tag{12}$$

**The first excited states ($\lambda = 1$) are degenerate and can be labeled by the momentum fraction, $y$, of one dressed photon/boson.**
**The 0-parton component of the first excited state**

$$f_0^{(1y)} = -Z\frac{g}{\sqrt{y}}, \tag{13}$$

**the one-parton component**

$$f_1^{(1y)}(x_1) = Z\left(-\frac{g}{\sqrt{y x_1}} + \delta(x_1 - y)\right), \tag{14}$$

**The two-parton wave function is**

$$f_2^{(1y)}(x_1,x_2) = Z\left(-\frac{g^3}{\sqrt{yx_1x_2}} + \frac{g}{\sqrt{x_1}}\delta(x_2-y) + \frac{g}{\sqrt{x_2}}\delta(x_1-y)\right), \quad x_1 \le x_2, \qquad (15)$$

**the three-parton wave function reads**

$$f_3^{(1y)}(x_1,x_2,x_3) = Z\left(-\frac{g^4}{\sqrt{yx_1x_2x_3}} + \frac{g^2}{\sqrt{x_1x_2}}\delta(x_3-y) + \frac{g^2}{\sqrt{x_1x_3}}\delta(x_2-y) + \frac{g^2}{\sqrt{x_2x_3}}\delta(x_1-y)\right),$$
$$x_1 \le x_2 \le x_3.$$

**Second excited family ($\lambda = 2$)**

$$f_2^{(2yz)}(x_1,x_2)/Z = \frac{g^4}{\sqrt{yzx_2x_2}}$$
$$-\frac{g^2}{\sqrt{x_1z}}\delta(x_2-y) - \frac{g^2}{\sqrt{x_1y}}\delta(x_2-z) - \frac{g^2}{\sqrt{x_2z}}\delta(x_1-y) - \frac{g^2}{\sqrt{x_2y}}\delta(x_1-z)$$
$$+\delta(x_1-y)\delta(x_2-z), \quad y \le z, \quad x_1 \le x_2 \qquad (16)$$

### 4.2 Numerics $\epsilon < x_i$

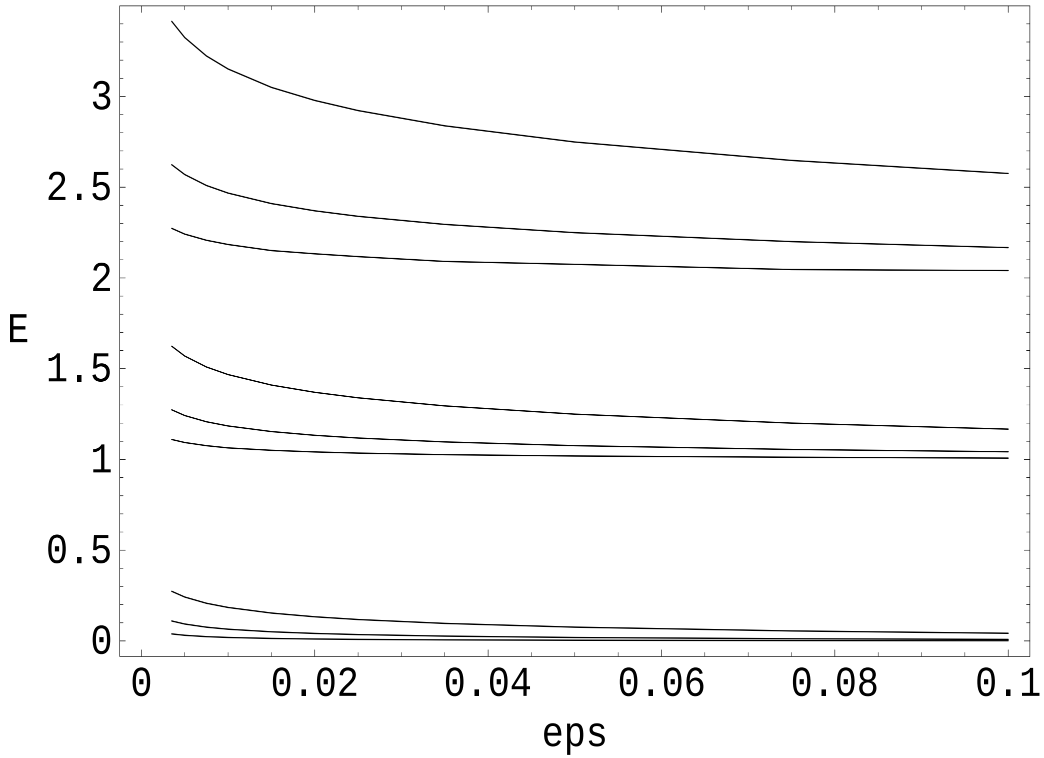



Figure 3: $\epsilon$ **dependence of the first three levels of the BN-toy for** $n_{max} = 2, 3, 4$**, at** $g = 0.5$

### 4.3 $QCD_2$ with an addjoined scalar matter

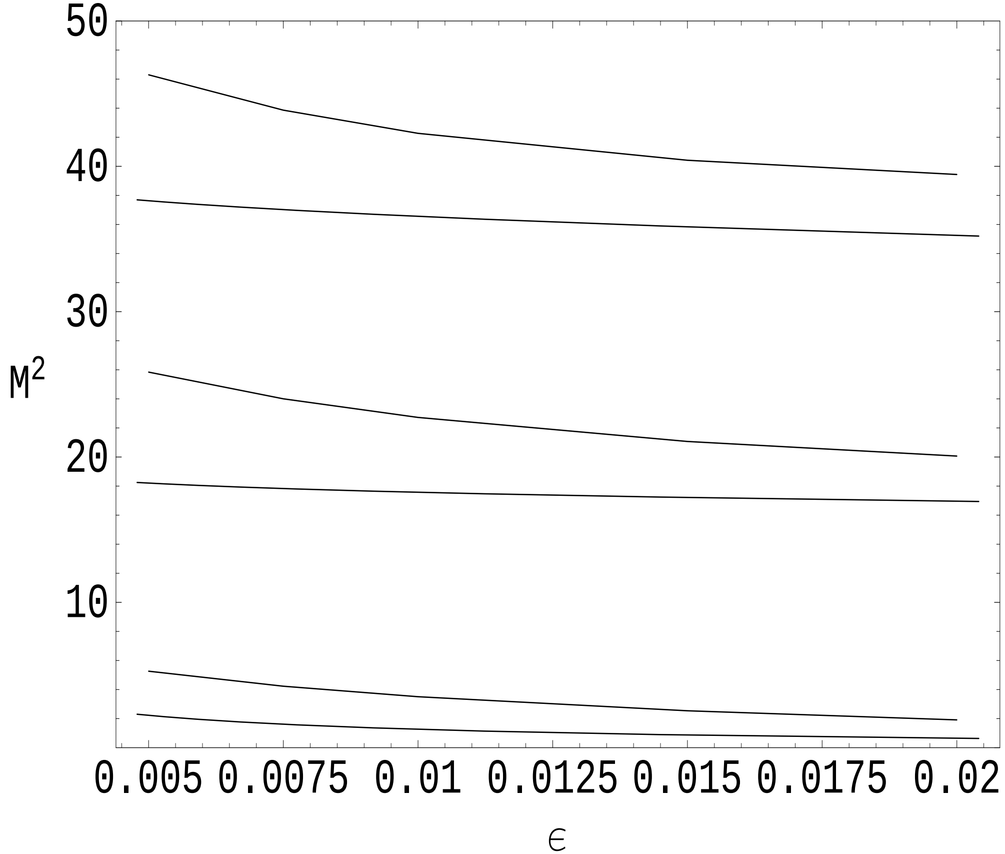



Figure 4: **As above, but comparing the LCD (lower) with the integral equations (upper). Two partons only.**

