Thermal Models - Life After Life

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50 Cracow School of Theoretical Physics

Developments in particle physics
from a 50 year perspective of the Cracow School

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Starting points

Die Mesonenausbeute beim Beschuß von leichten Kernen mit α-Teilchen

Von Heinz Koppe

Max-Planck-Institut für Physik, Göttingen

(Z. Naturforschg. 3a, 251—252 [1948]; eingeg. am 21. Juni 1948)

Mittels des neuen Berkeley-Betatrons ist es möglich gewesen, durch Beschuß von leichten Kernen (insbesondere C) mit α-Teilchen von etwa 380 MeV Mesonen zu erzeugen. Im folgenden soll eine einfache Methode angegeben werden, nach der sich die dabei zu erwartende Ausbeute abschätzen läßt.

Beim Stoß eines Kernes mit der Massenzahl M_1 und der kinetischen Energie E auf einen ruhenden Kern mit der Masse M_2 entsteht zunächst ein Zwischenkern mit der Masse $M = M_1 + M_2$, dem die Anregungsenergie pro Nucleon

$$U = \frac{M_2}{M^2} E \qquad (1)$$

zur Verfügung steht. Nach einer bekannten Beziehung[1] hat der Zwischenkern dann die Temperatur

$$T_0 = 3{,}8 \sqrt{U}. \qquad (2)$$

Dabei wird unter T das Produkt aus k und der absoluten Temperatur verstanden. Gl. (2) liefert T in MeV, wenn man U in MeV einsetzt.

Application of statistical physics to elementary particles is usually referred to Enrico Fermi (1950)

although it was Heinz Koppe(1948)who proposed this idea to production processes

Die Ausbeute an Mesonen ist dann gegeben durch

$$\eta = \int_0^\infty \nu(T)\, dt = \frac{O\,\mu}{\pi^2 \hbar^3} \int_0^\infty T^2 e^{-\mu c^2 \sqrt{1/T_0^2 + 2Bt}}\, dt\,.$$

Unter dem Integral kann man T^2 als langsam veränderlich durch T_0^2 ersetzen und außerdem die Wurzel nach t entwickeln. Es ergibt sich

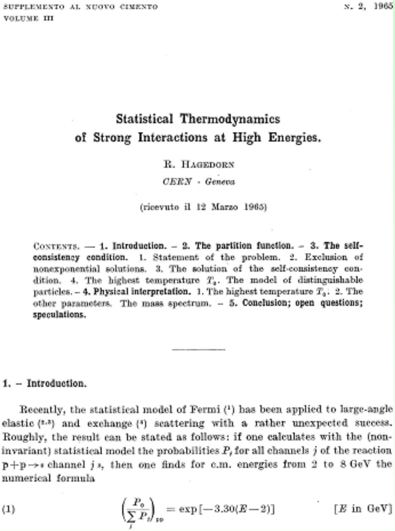
$$\eta = 0{,}031\, T_0\, M\, e^{-\mu c^2 / T_0}\,. \qquad (7)$$

Mit den oben angegebenen Werten liefert das Stoßausbeuten $\eta = 1{,}7 \cdot 10^{-4}$.

Limiting temperature

Rolf Hagedorn was the first who systematically analyzed high energy phenomena using all tools of statistical physics. He introduced the concept of the limiting temperature $\sim 140 MeV$ based on the statistical bootstrap model.

That was the origin of multiphase structure of hadronic matter.


SUPPLEMENTO AL NUOVO CIMENTO VOLUME III — N. 2, 1965

Statistical Thermodynamics of Strong Interactions at High Energies.

R. HAGEDORN
CERN - Geneva

(ricevuto il 12 Marzo 1965)

CONTENTS. — **1. Introduction.** – **2. The partition function.** – **3. The self-consistency condition.** 1. Statement of the problem. 2. Exclusion of nonexponential solutions. 3. The solution of the self-consistency condition. 4. The highest temperature T_0. The model of distinguishable particles. – **4. Physical interpretation.** 1. The highest temperature T_0. 2. The other parameters. The mass spectrum. – **5. Conclusion; open questions; speculations.**

1. – Introduction.

