BPS Skyrme models

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Skyrme models

- Low energy effective theory of hadrons - currently unknown
- Degrees of freedom of QCD:
  - high energy: quarks and gluons
  - low energy: hadrons
- One proposal: Skyrme model Skyrme (61), Adkins, Nappi, Witten (83)
  - primary fields are mesons
  - baryons (hadrons and nuclei) are realized as solitons
  - realizes unbroken symmetries
  - simplest case (two flavors): target space = SU(2) (isospin) matrix U
  - topological charge = baryon number

Original Skyrme model:

- Skyrme field U:

$$x^\mu \to U(x) : \quad \mathbb{R}^3 \times \mathbb{R} \to SU(2)$$

- Lagrangian:

$$L = L_2 + L_4 + L_0$$

- Sigma model term:

$$L_2 = -\frac{f_\pi^2}{4} \,\mathrm{Tr}\,(U^\dagger \partial_\mu U \, U^\dagger \partial^\mu U)$$

pion kinetic term

- quartic Skyrme term:

$$L_4 = -\frac{1}{32e^2}\,\mathrm{Tr}\,([U^\dagger\partial_\mu U, U^\dagger\partial_\nu U]^2)$$

necessary to circumvent the Derrick argument

L_2, L_4 symmetric under SU(2) $\times$ SU(2), $U \to VUW^\dagger$

- Potential term:

$$L_0 (= -\mu^2 V(U, U^\dagger)) = -\mu^2 V(\mathrm{tr}\, U)$$

Symmetric under diagonal SU(2), $U \to VUV^\dagger$

Generalized Skyrme models:

QCD $\Rightarrow$ Chiral effective meson/baryon theory (true at $N \to \infty$) t'Hooft (74), Witten (79)

$\Downarrow$ derivative expansion Simic (85)

$\mathcal{L}_{sk}$ + higher order terms Jackson et. al. (85), Oka (87), Li, Yan (92)...

$\Downarrow$ (?)

Integrable (BPS) soliton model physical interpretation

Need for BPS soliton model

- mathematical reasons
    - solvable model, exact solutions
    - analytical calculations: impact of values of parameters, testing potentials
    - approximate methods for non-integrable models
- physical reasons
    - too big binding energy
    - shell or crystal like densities

- Simplest proposals
  - $L_2 + L_0$ - excluded by the Derrick
  - $L_4 + L_0$ -excluded by dynamics: no topological solitons

Among higher order terms is the following sextic term

$$L_6 = \frac{\lambda^2}{24^2} \left[\mathrm{Tr} \left(\epsilon^{\mu\nu\rho\sigma} U^\dagger \partial_\mu U \, U^\dagger \partial_\nu U \, U^\dagger \partial_\rho U \right) \right]^2$$

- Quadratic in time derivatives $\Rightarrow$ standard hamiltonian formulation

- Square of the topological (baryon) current

$$L_6 = \lambda^2 \pi^4 \mathbb{B}_\mu \mathbb{B}^\mu$$

where

$$\mathbb{B}^\mu = \frac{1}{24\pi^2} \mathrm{Tr}\left(\epsilon^{\mu\nu\rho\sigma} U^\dagger \partial_\nu U \, U^\dagger \partial_\rho U \, U^\dagger \partial_\sigma U\right)$$

is the topological (baryon) current

- Phenomenologically induced by a massive vector meson coupled to the baryon density. Adkins, Nappi (84), Sutcliffe (09)
- Improves phenomenological results of Skyrme model

BPS Skyrme model:

Propose following limit of generalized Skyrme models

$$L_{06} = L_6 + L_0$$

- ∞ many symmetries
- Integrable: ∞ many conservation laws
- BPS (Bogomolny) bound
- ∞ many exact solutions saturating the BPS bound

Parametrization for U

$$U = e^{i\xi\vec{n}\cdot\vec{\sigma}} = \cos\xi + i\sin\xi\vec{n}\cdot\vec{\sigma} \qquad \vec{n}^2 = 1$$

and stereographic projection

$$\vec{n} = \frac{1}{1+|u|^2}\left(u+\bar{u}, -i(u-\bar{u}), 1-|u|^2\right)$$

$$\Rightarrow \quad L_{06} = -\frac{\lambda^2 \sin^4\xi}{(1+|u|^2)^4}\,(\epsilon^{\mu\nu\rho\sigma}\xi_\nu u_\rho \bar{u}_\sigma)^2 - \mu^2 V(\xi)$$