## 5 Coulomb divergences

- **IR divergencies (logarithmic) couple different multiplicity sectors**
- **Coulomb divergencies (linear), but they cancel within one multiplicity**
- **Can be done independently for each parton multiplicity** $p$

**A possibility**

- $\longrightarrow$ **Solve Coulomb problem first, and then successively add radiation**

**Simplified Hamiltonian,** $SYM_2$ **reduced from** $SYM_4$**, keeping only Coulomb terms**

$$H_C^{quad} = const \int_0^\infty dk \int_0^k \frac{dq}{q^2} \mathbf{Tr}[a_k^\dagger a_k] \tag{17}$$

$$H_C^{quartic} = -const \int_0^\infty dp_1 dp_2 \left[ \int_0^{p_1} \frac{dq}{q^2} \mathbf{Tr}[a_{p_1}^\dagger b_{p_2}^\dagger b_{p_2+q} a_{p_1-q}] + \int_0^{p_2} \frac{dq}{q^2} \mathbf{Tr}(a_{p_2}^\dagger b_{p_1}^\dagger b_{p_1+q} a_{p_2-q}) \right] \tag{18}$$

### 5.1 Two partons (a,b)

$$|k, K-k\rangle, \quad k = 1, .., K-1 \tag{19}$$

$$\langle k|H|k'\rangle \Rightarrow |\Phi_n\rangle \Rightarrow \Phi_n(k) \overset{FT}{\Rightarrow} \Phi_n(d_{12}) \tag{20}$$

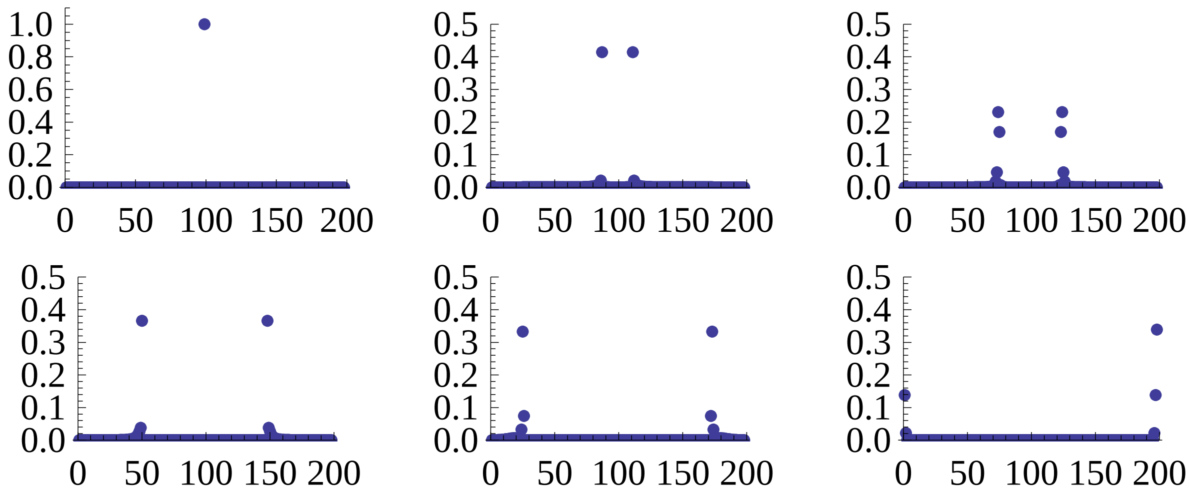



Figure 5: $\rho_n(d_{12}), p = 2, K = 200, n = 1, 25, 50, 100, 150, 199.$

## 6 Three partons

$$|k_1, k_2, K - k_1 - k_2\rangle, \quad k_1 = 1, .., K - 2, \quad k_2 = 1, .., K - k_1 - 1 \tag{21}$$

$$\langle k_1, k_2 | H | k_1', k_2' \rangle \Rightarrow |\Phi_n\rangle \Rightarrow \Phi_n(k_1, k_2) \stackrel{FT}{\Rightarrow} \Phi_n(d_{13}, d_{23}) \tag{22}$$

