Skyrme models

- Low energy effective theory of hadrons - currently unknown
- Degrees of freedom of QCD:
  - high energy: quarks and gluons
  - low energy: hadrons
- One proposal: Skyrme model
  - primary fields are mesons
  - baryons (hadrons and nuclei) are realized as solitons
  - realizes unbroken symmetries
  - simplest case (two flavors): target space = SU(2) (isospin) matrix $U$
  - topological charge = baryon number

Skyrme (61), Adkins, Nappi, Witten (83)
Original Skyrme model:

- **Skyrme field** $U$:
  \[ x^\mu \rightarrow U(x) : \mathbb{R}^3 \times \mathbb{R} \rightarrow SU(2) \]

- **Lagrangian**:
  \[ L = L_2 + L_4 + L_0 \]

- **Sigma model term**:
  \[ L_2 = -\frac{f_\pi^2}{4} \text{Tr} (U^\dagger \partial_\mu U U^\dagger \partial^\mu U) \]

  pion kinetic term
- **quartic Skyrme term:**

\[
L_4 = -\frac{1}{32e^2} \text{Tr} \left( [U^\dagger \partial_\mu U, U^\dagger \partial_\nu U]^2 \right)
\]

necessary to circumvent the Derrick argument

\(L_2, L_4\) symmetric under \(\text{SU}(2) \times \text{SU}(2), \ U \rightarrow VUV^\dagger\)

- **Potential term:**

\[
L_0(= -\mu^2 V(U, U^\dagger)) = -\mu^2 V(\text{tr} \ U)
\]

Symmetric under diagonal \(\text{SU}(2), \ U \rightarrow VUV^\dagger\)
Generalized Skyrme models:

QCD $\Rightarrow$ Chiral effective meson/baryon theory $\ (true \ at \ N \to \infty)$ t’Hooft (74), Witten (79)

$\Downarrow$ derivative expansion Simic (85)

$\mathcal{L}_{sk} + \text{higher order terms}$ Jackson et. al. (85), Oka (87), Li, Yan (92)...

$\Downarrow$ (?)

Integrable (BPS) soliton model physical interpretation

Need for BPS soliton model

- **Mathematical reasons**
  - solvable model, exact solutions
  - analytical calculations: impact of values of parameters, testing potentials
  - approximate methods for non-integrable models

- **Physical reasons**
  - too big binding energy
  - shell or crystal like densities
Simplest proposals

- $L_2 + L_0$ - excluded by the Derrick
- $L_4 + L_0$ - excluded by dynamics: no topological solitons

Among higher order terms is the following sextic term

$$L_6 = \frac{\lambda^2}{24^2} \left[ \text{Tr} \left( \epsilon^{\mu\nu\rho\sigma} U^\dagger \partial_{\mu} U U^\dagger \partial_{\nu} U U^\dagger \partial_{\rho} U \right) \right]^2$$

- Quadratic in time derivatives $\Rightarrow$ standard hamiltonian formulation
Square of the topological (baryon) current

\[ L_6 = \lambda^2 \pi^4 B_\mu B^\mu \]

where

\[ B_\mu = \frac{1}{24\pi^2} \text{Tr} \left( \epsilon^{\mu\nu\rho\sigma} U^\dagger \partial_\nu U U^\dagger \partial_\rho U U^\dagger \partial_\sigma U \right) \]

is the topological (baryon) current

Phenomenologically induced by a massive vector meson coupled to the baryon density.  Adkins, Nappi (84), Sutcliffe (09)

Improves phenomenological results of Skyrme model
BPS Skyrme model:

Propose following limit of generalized Skyrme models

\[ L_{06} = L_6 + L_0 \]