Symmetries

- Poincare Symmetries
- ∞ many target space diffeomorphisms
  - L_6 is square of pullback of the volume form in target space $\mathrm{SU}(2) \sim S^3$,

$$dV = -i \frac{\sin^2 \xi}{(1+|u|^2)^2} d\xi du d\bar{u}$$

    $\Rightarrow$ has all volume-preserving diffeomorphisms on target space S^3 as symmetries.
  - $L_0 = -\mu^2 V(\xi)$ respects some of them: the ones that act nontrivially only on u, $\bar{u}$ $\Rightarrow$ area-preserving diffeos on target space S^2 spanned by u, but may depend on ξ as a parameter:

$$\xi \to \xi\,, \quad u \to \tilde{u}(u, \bar{u}, \xi)\,, \quad (1+|\tilde{u}|^2)^{-2} d\xi d\tilde{u} d\bar{\tilde{u}} = (1+|u|^2)^{-2} d\xi d\tilde{u} d\bar{u}$$

- Volume-preserving base space diffeos.
  Energy functional for static fields:

$$E = \int d^3x \left(\frac{\lambda^2 \sin^4 \xi}{(1+|u|^2)^4} (\epsilon^{mnl} i \xi_m u_n \bar{u}_l)^2 + \mu^2 V \right)$$

  Both d^3x and $\epsilon^{ijk}\partial_i\partial_j\partial_k$ invariant under volume preserving diffeos on base space $\mathbb{R}^3$. NOT a Noether symmetry.

Integrability - ∞ many conservation laws

Target space symmetries $\Rightarrow$ conserved currents

$$
\begin{aligned}
J^\mu_G &= G_{\bar{u}} \bar{\mathcal{K}}^\mu - G_u \mathcal{K}^\mu, \quad G = G(u, \bar{u}, \xi) \\
\mathcal{K}^\mu &\equiv \frac{1}{(1+|u|^2)^2} \frac{\partial (\epsilon^{\alpha\nu\rho\sigma} \xi_\nu u_\rho \bar{u}_\sigma)^2}{\partial \bar{u}_\mu}
\end{aligned}
$$

$$
\begin{aligned}
\partial_\mu J^\mu_G = {} & G_{\bar{u}\bar{u}} \bar{u}_\mu \bar{\mathcal{K}}^\mu + G_{\bar{u}u} u_\mu \bar{\mathcal{K}}^\mu - G_{u\bar{u}} \bar{u}_\mu \mathcal{K}^\mu - G_{uu} u_\mu \mathcal{K}^\mu \\
& + G_{\bar{u}} \partial_\mu \bar{\mathcal{K}}^\mu - G_u \partial_\mu \mathcal{K}^\mu + G_{\bar{u}\xi} \xi_\mu \bar{\mathcal{K}}^\mu - G_{u\xi} \xi_\mu \mathcal{K}^\mu = 0
\end{aligned}
$$

where $u_\mu \mathcal{K}^\mu \equiv 0$, $\xi_\mu \mathcal{K}^\mu \equiv 0$, $\bar{u}_\mu \mathcal{K}^\mu \equiv u_\mu \bar{\mathcal{K}}^\mu$

and $\partial_\mu \mathcal{K}^\mu = 0 \ldots$ field eq. for u

Bogomolny (BPS) bound

- Derrick scaling $\Rightarrow E_6 = E_0 \ldots$ compatible with BPS
- BPS bound $B \ldots$ top. charge (baryon number)

$$\begin{aligned}
E &= \int d^3x \left(\frac{\lambda^2 \sin^4 \xi}{(1+|u|^2)^4} (\epsilon^{mnl} i \xi_m u_n \bar{u}_l)^2 + \mu^2 V \right) \\
&= \int d^3x \left(\frac{\lambda \sin^2 \xi}{(1+|u|^2)^2} \epsilon^{mnl} i \xi_m u_n \bar{u}_l \pm \mu \sqrt{V} \right)^2 \\
&\mp \int d^3x \frac{2\mu\lambda \sin^2 \xi \sqrt{V}}{(1+|u|^2)^2} \epsilon^{mnl} i \xi_m u_n \bar{u}_l \\
&\geq \pm (2\lambda\mu\pi^2) \left[\frac{-i}{\pi^2} \int d^3x \frac{\sin^2 \xi \sqrt{V}}{(1+|u|^2)^2} \epsilon^{mnl} \xi_m u_n \bar{u}_l \right] \\
&\equiv 2\lambda\mu\pi^2 C_1 |B|
\end{aligned}$$