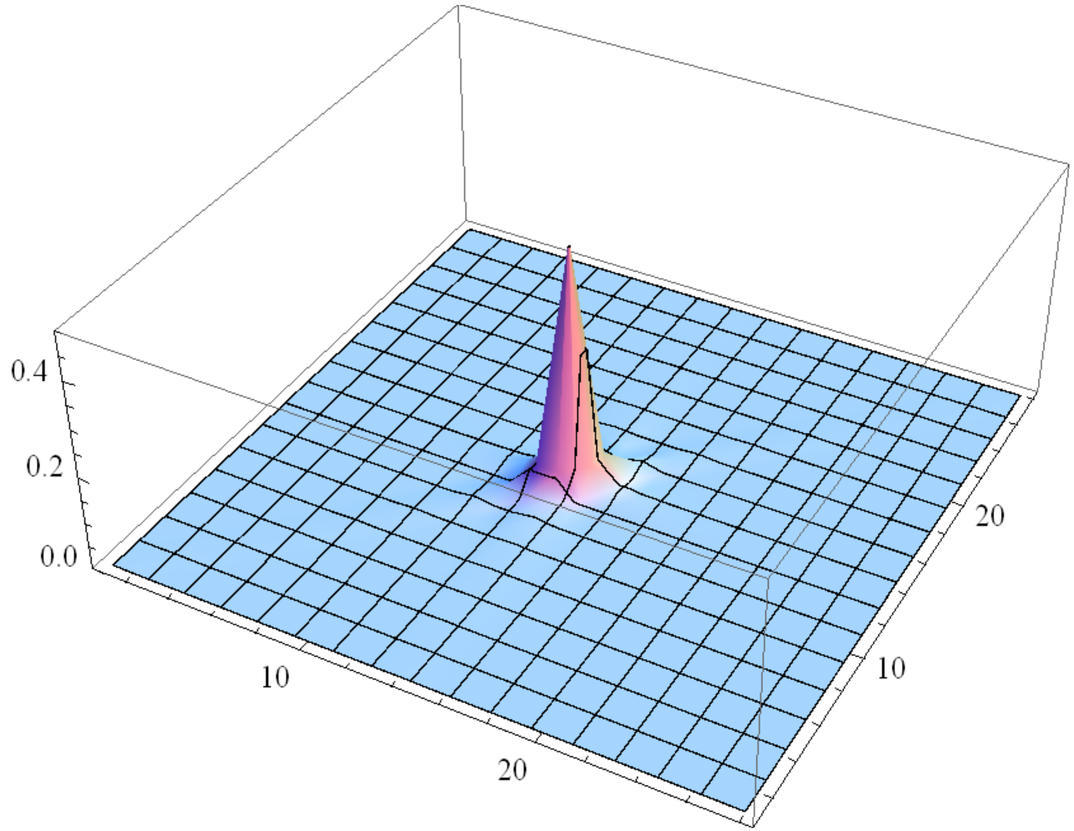



Figure 6: $\rho_1(d_{13}, d_{23})$

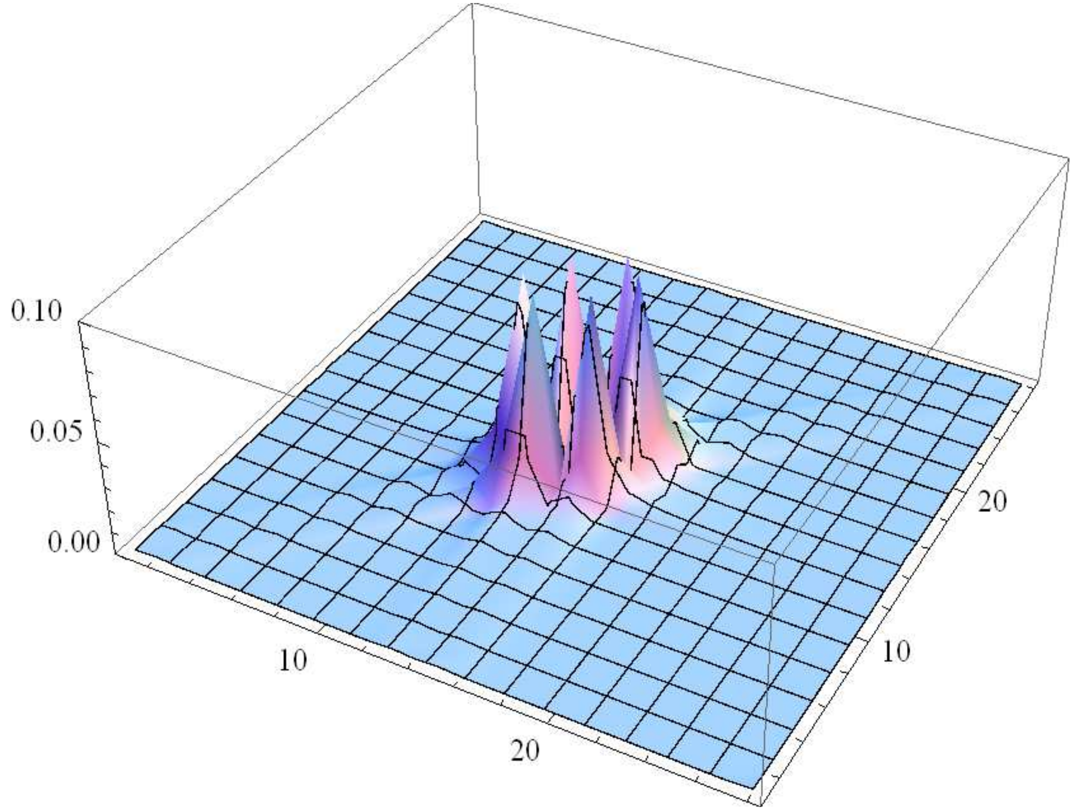



Figure 7: $|\rho_{10}(d_{13}, d_{23})$

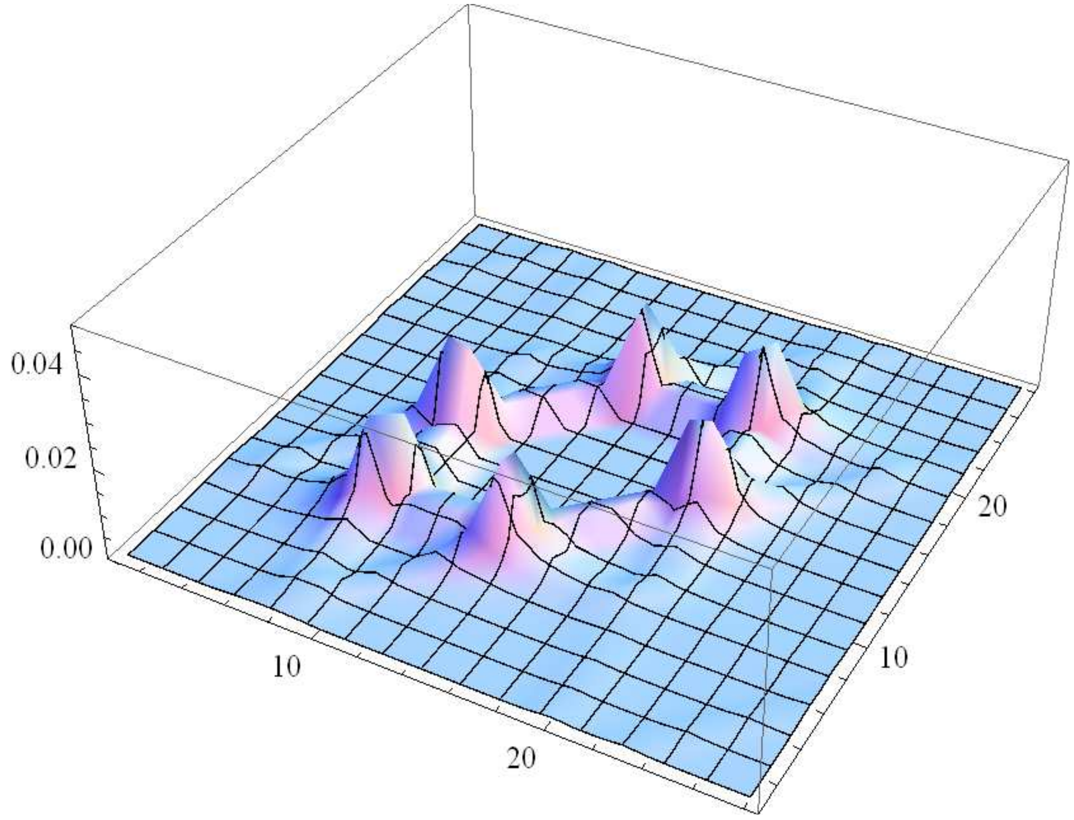



Figure 8: $\rho_{50}(d_{13}, d_{23})$

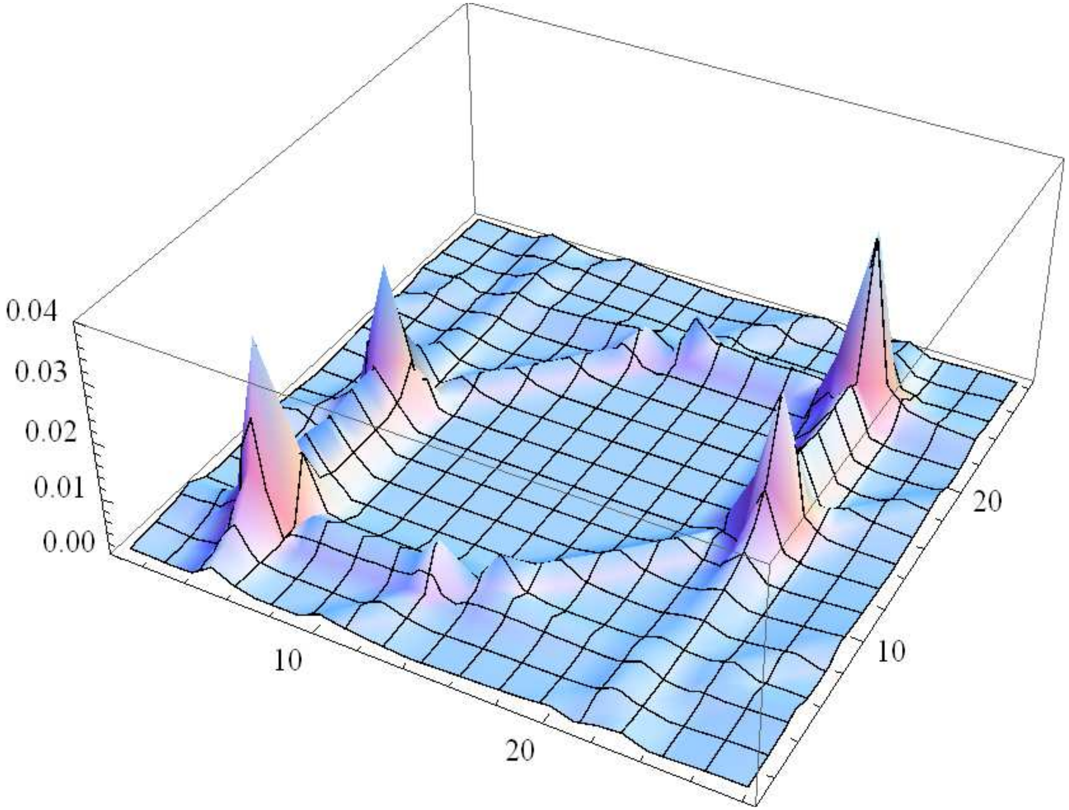



Figure 9: $\rho_{100}(d_{13}, d_{23})$

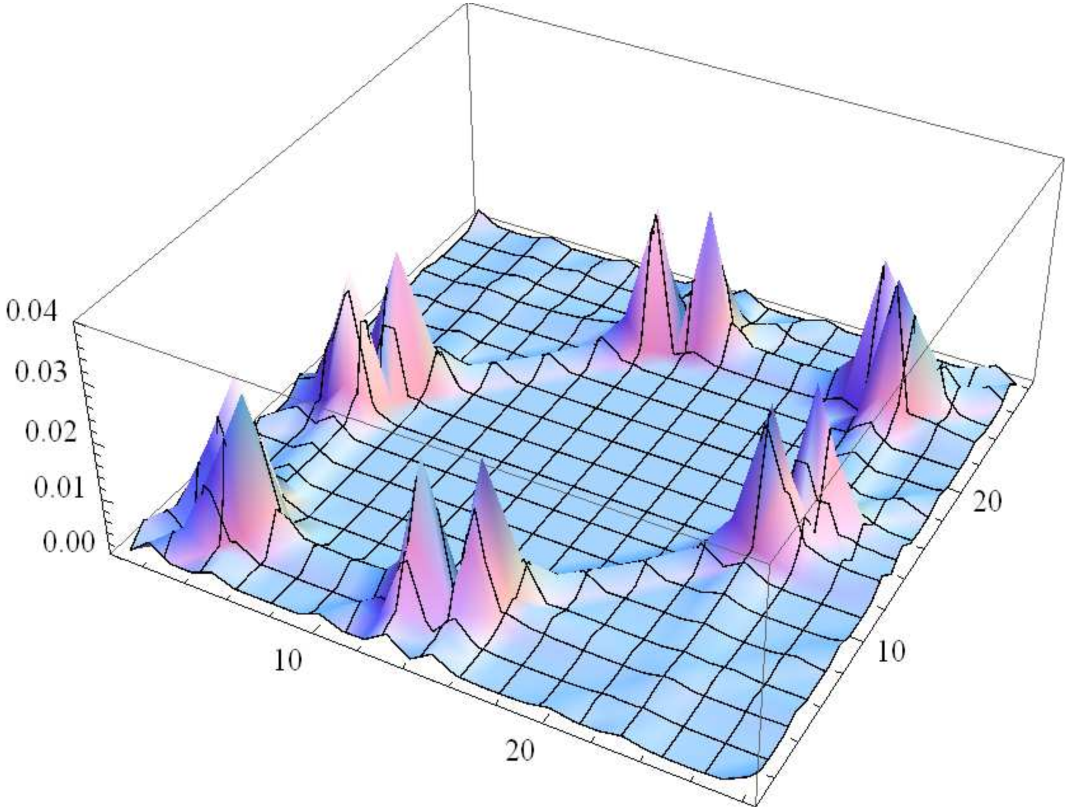