- \( \infty \) many symmetries
- Integrable: \( \infty \) many conservation laws
- BPS (Bogomolny) bound
- \( \infty \) many exact solutions saturating the BPS bound
Parametrization for $U$

$$U = e^{i\xi\vec{n} \cdot \vec{\sigma}} = \cos \xi + i \sin \xi \vec{n} \cdot \vec{\sigma} \quad \vec{n}^2 = 1$$

and stereographic projection

$$\vec{n} = \frac{1}{1 + |u|^2} \left( u + \bar{u}, -i(u - \bar{u}), 1 - |u|^2 \right)$$

$$\Rightarrow \quad L_{06} = -\frac{\lambda^2 \sin^4 \xi}{(1 + |u|^2)^4} \left( \epsilon^{\mu\nu\rho\sigma} \xi_{\nu} u_{\rho} \bar{u}_{\sigma} \right)^2 - \mu^2 V(\xi)$$
Symmetries

- Poincare Symmetries
- $\infty$ many target space diffeomorphisms
  - $L_6$ is square of pullback of the volume form in target space $SU(2) \sim S^3$,

\[
dV = -i \frac{\sin^2 \xi}{(1 + |u|^2)^2} d\xi dud\bar{u}
\]

$\Rightarrow$ has all volume-preserving diffeomorphisms on target space $S^3$ as symmetries.

- $L_0 = -\mu^2 V(\xi)$ respects some of them: the ones that act nontrivially only on $u, \bar{u}$ $\Rightarrow$ area-preserving diffeos on target space $S^2$ spanned by $u$, but may depend on $\xi$ as a parameter:

\[
\xi \rightarrow \xi, \quad u \rightarrow \tilde{u}(u, \bar{u}, \xi), \quad (1 + |\tilde{u}|^2)^{-2} d\xi d\tilde{u}d\bar{u} = (1 + |u|^2)^{-2} d\xi du d\bar{u}
\]
Volume-preserving base space diffeos.
Energy functional for static fields:

\[ E = \int d^3 x \left( \frac{\lambda^2 \sin^4 \xi}{(1 + |u|^2)^4} (\epsilon^{mnl}_i \xi_m u_n \bar{u}_l)^2 + \mu^2 V \right) \]

Both \( d^3 x \) and \( \epsilon^{ijk} \partial_i \partial_j \partial_k \) invariant under volume preserving diffeos on base space \( \mathbb{R}^3 \). NOT a Noether symmetry.
Integrability - \( \infty \) many conservation laws

Target space symmetries \( \Rightarrow \) conserved currents

\[
\mathcal{J}_G^\mu = G\bar{u}\bar{K}^\mu - G_u\mathcal{K}^\mu, \quad G = G(u, \bar{u}, \xi)
\]

\[
\mathcal{K}^\mu \equiv \frac{1}{(1 + |u|^2)^2} \frac{\partial (\epsilon^{\alpha\nu\rho\sigma} \xi_\nu u_\rho \bar{u}_\sigma)^2}{\partial \bar{u}_\mu}
\]

\[
\partial_\mu \mathcal{J}_G^\mu = G_{\bar{u}\bar{u}} \bar{u}_\mu \bar{K}^\mu + G_{\bar{u}u} u_\mu \bar{K}^\mu - G_{u\bar{u}} \bar{u}_\mu \mathcal{K}^\mu - G_{uu} u_\mu \mathcal{K}^\mu
\]
\[
+ G_{\bar{u}} \partial_\mu \bar{K}^\mu - G_u \partial_\mu \mathcal{K}^\mu + G_{\bar{u}\xi} \xi_\mu \bar{K}^\mu - G_{u\xi} \xi_\mu \mathcal{K}^\mu = 0
\]

where \( u_\mu \mathcal{K}^\mu = 0, \xi_\mu \mathcal{K}^\mu = 0, \bar{u}_\mu \mathcal{K}^\mu \equiv u_\mu \bar{K}^\mu \)

and \( \partial_\mu \mathcal{K}^\mu = 0 \ldots \) field eq. for \( u \)
Bogomolny (BPS) bound