- for $V(\xi) \to 1$ ... standard expression for top. charge (and $C_1 \to 1$)

$$B \equiv \frac{-i}{\pi^2} \int d^3x \frac{\sin^2 \xi}{(1+|u|^2)^2} \epsilon^{mnl} \xi_m u_n \bar{u}_l$$

equivalently: integral of pull-back of volume form on target space S^3

- Remains true with $\sqrt{V(\xi)}$ present, in terms of the new target space coordinate $\bar{\xi}(\xi)$ with

$$\sin^2 \xi \sqrt{V(\xi)}\, d\xi = C_1 \sin^2 \bar{\xi}\, d\bar{\xi}$$

C_1 and second integration constant C_2 ... necessary to implement $\bar{\xi}(\xi = 0) = 0$, $\bar{\xi}(\xi = \pi) = \pi$
e.g. $V = 1 - \cos \xi \Rightarrow C_1 = \frac{32\sqrt{2}}{15\pi}$

Energy density profiles

- BPS solution $\Rightarrow \mathcal{E}_6 = \mathcal{E}_0 = \frac{1}{2}\mathcal{E} \ldots$ density profile detemined by potential
- One-vacuum potentials: $V(r=\infty)=0$, $V(r=0)\neq 0 \Rightarrow$ density profile of the ball type
- Two-vacua potentials: $V(r=\infty)=0$, $V(r=0)=0 \Rightarrow$ density profile of the shell type
- More complicated vacuum manifolds $\Rightarrow$ more complicated profiles (may be determined by analytical arguments)
- In standard Skyrme model: similar behaviour, but results from complicated numerics Houghton, Manton, Sutcliffe (98), Atiyah, Manton (89), Battye, Sutcliffe (97), (01), (02), (05), (06), (09)

Exact solutions

- non-trivial configuration: u covers full $\mathbb{C}$, and $\xi \in [0, \pi]$
- symmetric (hedgehog) ansatz

$$\xi = \xi(r), \quad u(\theta, \phi) = g(\theta) e^{in\phi}$$

  compatible with field eq.

- Equation for $g(\theta)$

$$\partial_\theta \left(\frac{g g_\theta}{(1 + g^2)^2 \sin\theta} \right) = 0$$

  with solution

$$g(\theta) = \tan\frac{\theta}{2}$$

  for all n

- Equation for ξ

$$\frac{n^2\lambda^2\sin^2\xi}{2r^2}\partial_r\left(\frac{\sin^2\xi\,\xi_r}{r^2}\right)-\mu^2 V_\xi=0$$

or with

$$z=\frac{\sqrt{2}\mu}{3|n|\lambda}r^3$$

$$\Rightarrow\quad \sin^2\xi\,\partial_z\left(\sin^2\xi\,\xi_z\right)-V_\xi=0$$

first integral

$$\frac{1}{2}\sin^4\xi\,\xi_z^2=V$$

dim. reduced version of BPS

Compact and non-compact skyrmions

- Reduced energy functional (example $V = (1 - \cos\xi)^a$)

$$E = C \int dz \left(\frac{1}{2} \sin^4 \xi \xi_z^2 + (1 - \cos\xi)^a \right)$$

where $C = 2\sqrt{2}\lambda\mu|n|$
with $\chi = \sqrt{1 - \cos\xi}$, $\Phi = \chi^3$

$$\begin{aligned} E &= C \int dz \left(2(2 - \chi^2)\chi^4 \chi_z^2 + \chi^{2a} \right) \\ &= C \int dz \left(\frac{2}{9}(2 - \Phi^{\frac{2}{3}})\Phi_z^2 + \Phi^{\frac{2a}{3}} \right) \end{aligned}$$

for $a < 3 \ldots$ sublinear force law $\Rightarrow$ compactons

Example: Skyrme potential

$$V = \frac{1}{2}\mathrm{Tr}(1-U) \quad \rightarrow \quad V(\xi) = 1 - \cos\xi$$

- Solution

$$\xi = \begin{cases} 2\arccos\sqrt[3]{\frac{3z}{4}} & z \in \left[0, \frac{4}{3}\right] \\ 0 & z \geq \frac{4}{3}. \end{cases}$$

compacton Rosenau et. al. (93), Arodź (02)