Recently, the statistical model of Fermi (1) has been applied to large-angle elastic (2,3) and exchange (4) scattering with a rather unexpected success. Roughly, the result can be stated as follows: if one calculates with the (non-invariant) statistical model the probabilities P_j for all channels j of the reaction p+p→«channel j», then one finds for c.m. energies from 2 to 8 GeV the numerical formula

$$\left(\frac{P_0}{\sum_j P_j}\right)_{pp} = \exp[-3.30(E-2)] \qquad [E \text{ in GeV}] \qquad (1)$$

Density of states

$$\sigma(E,V) = \sum_{n=1}^{\infty} \frac{V^n}{n!} \int \delta(E - \sum_{i=1}^{n} E_i) \prod_{i=1}^{n} \varrho(m_i) dm_i d^3p_i$$

$$\sigma(E, V) = \sum_{n=1}^{\infty} \frac{V_0^n}{n!} \int \delta(m - \sum_{i=1}^{n} E_i) \prod_{i=1}^{n} \varrho(m_i) dm_i d^3 p_i$$

The bootstrap equation

$$\varrho(m) = \delta(m - m_0) + \sum_{n=2}^{\infty} \frac{V_0^n}{n!} \int \delta(m - \sum_{i=1}^{n} E_i) \prod_{i=1}^{n} \varrho(m_i) dm_i d^3p_i$$

$$\varrho(m) = \delta(m - m_0) + \sum_{n=2}^{\infty} \frac{V_0^n}{n!} \int \delta(m - \sum_{i=1}^{n} E_i) \prod_{i=1}^{n} \varrho(m_i) dm_i d^3 p_i$$

$\Downarrow$ $\Downarrow$ $\Downarrow$ $\Downarrow$ $\Downarrow$ $\Downarrow$ $\Downarrow$

$$\varrho(m) = f(m) e^{m/T_0}$$

$+$ limiting temperature

Hagedorn spectrum fit

$$f_{FIT}(m) = log_{10}\left(\int_0^m \frac{c}{(x^2+m_0^2)^{5/4}} exp(x/T_H)\right)$$

$$\rho(m) = \frac{c}{(m^2+m_0^2)^{5/4}} exp(m/T_H)$$

$$N_{exp}(m) = \sum_i g_i \Theta(m - m_i)$$

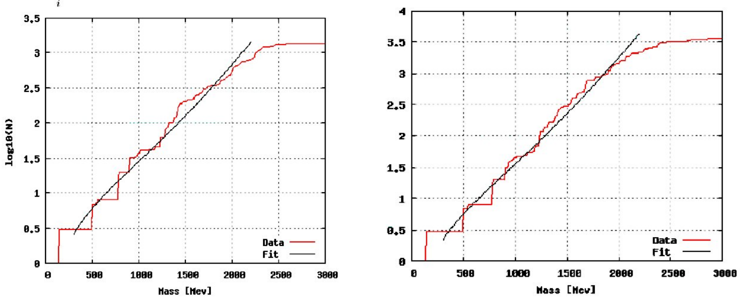
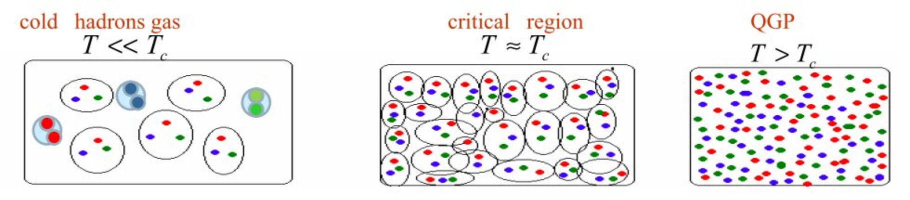
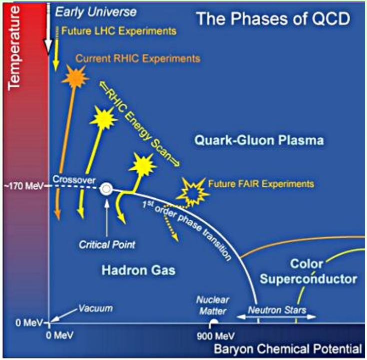


Figure 2: All mesons $T_H = 203.315$, $c = 25132.674$, range: $300 - 2200$ MeV All hadrons $T_H = 177.086$, $c = 18726.494$, range: $300 - 2200$ MeV