- Derrick scaling $\Rightarrow E_6 = E_0 \ldots$ compatible with BPS
- BPS bound $B \ldots$ top. charge (baryon number)

\[
E = \int d^3x \left( \frac{\lambda^2 \sin^4 \xi}{(1 + |u|^2)^4} \left( \epsilon^{mnl} i \xi_m u_n \bar{u}_l \right)^2 + \mu^2 V \right)
\]

\[
\geq \left( \frac{2 \lambda \mu \pi^2}{\pi^2} \right) \left[ \frac{-i}{\pi^2} \int d^3x \frac{\sin^2 \xi \sqrt{V}}{(1 + |u|^2)^2} \epsilon^{mnl} \xi_m u_n \bar{u}_l \right]
\]

\[
\equiv 2 \lambda \mu \pi^2 C_1 |B|
\]
for $V(\xi) \to 1$ \ldots standard expression for top. charge (and $C_1 \to 1$)

\[ B \equiv \frac{-i}{\pi^2} \int d^3 x \frac{\sin^2 \xi}{(1 + |u|^2)^2} \epsilon^{mnl} \xi_m u_n \bar{u}_l \]

equivalently: integral of pull-back of volume form on target space $S^3$

Remains true with $\sqrt{V(\xi)}$ present, in terms of the new target space coordinate $\bar{\xi}(\xi)$ with

\[ \sin^2 \xi \sqrt{V(\xi)} d\xi = C_1 \sin^2 \bar{\xi} d\bar{\xi} \]

$C_1$ and second integration constant $C_2$ \ldots necessary to implement $\bar{\xi}(\xi = 0) = 0$, $\bar{\xi}(\xi = \pi) = \pi$

e.g. $V = 1 - \cos \xi \Rightarrow C_1 = \frac{32\sqrt{2}}{15\pi}$
Energy density profiles

- BPS solution $\Rightarrow \mathcal{E}_6 = \mathcal{E}_0 = \frac{1}{2} \mathcal{E}$ ... density profile determined by potential
- One-vacuum potentials: $V(r = \infty) = 0$, $V(r = 0) \neq 0$ $\Rightarrow$ density profile of the ball type
- Two-vacua potentials: $V(r = \infty) = 0$, $V(r = 0) = 0$ $\Rightarrow$ density profile of the shell type
- More complicated vacuum manifolds $\Rightarrow$ more complicated profiles (may be determined by analytical arguments)
- In standard Skyrme model: similar behaviour, but results from complicated numerics

Houghton, Manton, Sutcliffe (98), Atiyah, Manton (89), Battye, Sutcliffe (97), (01), (02), (05), (06), (09)
Exact solutions

- non-trivial configuration: $u$ covers full $\mathbb{C}$, and $\xi \in [0, \pi]$
- symmetric (hedgehog) ansatz

$$\xi = \xi(r), \quad u(\theta, \phi) = g(\theta)e^{i n \phi}$$

compatible with field eq.

- Equation for $g(\theta)$

$$\partial_\theta \left( \frac{gg_\theta}{(1 + g^2)^2 \sin \theta} \right) = 0$$

with solution

$$g(\theta) = \tan \frac{\theta}{2}$$

for all $n$
Equation for $\xi$

\[
\frac{n^2 \lambda^2 \sin^2 \xi}{2r^2} \partial_r \left( \frac{\sin^2 \xi}{r^2} \xi_r \right) - \mu^2 V_{\xi} = 0
\]

or with

\[
z = \frac{\sqrt{2} \mu}{3|n|\lambda} r^3
\]

\[\Rightarrow \sin^2 \xi \partial_z \left( \sin^2 \xi \xi_z \right) - V_{\xi} = 0\]

first integral

\[
\frac{1}{2} \sin^4 \xi \xi_z^2 = V
\]

dim. reduced version of BPS
Compact and non-compact skyrmions

- Reduced energy functional (example $V = (1 - \cos \xi)^a$)

\[
E = C \int dz \left( \frac{1}{2} \sin^4 \xi \xi_z^2 + (1 - \cos \xi)^a \right)
\]

where $C = 2 \sqrt{2} \lambda \mu |n|$

with $\chi = \sqrt{1 - \cos \xi}$, $\Phi = \chi^3$

\[
E = C \int dz \left( 2(2 - \chi^2)\chi^4 \chi_z^2 + \chi^{2a} \right) = C \int dz \left( \frac{2}{9}(2 - \Phi^2)\Phi_z^2 + \Phi^{2a} \right)
\]

for $a < 3 \ldots$ sublinear force law $\Rightarrow$ compactons
Example: Skyrme potential

\[ V = \frac{1}{2} \text{Tr}(1 - U) \quad \rightarrow \quad V(\xi) = 1 - \cos \xi \]