Figure 10: $\rho_{200}(d_{13}, d_{23})$

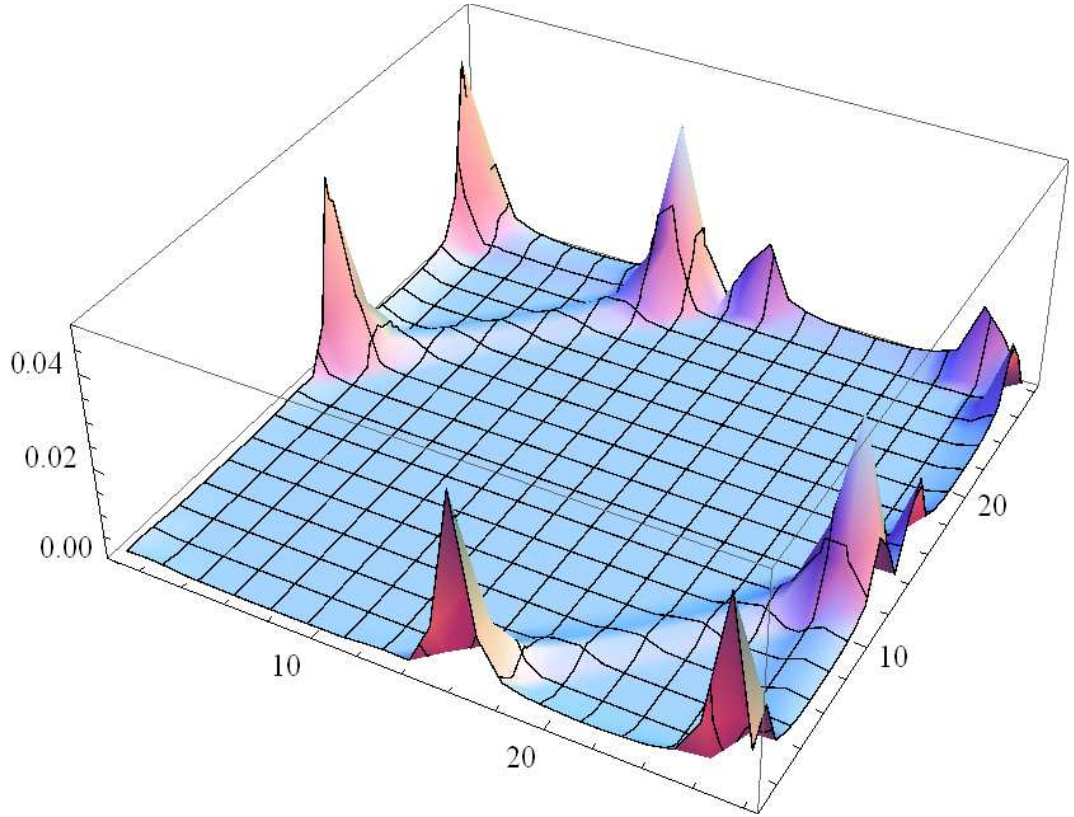



Figure 11: $\rho_{300}(d_{13}, d_{23})$

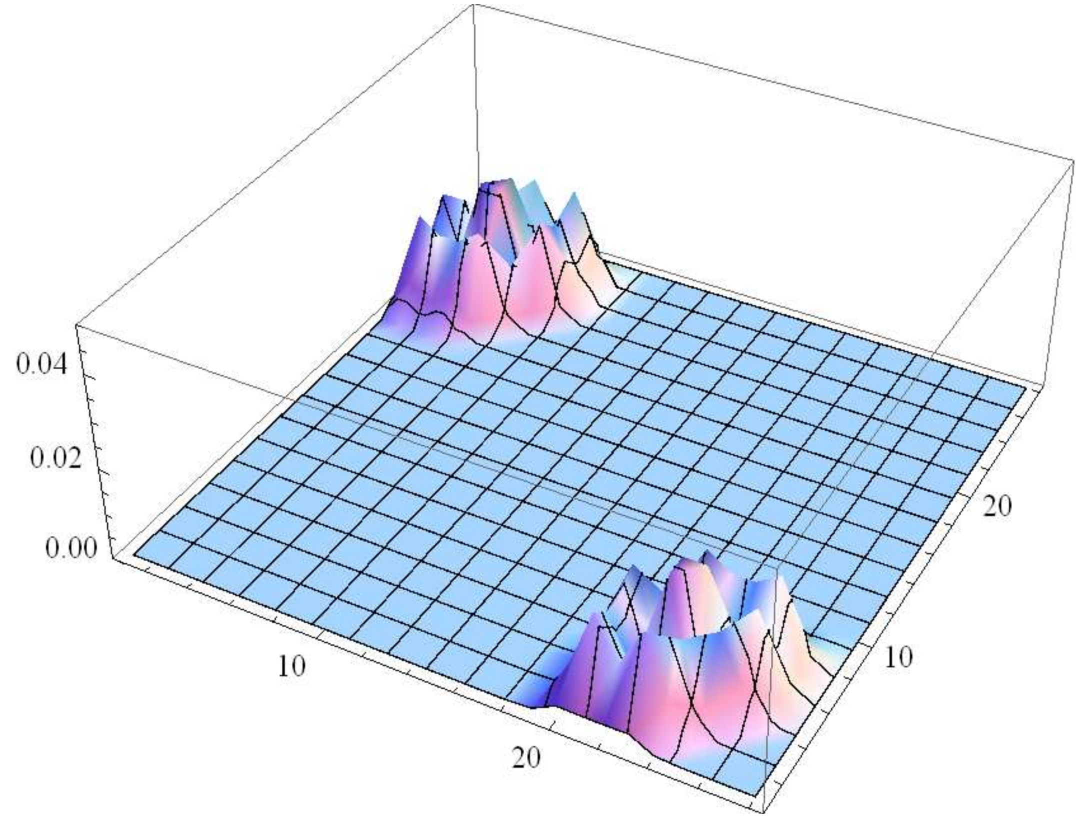



Figure 12: $\rho_{400}(d_{13}, d_{23})$

## The highest state

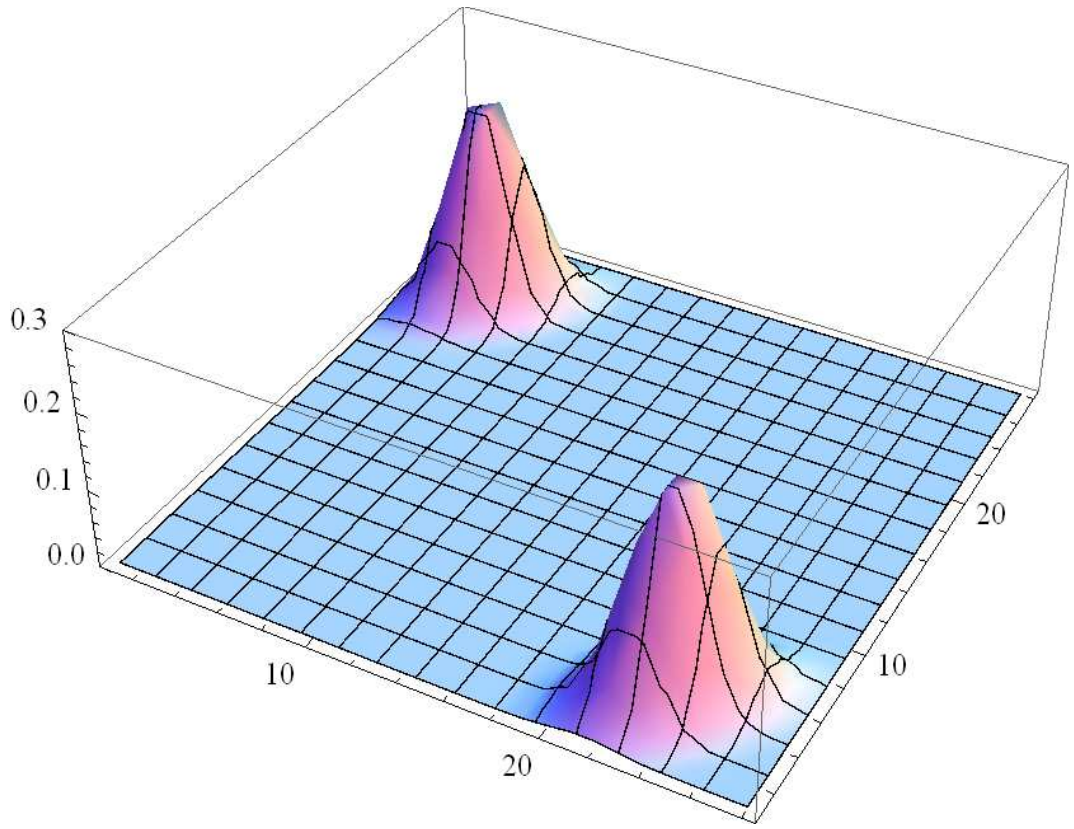



Figure 13: $\rho_{406}(d_{13}, d_{23})$