- Energy

$$E = \frac{64\sqrt{2}\pi}{15}\mu\lambda|n|$$

linear in $|B| = |n|$

- energy density

$$\begin{aligned}\mathcal{E} &= 8\sqrt{2}\mu\lambda(1-|n|^{-\frac{2}{3}}\tilde{r}^2) \quad \text{for} \quad 0 \le \tilde{r} \le |n|^{\frac{1}{3}} \\ &= 0 \quad \text{for} \quad \tilde{r} > |n|^{\frac{1}{3}}\end{aligned}$$

where

$$\tilde{r} = \left(\frac{\mu}{4\sqrt{2}\lambda}\right)^{\frac{1}{3}} r \equiv \frac{r}{R_0}$$

rescaled radius $\tilde{r}$ $R_0 \ldots$ compacton radius for $n = 1$

- baryon density

$$\begin{aligned}\mathcal{B} &= \text{sign}(n)\frac{4}{\pi^2}(1-|n|^{-\frac{2}{3}}\tilde{r}^2)^{\frac{1}{2}} \quad \text{for} \quad 0 \le \tilde{r} \le |n|^{\frac{1}{3}} \\ &= 0 \quad \text{for} \quad \tilde{r} > |n|^{\frac{1}{3}}\end{aligned}$$

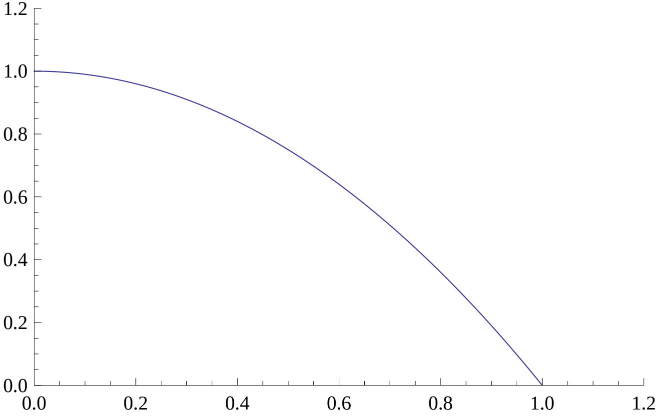

Figure: Normalized energy density as a function of the rescaled radius $\tilde{r}$, for topological charge n=1. For $|n| > 1$, the height of the density remains the same, whereas the radius grows like $|n|^{\frac{1}{3}}$

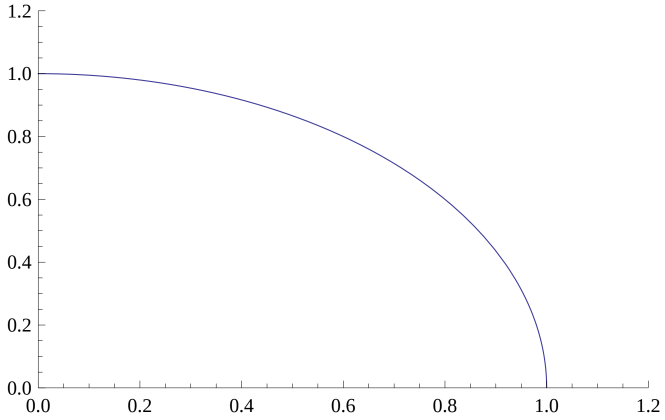

Figure: Normalized topological charge density as a function of the rescaled radius $\tilde{r}$, for topological charge n=1. For $|n| > 1$, the height of the density remains the same, whereas the radius grows like $|n|^{\frac{1}{3}}$

Generalized Skyrme potentials Zakrzewski (04), Karliner et. al. (08),(09)

$$V = (\frac{1}{2}\text{Tr}(1 - U))^a \rightarrow V(\xi) = (1 - \cos\xi)^a$$

- Solutions:
  - Compactons for $a < 3$
  - exponentially localized solution for $a = 3$
  - power-like localized solutions for $a > 3$