- Solution

\[ \xi = \begin{cases} 
2 \arccos \sqrt{\frac{3z}{4}} & z \in [0, \frac{4}{3}] \\
0 & z \geq \frac{4}{3}.
\end{cases} \]

compacton Rosenau et. al. (93), Arodz (02)

- Energy

\[ E = \frac{64\sqrt{2}\pi}{15} \mu \lambda |n| \]

linear in \( |B| = |n| \)
energy density

\[ \mathcal{E} = 8 \sqrt{2} \mu \lambda (1 - |n|^{-\frac{2}{3}} \tilde{r}^2) \quad \text{for} \quad 0 \leq \tilde{r} \leq |n|^{\frac{1}{3}} \]
\[ = 0 \quad \text{for} \quad \tilde{r} > |n|^{\frac{1}{3}} \]

where

\[ \tilde{r} = \left( \frac{\mu}{4 \sqrt{2} \lambda} \right)^{\frac{1}{3}} \frac{r}{R_0} \]

rescaled radius \( \tilde{r} \)

\( R_0 \) \ldots compacton radius for \( n = 1 \)

baryon density

\[ \mathcal{B} = \text{sign}(n) \frac{4}{\pi^2} (1 - |n|^{-\frac{2}{3}} \tilde{r}^2)^{\frac{1}{2}} \quad \text{for} \quad 0 \leq \tilde{r} \leq |n|^{\frac{1}{3}} \]
\[ = 0 \quad \text{for} \quad \tilde{r} > |n|^{\frac{1}{3}} \]
Figure: Normalized energy density as a function of the rescaled radius $\tilde{r}$, for topological charge $n=1$. For $|n| > 1$, the height of the density remains the same, whereas the radius grows like $|n|^{\frac{1}{3}}$. 
Figure: Normalized topological charge density as a function of the rescaled radius $\tilde{r}$, for topological charge $n=1$. For $|n| > 1$, the height of the density remains the same, whereas the radius grows like $|n|^{1/3}$
Generalized Skyrme potentials

\[ V = \left( \frac{1}{2} \text{Tr}(1 - U) \right)^a \rightarrow V(\xi) = (1 - \cos \xi)^a \]

**Solutions:**
- Compactons for \( a < 3 \)
- Exponentially localized solution for \( a = 3 \)
- Power-like localized solutions for \( a > 3 \)
Two-vacua potential - example:  

\[ V = 1 - \cos^2 \xi \]

vacua at \( \xi = 0, \pi \)

Solution

\[ \xi = \begin{cases} \arccos \sqrt{2}z - 1 & z \in [0, \sqrt{2}] \\ 0 & z \geq \sqrt{2} \end{cases} \]

compacton; \( E \propto |B| \)

Energy density

\[ \epsilon(z) \sim \sqrt{2}z(1 - \sqrt{2}z) \]
Figure: Two-vacua potential - shell structure: normalized energy density as a function of the rescaled radius $\tilde{r}$, for topological charge $n=1$. For $|n| > 1$, the height of the maximum remains the same, whereas the radius grows like $|n|^{\frac{1}{3}}$. 
Sutcliffe’s BPS Skyrme model

- original Skyrme model coupled with an infinite tower of the vector mesons

\[ \Leftrightarrow \]

- dimensional reduction of 5dim YM

- nice realization of the vector meson domination

- no potential term
Some phenomenology of nuclei for standard Skyrme potential

\[ V = 1 - \cos \xi \]
A) Classical solitons

- Linear mass - baryon number relation:
  - BPS bound \( E = 64\sqrt{2}\mu\lambda |B|/15 \equiv E_0 |B| \)
  - well-known relation for nuclei