## A "mercedes" configuration

## This is in fact a generalization of the 't Hooft solution to many bodies.

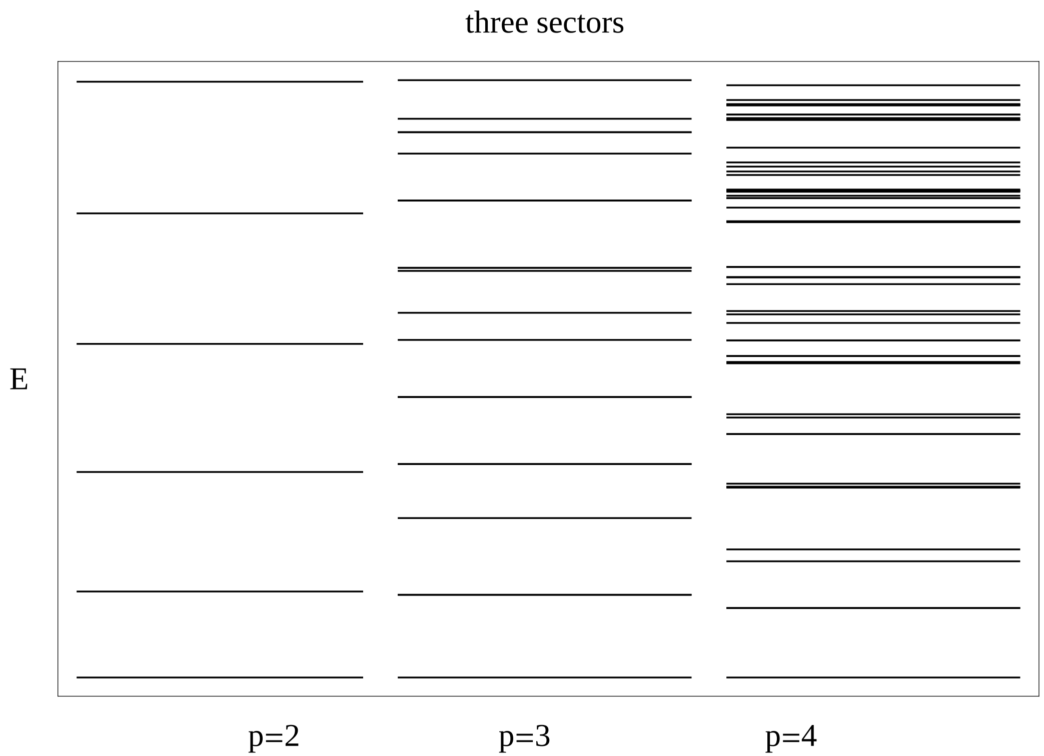



Figure 14: **Lowest spectra in sectors with 2,3 and 4 partons.**

## Stringy plot for two partons

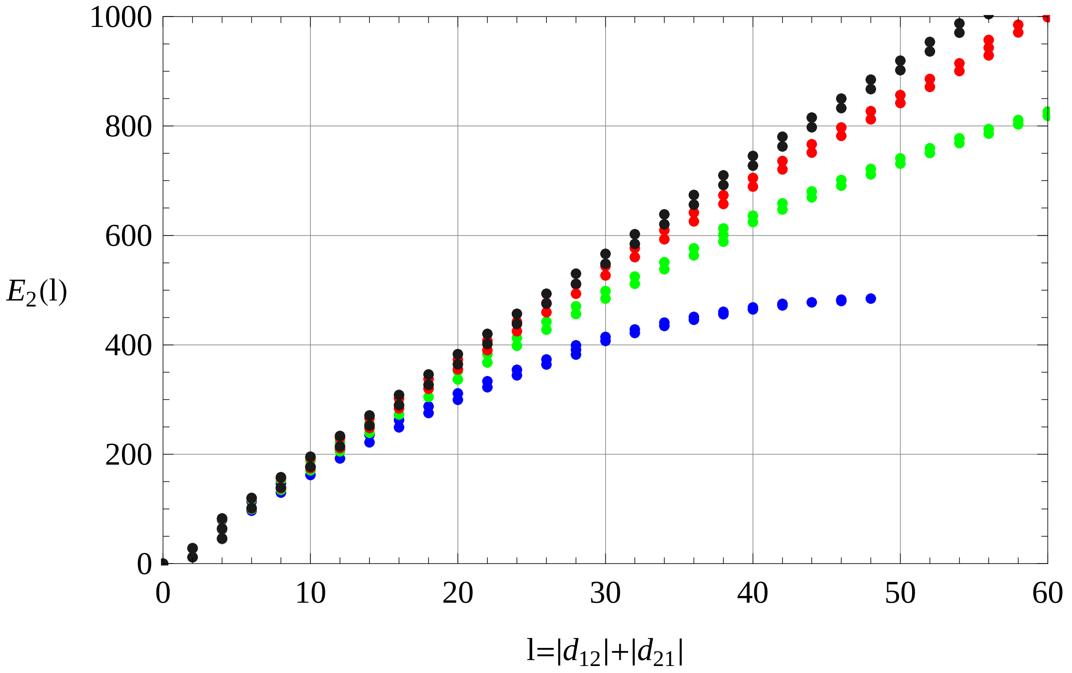



Figure 15: **Eigenenergies of the, p=2, excited states as a function of the relative separation between two partons,** $K = 30, 50, 100, 200$**.**