Two-vacua potential - example: Zakrzewski (04)

$$V = 1 - \cos^2 \xi$$

vacua at $\xi = 0, \pi$

- Solution

$$\xi = \begin{cases} \arccos \sqrt{2}z - 1 & z \in \left[0, \sqrt{2}\right] \\ 0 & z \geq \sqrt{2} \end{cases}$$

compacton; $E \propto |B|$

- Energy density

$$\epsilon(z) \sim \sqrt{2}z(1 - \sqrt{2}z)$$

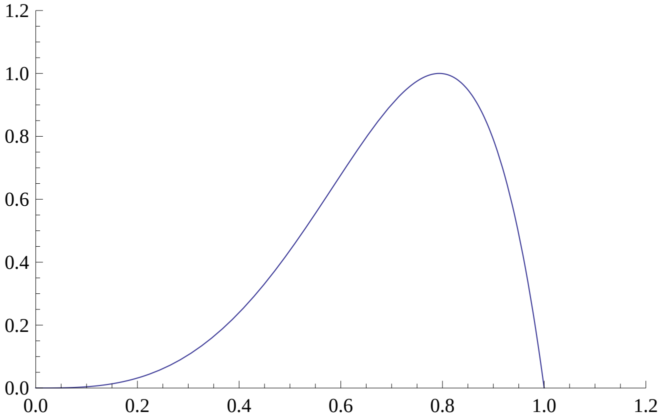

Figure: Two-vacua potential - shell structure: normalized energy density as a function of the rescaled radius $\tilde{r}$, for topological charge n=1. For $|n| > 1$, the height of the maximum remains the same, whereas the radius grows like $|n|^{\frac{1}{3}}$

Sutcliffe's BPS Skyrme model

- original Skyrme model coupled with an infinite tower of the vector mesons

  $$\Updownarrow$$

  dimensional reduction of 5dim YM
- nice realization of the vector meson domination
- no potential term

Some phenomenology of nuclei

for standard Skyrme potential

$$V = 1 - \cos \xi$$

A) Classical solitons

- Linear mass - baryon number relation:
  - BPS bound $\Rightarrow E = 64\sqrt{2}\mu\lambda|B|/15 \equiv E_0|B|$
  - well-known relation for nuclei

B	E_{BPS}	$E_{vec\ \ Skyrme}$	E_{Skyrme}	$E_{experiment}$
1	931.75	996	1024	939
2	1863.5	1999	1937	1876
3	2795.25	2913	2836	2809
4	3727	3727	3727	3727
6	5590.5	-	5520	5601
8	7454	-	7327	7455
10	9317.5	-	9113	9327

Table: Energies of the solutions in the BPS Skyrme model, compared with masses for the vector-Skyrme and massive Skyrme models, as well as with the experimental values. Numbers in MeV

- No binding energy
  - BPS solutions $\Rightarrow$ zero binding energy
  - binding energies for physical nuclei small $\leq 1\%$
  - in standard Skyrme model significantly bigger
- No forces between solitons
  - BPS and compactons $\Rightarrow$ no forces
  - physical nuclei: very short range interaction
- radii of nuclei
  - Compacton: definite radius
    $R = (2\sqrt{2}\lambda|B|/\mu)^{(1/3)} \equiv R_0 \sqrt[3]{|B|}$
  - reproduces phenomenological relation $R \sim 1.25 \sqrt[3]{|B|}\text{fm}$

- Incompressible fluid
  - Static energy functional: volume-preserving diffeos (VPDs)
  - Symmetries of an incompressible fluid where surface energy is neglected
  - Physical nuclei do not have this symmetry: have definite shape, deformations cost energy
  - But: volume-preserving deformations cost less energy . . . liquid drop model of nuclei

$\Rightarrow$ classical model reproduces some features of the nuclear liquid drop model (mass prop to volume, strictly finite size, VPDs)

- Clearly too naive (no pions $\Rightarrow$ no interactions, no pion cloud; no quantum corrections)

B) Soliton quantization

work in progress; top. charge $Q = 1$ (nucleon)

- Collective coordinate quantization: Quantize symmetry transformations which are NOT symmetries of the soliton (=lightest d.o.f.)
- Example: rotations (equivalent to isospin = diagonal target space SU(2), because $\mathrm{SO(3)_b} \times \mathrm{SU(2)_{diag}}$ IS symmetry of hedgehog

⇒ Rigid rotor quantization (=undeformed rotating soliton, quantized angular momentum)

- Well-known procedure

- E.g. for isospin: Insert $U(t) = A(t)U_0A^\dagger(t)$ into action (U_0 = $Q = 1$ soliton (hedgehog)
- Promote the three time-dependent parameters in $A(t)$ to quantum mechanical variables
- Result:

$$E = E_0 + \frac{1}{2I}\vec{J}^2 = E_0 + \frac{1}{2I}\vec{S}^2$$

$\vec{J}$ ... isospin, $\vec{S}$ ... spin, E_0 ... soliton energy (mass)
I ... moment of inertia of the soliton (rigid rotor)

$$I = \frac{4\pi}{3}\int dr \sin^4\xi\, \xi_r^2$$

- Classical result: $E = E_0 + \frac{1}{2I}\vec{L}^2$
$\vec{L}$ ... angular momentum