<table>
<thead>
<tr>
<th>B</th>
<th>( E_{BPS} )</th>
<th>( E_{vec\ Skyrme} )</th>
<th>( E_{Skyrme} )</th>
<th>( E_{experiment} )</th>
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<td>1</td>
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<td>3</td>
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<td>3727</td>
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<td>3727</td>
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<tr>
<td>6</td>
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<td>8</td>
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<td>-</td>
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<td>7455</td>
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<tr>
<td>10</td>
<td>9317.5</td>
<td>-</td>
<td>9113</td>
<td>9327</td>
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**Table:** Energies of the solutions in the BPS Skyrme model, compared with masses for the vector-Skyrme and massive Skyrme models, as well as with the experimental values. Numbers in MeV
No binding energy
- BPS solutions $\Rightarrow$ zero binding energy
- binding energies for physical nuclei small $\leq 1$
- in standard Skyrme model significantly bigger

No forces between solitons
- BPS and compactons $\Rightarrow$ no forces
- physical nuclei: very short range interaction

radii of nuclei
- Compacton: definite radius
  \[ R = (2\sqrt{2}\lambda |B|/\mu)^{1/3} \equiv R_0 \sqrt[3]{|B|} \]
- reproduces phenomenological relation $R \sim 1.25 \sqrt[3]{|B|} \text{fm}$
Incompressible fluid

- Static energy functional: volume-preserving diffeos (VPDs)
- Symmetries of an incompressible fluid where surface energy is neglected
- Physical nuclei do not have this symmetry: have definite shape, deformations cost energy
- But: volume-preserving deformations cost less energy . . .

⇒ classical model reproduces some features of the nuclear liquid drop model (mass prop to volume, strictly finite size, VPDs)

- Clearly too naive (no pions ⇒ no interactions, no pion cloud; no quantum corrections)
B) Soliton quantization

work in progress; top. charge $Q = 1$ (nucleon)

- Collective coordinate quantization: Quantize symmetry transformations which are NOT symmetries of the soliton (=lightest d.o.f.)

- Example: rotations (equivalent to isospin = diagonal target space $SU(2)$, because $SO(3)_b \times SU(2)_{\text{diag}}$ IS symmetry of hedgehog

⇒ Rigid rotor quantization (=undeformed rotating soliton, quantized angular momentum)

- Well-known procedure
E.g. for isospin: Insert $U(t) = A(t)U_0A^\dagger(t)$ into action ($U_0 = Q = 1$ soliton (hedgehog))

Promote the three time-dependent parameters in $A(t)$ to quantum mechanical variables

Result:

$$E = E_0 + \frac{1}{2I} \vec{J}^2 = E_0 + \frac{1}{2I} \vec{S}^2$$

$\vec{J}$ ... isospin, $\vec{S}$ ... spin, $E_0$ ... soliton energy (mass) $I$ ... moment of inertia of the soliton (rigid rotor)

$$I = \frac{4\pi}{3} \int dr \sin^4 \xi \xi_r^2$$

Classical result: $E = E_0 + \frac{1}{2I} \vec{L}^2$

$\vec{L}$ ... angular momentum
Concretely for $Q = 1$ soliton (compacton)

\[ l = \frac{2^8 \sqrt{2}\pi}{15 \cdot 7} \lambda \mu \left( \frac{\lambda}{\mu} \right)^\frac{2}{3}, \quad E_0 = \frac{64 \sqrt{2}\pi}{15} \lambda \mu \]

\[ E = E_0 + \frac{1}{2l} \hbar^2 s(s + 1) \]

Now fit to the proton mass (spin $s = \frac{1}{2}$, $M_p = 938.9$ MeV) and $\Delta$ resonance mass ($s = \frac{3}{2}$, $M_\Delta = 1232$ MeV) and use $\hbar = 197.3$ MeV fm then

\[ \lambda \mu = 45.70\text{MeV}, \quad \frac{\lambda}{\mu} = 0.1523\text{fm}^3 \]

may now be used to "predict" other nucleon quantities
Charge radii

- Isoscalar and isovector charge densities (per unit length)

\[ \rho_0 = \frac{2}{\pi} \sin^2 \xi \xi_r, \quad \rho_1 = \frac{1}{7} \frac{4\pi}{3} \lambda^2 \sin^4 \xi \xi_r^2 \]