# Families of states with three partons

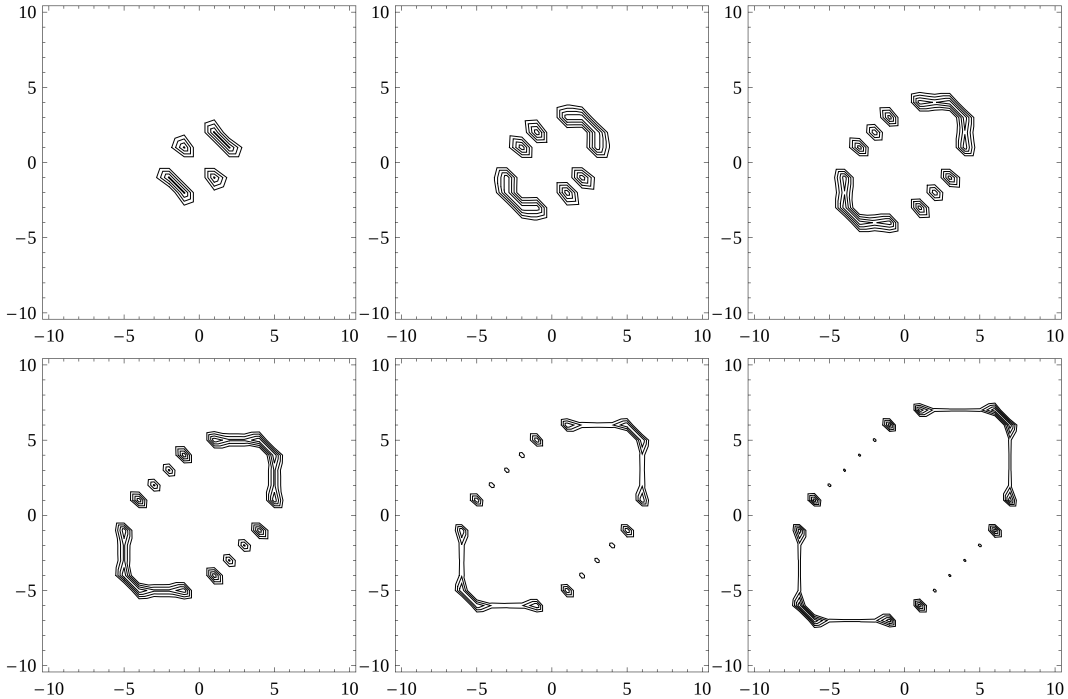



Figure 16: **Contour plots of** $\rho_n(d_{13}, d_{23})$**, as partons are moved further away.**
**Series A :** $n = 10, 19, 28, 41, 54, 72$**,** $4 \leq l = |d_{12}| + |d_{23}| + |d_{31}| \leq 14$**.**
**The minimal distance between partons = 1.**