- Concretely for $Q = 1$ soliton (compacton)

$$I = \frac{2^8\sqrt{2}\pi}{15\cdot 7}\lambda\mu\left(\frac{\lambda}{\mu}\right)^{\frac{2}{3}}, \quad E_0 = \frac{64\sqrt{2}\pi}{15}\lambda\mu$$

$$E = E_0 + \frac{1}{2I}\hbar^2 s(s+1)$$

- Now fit to the proton mass (spin $s = \frac{1}{2}$, $M_p = 938.9$ MeV) and Δ resonance mass ($s = \frac{3}{2}$, $M_\Delta = 1232$ MeV) and use $\hbar = 197.3$ MeV fm then

$$\lambda\mu = 45.70 MeV, \quad \frac{\lambda}{\mu} = 0.1523 fm^3$$

may now be used to "predict" other nucleon quantities

- Charge radii
  - Isoscalar and isovector charge densities (per unit length)

$$\rho_0 = \frac{2}{\pi} \sin^2 \xi \, \xi_r \, , \quad \rho_1 = \frac{1}{I} \frac{4\pi}{3} \lambda^2 \sin^4 \xi \, \xi_r^2$$

  - Corresponding proton (neutron) electric charge densities are

$$\rho_{p(n)} = \frac{1}{2}(\rho_0 + (-)\rho_1)$$

  - Resulting isoscalar and isovector mean square electric radii

$$\begin{aligned} < r^2 >_{e,0} \equiv \int drr^2 \rho_0 &= \left(\frac{\lambda}{\mu}\right)^{\frac{2}{3}} \\ < r^2 >_{e,1} \equiv \int drr^2 \rho_1 &= \frac{10}{9} \left(\frac{\lambda}{\mu}\right)^{\frac{2}{3}} \end{aligned}$$

- Isoscalar magnetic mean square radius

$$< r^2 >_{m,0} = \frac{\int drr^4 \sin^2 \xi \, \xi_r}{\int drr^2 \sin^2 \, \xi \xi_r} = \frac{5}{2} \left(\frac{\lambda}{\mu} \right)^{\frac{2}{3}}$$

- Numerical values

radius	BPS Skyrme	massive Skyrme	experiment
compacton	0.755	-	-
$r_{e,0}$	0.534	0.68	0.72
$r_{e,1}$	0.563	1.04	0.88
$r_{m,0}$	0.597	0.95	0.81

Table: Charge radii. Numbers in fm

$\Rightarrow$ Radii too small; not so surprising (no pion cloud)

- should be better for ratios of radii

ratios	BPS Skyrme	massive Skyrme	experiment
$r_{e,1}/r_{e,0}$	1.054	1.529	1.222
$r_{m,0}/r_{e,0}$	1.118	1.397	1.125
$r_{e,1}/r_{m,0}$	0.943	1.095	1.086

Table: Charge ratios. Numbers in fm

- Magnetic moments

$$\mu_{p(n)} = 2M_N \left(\frac{1}{12I} < r^2 >_{e,0} +(-) \frac{I}{6\hbar^2} \right)$$

- Numerical values

	BPS Skyrme	massive Skyrme	experiment
μ_p	1.827	1.97	2.79
μ_n	-1.379	-1.24	-1.91
$\lvert \mu_p/\mu_e \rvert$	1.325	1.59	1.46

Table: Magnetic moments

Conclusions

- Interesting field theory: ∞ symmetries, integrable, BPS bound, ∞ exact solutions
- limit of generalized Skyrme models
  - integrable
  - nontrivial "$m_\pi \to \infty$" or liquid droplet limit
- Phenomenology of nuclei
  - reproduces qualitatively some properties of nuclei $\Rightarrow$ reasonable approximation under some conditions
  - essentially topological, no pion propagation, no two-body interaction $\Rightarrow$ in some circumstances collective contributions seem to be the most important ones
  - Quantization: some first results; much more detailed study necessary

Future

- more applications
  - topological d.o.f. dominate
  - perfect (incompressible) fluid
- mesons
  - nonlinear perturbations
  - integrability $\Rightarrow$ stable non-topological (oscillating) solutions
- derivation of the BPS Skyrme model form QCD