- Corresponding proton (neutron) electric charge densities are

\[ \rho_{p(n)} = \frac{1}{2} (\rho_0 + (-)\rho_1) \]

- Resulting isoscalar and isovector mean square electric radii

\[ \langle r^2 \rangle_{e,0} = \int drr^2 \rho_0 = \left( \frac{\lambda}{\mu} \right)^{\frac{2}{3}} \]
\[ \langle r^2 \rangle_{e,1} = \int drr^2 \rho_1 = \frac{10}{9} \left( \frac{\lambda}{\mu} \right)^{\frac{2}{3}} \]
Isoscalar magnetic mean square radius

\[ \langle r^2 \rangle_{m,0} = \frac{\int dr r^4 \sin^2 \xi \xi_r}{\int dr r^2 \sin^2 \xi \xi_r} = \frac{5}{2} \left( \frac{\lambda}{\mu} \right)^{\frac{2}{3}} \]
Numerical values

<table>
<thead>
<tr>
<th>radius</th>
<th>BPS Skyrme</th>
<th>massive Skyrme</th>
<th>experiment</th>
</tr>
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<tbody>
<tr>
<td>compacton</td>
<td>0.755</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>$r_{e,0}$</td>
<td>0.534</td>
<td>0.68</td>
<td>0.72</td>
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<tr>
<td>$r_{e,1}$</td>
<td>0.563</td>
<td>1.04</td>
<td>0.88</td>
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<tr>
<td>$r_{m,0}$</td>
<td>0.597</td>
<td>0.95</td>
<td>0.81</td>
</tr>
</tbody>
</table>

Table: Charge radii. Numbers in fm

⇒ Radii too small; not so surprising (no pion cloud)
- should be better for ratios of radii

<table>
<thead>
<tr>
<th>ratios</th>
<th>BPS Skyrme</th>
<th>massive Skyrme</th>
<th>experiment</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r_{e,1}/r_{e,0}$</td>
<td>1.054</td>
<td>1.529</td>
<td>1.222</td>
</tr>
<tr>
<td>$r_{m,0}/r_{e,0}$</td>
<td>1.118</td>
<td>1.397</td>
<td>1.125</td>
</tr>
<tr>
<td>$r_{e,1}/r_{m,0}$</td>
<td>0.943</td>
<td>1.095</td>
<td>1.086</td>
</tr>
</tbody>
</table>

Table: Charge ratios. Numbers in fm
Magnetic moments

\[ \mu_{p(n)} = 2M_N \left( \frac{1}{12I} < r^2 >_{e,0} + (-) \frac{l}{6\hbar^2} \right) \]

Numerical values

<table>
<thead>
<tr>
<th></th>
<th>BPS Skyrme</th>
<th>massive Skyrme</th>
<th>experiment</th>
</tr>
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<tbody>
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<td>( \mu_p )</td>
<td>1.827</td>
<td>1.97</td>
<td>2.79</td>
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<tr>
<td>( \mu_n )</td>
<td>-1.379</td>
<td>-1.24</td>
<td>-1.91</td>
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<tr>
<td></td>
<td>1.325</td>
<td>1.59</td>
<td>1.46</td>
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Table: Magnetic moments
Conclusions

- Interesting field theory: $\infty$ symmetries, integrable, BPS bound, $\infty$ exact solutions
- Limit of generalized Skyrme models
  - Integrable
  - Nontrivial "$m_\pi \to \infty"$ or liquid droplet limit
- Phenomenology of nuclei
  - Reproduces qualitatively some properties of nuclei $\Rightarrow$ reasonable approximation under some conditions
  - Essentially topological, no pion propagation, no two-body interaction $\Rightarrow$ in some circumstances collective contributions seem to be the most important ones
  - Quantization: some first results; much more detailed study necessary
Future

- more applications
  - topological d.o.f. dominate
  - perfect (incompressible) fluid
- mesons
  - nonlinear perturbations
  - integrability $\Rightarrow$ stable non-topological (oscillating) solutions
- derivation of the BPS Skyrme model form QCD