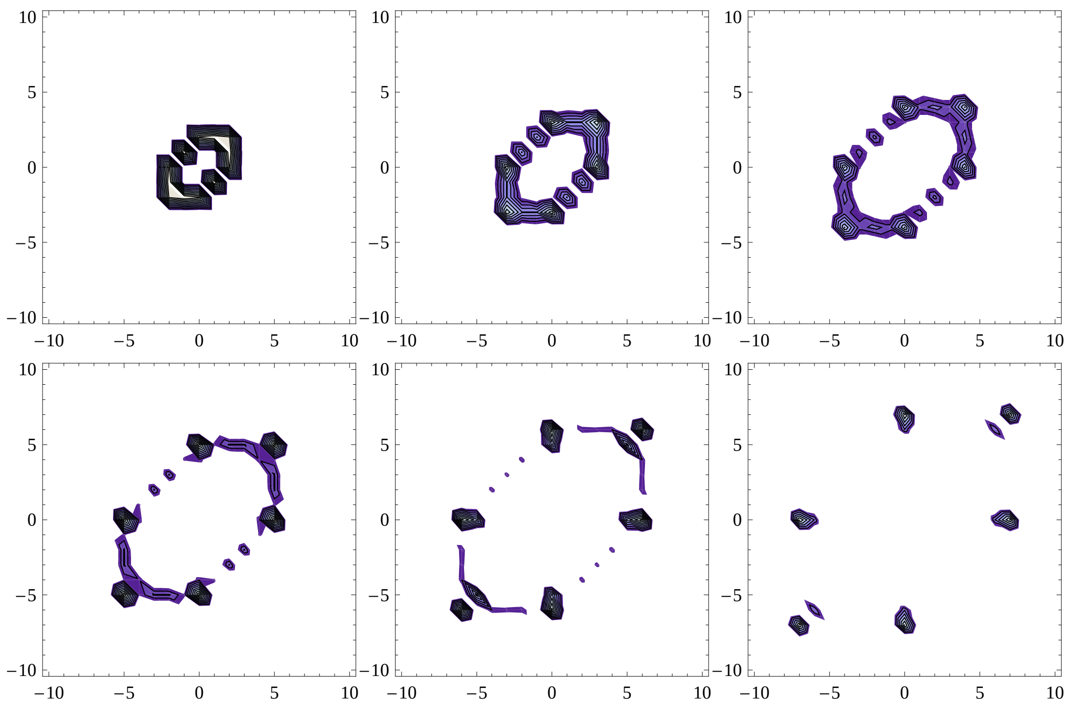



Figure 17: **Series B. As above but now diquarks are allowed,** $d_{min} = 0$

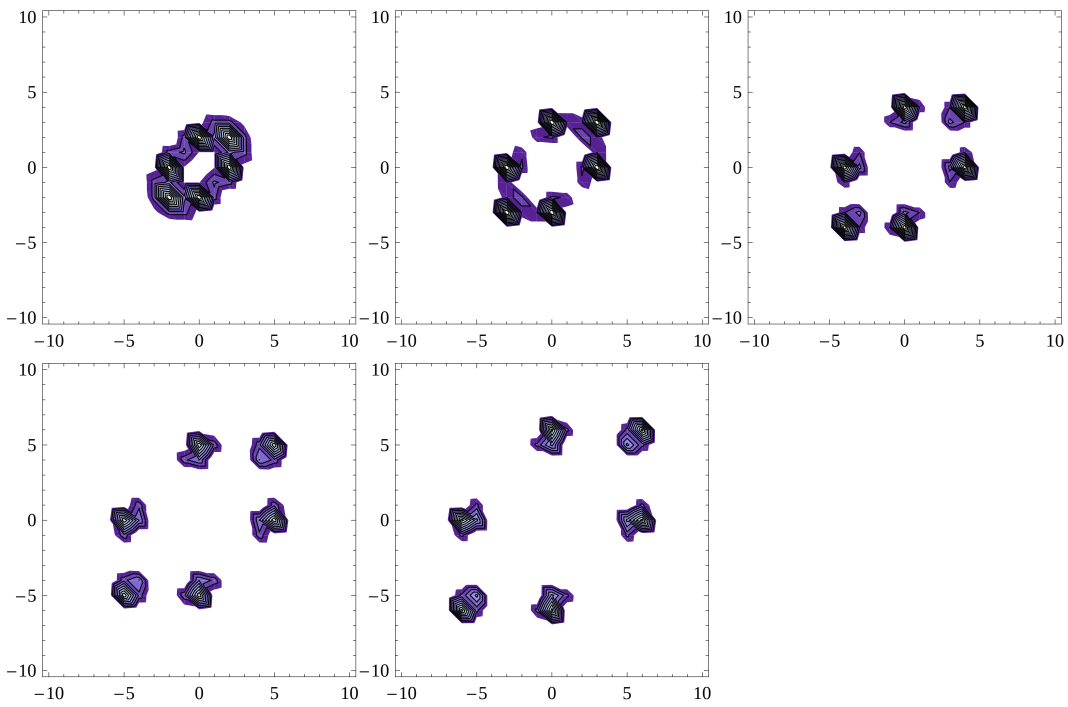



Figure 18: **Series D: diquarks only but somewhat dressed,** $d_{min} = 0$

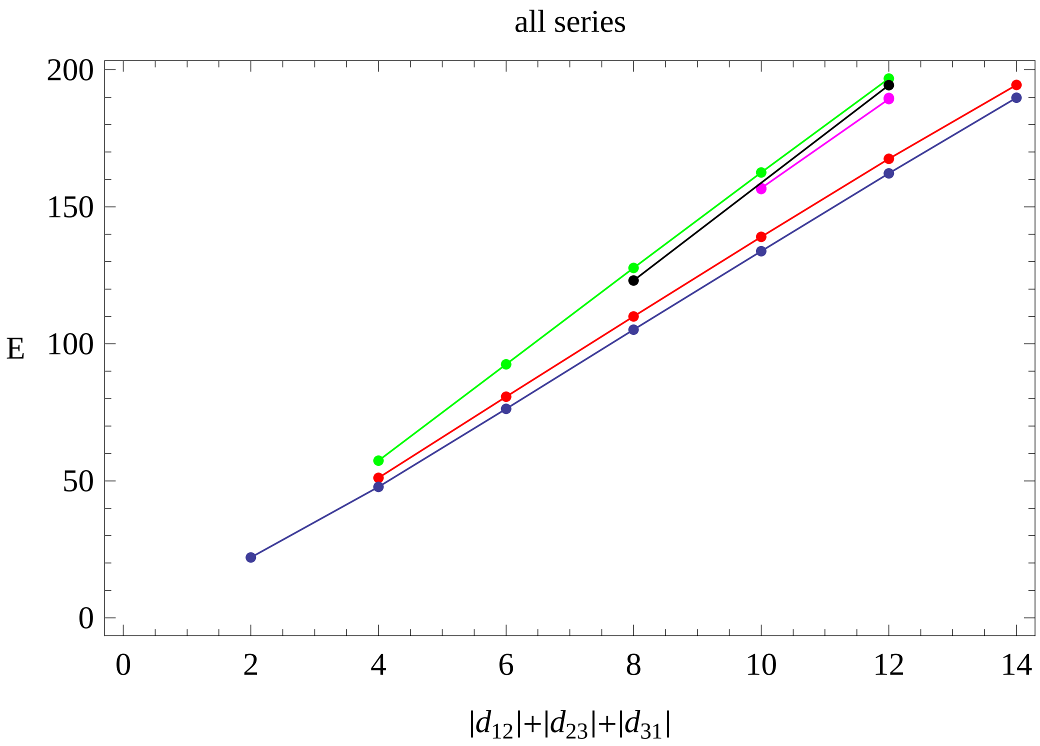



Figure 19: **Energies of above series vs.** $l = |q_{12}| + |d_{23}| + |d_{31}|$

# Stringy plot for three partons

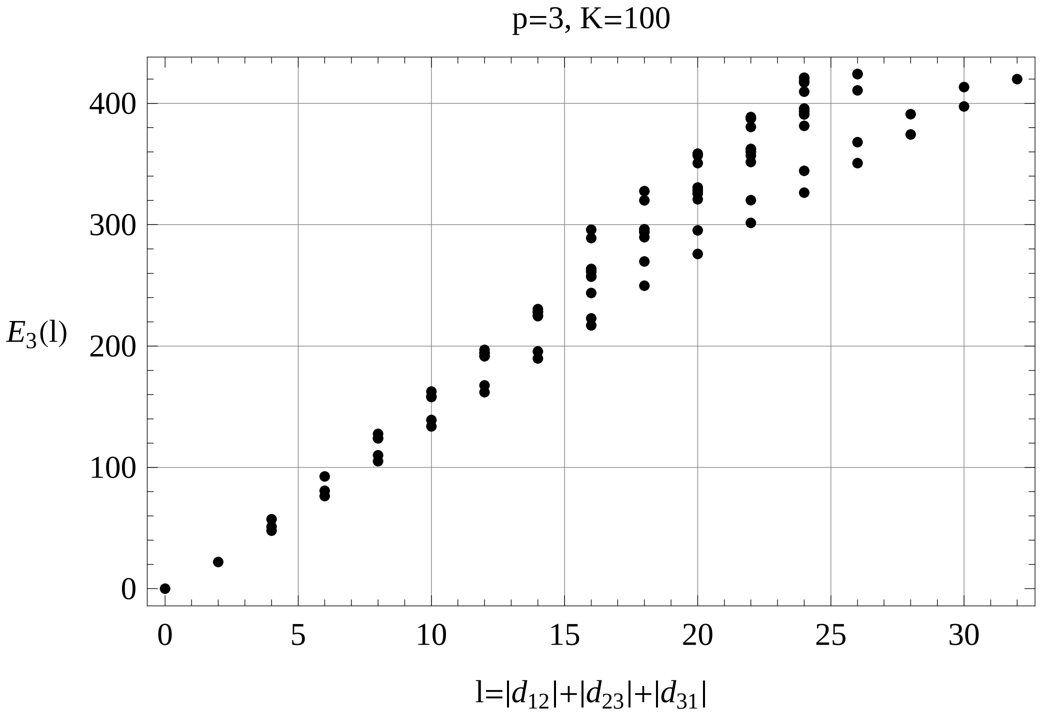



Figure 20: **Eigenenergies of the, p=3, excited states as a function of the total length of strings stretching between three partons.**

**The string tensions extracted from $E_2(l)$ and $E_3(l)$ are consistent !**

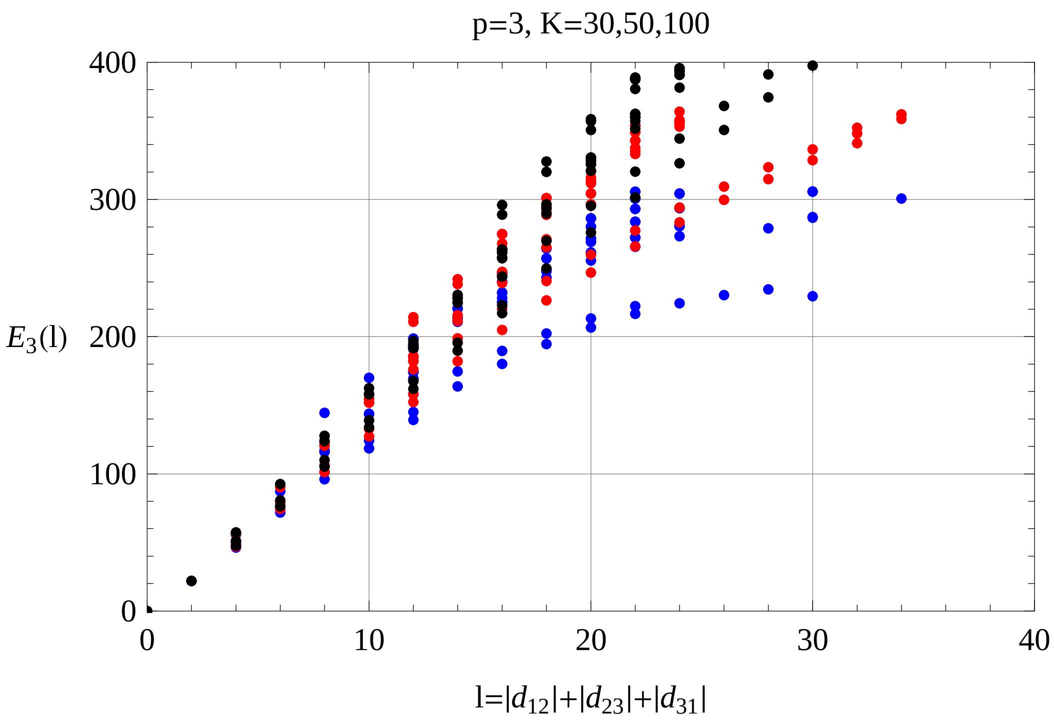



Figure 21: **As above, but for different K:** $K = 30, 50, 100$

# Four partons

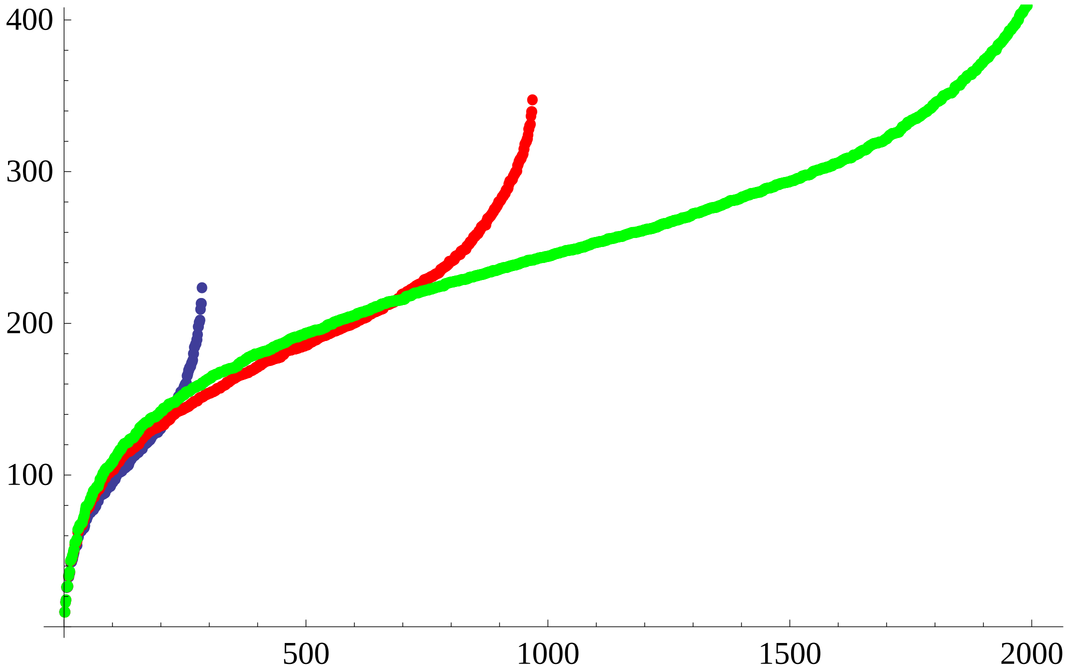



Figure 22: **Convergence of eigenenergies for the four parton case as we increase $K = 30, 40, 50$.**

## 7 Summary and the future

- 't Hooft solutions have a very simple interpretation in the configuration space.
- Generalization to more partons

  a) is readily possible, and

  b) also confirms a simple string picture.
- Future: generalizations of the (1+1) Coulomb problem

  identical particles

  fermions

  supersymmetry

  high multiplicities
- Add radiation
- Mass gap in the 1+1 supersymmetric theory
- 3+1