Done by M. Sobczak according to states in PDG2008

Phase structure





Statistical ensembles of high energy physics

The thermodynamic system of volume V and temperature T composed of charged particles and their antiparticles carrying charge ± 1.
The partition functions of the canonical and grand canonical statistical system

$$\mathcal{Z}_Q^C(V,T) = \mathrm{Tr}_Q\, \mathrm{e}^{-\beta \hat{H}} = \sum_{N_+ - N_- = Q}^{\infty} \frac{z^{N_- + N_+}}{N_-! N_+!} = I_Q(2Vz_0)\,,$$

$$\mathcal{Z}^{GC}(V,T) = \mathrm{Tr}\, \mathrm{e}^{-\beta(\hat{H} - \mu \hat{Q})} = \exp\left(2Vz_0 \cosh\frac{\mu}{T}\right).$$

Vz_0 is the sum over all one-particle partition functions

$$z_0^{(i)}(T) = \frac{1}{V}\frac{V}{(2\pi)^3} g_i \int d^3p\, e^{-\beta\sqrt{p^2+m_i^2}} = \frac{1}{2\pi^2} T g_i\, m_i^2\, K_2\left(\frac{m_i}{T}\right)\,,$$

g_i – the spin degeneracy factor.

Archive

SOVIET PHYSICS JETP VOLUME 2, NUMBER 1 JANUARY, 1956

The Statistics of Charge-Conserving Systems and Its Application to the Theory of Multiple Production

V. B. MAGALINSKII AND IA. P. TERLETSKII
Moscow State University

Submitted to JETP editor November 9, 1954
J. Exper. Theoret. Phys. USSR 29, 151-157 (August, 1955)

The quantum statistics of systems with a variable number of non-interacting particles is generalized to the case of an aggregate of oppositely charged particles, which obey the law of charge conservation. Formulas which differ from the corresponding formulas of ordinary quantum statistics are derived for the total number of particles and the total energy. The results obtained are applied to the theory of multiple production of mesons. The following questions are studied: the dependence of the energy on the relative proportions of neutral and charged mesons, the formation of nucleon-antinucleon pairs, and the relation between the yield and the primary energy. The theory is compared with the available experimental data.

1. INTRODUCTION

IN the statistical treatment of the phenomenon of multiple production of particles at high energies, proposed by Fermi[1], the total number of particles, the total energy of the system, and also the relation charge-conserving systems, a more detailed examination of processes of multiple production in the framework of the "thermodynamic" approximation is possible.

We make this generalization in the present paper, and as a result obtain new formulas for the total

Power of XIX century mathematics

The universal formula

$$e^{y\left(at+\frac{b}{t}\right)} = \sum_{n=-\infty}^{\infty} t^n \left(\frac{a}{b}\right)^{n/2} I_n(2y\sqrt{ab}\,)$$

allows to write a concise expression for the generating function.
Nucleons - pions:

$$\begin{aligned}
&\tilde{\mathcal{Z}}(\varphi,\theta) \\
&= \exp\left[Z_\pi^{(1)}\right] \exp\left[Z_\pi^{(1)}\left(e^{i\theta}+e^{-i\theta}\right)\right] \\
&\times \exp\left[Z_N^{(1)}\left(e^{i(\varphi+\theta/2)}+e^{-i(\varphi+\theta/2)}\right)\right] \exp\left[Z_N^{(1)}\left(e^{i(\varphi-\theta/2)}+e^{-i(\varphi-\theta/2)}\right)\right] \\
&= \exp\left[Z_\pi^{(1)}\right] \sum_{n,k,r=-\infty}^{\infty} I_n(2Z_N^{(1)}) I_k(2Z_N^{(1)}) I_r(2Z_\pi^{(1)})\, e^{i(n+k)\varphi}\, e^{i(n/2-k/2+r)\theta}
\end{aligned}$$

Thermal models calculations - in priciple

$$\epsilon = \frac{1}{2\pi^2}\sum_{i=1}^{l}(2s_i+1)\int_0^\infty dp\,\frac{p^2 E_i}{\exp\left\{\frac{E_i-\mu_i}{T}\right\}+g_i}\,,$$

$$n_B = \frac{1}{2\pi^2}\sum_{i=1}^{l}(2s_i+1)\int_0^\infty dp\,\frac{p^2 B_i}{\exp\left\{\frac{E_i-\mu_i}{T}\right\}+g_i}\,,$$

$$n_S = \frac{1}{2\pi^2}\sum_{i=1}^{l}(2s_i+1)\int_0^\infty dp\,\frac{p^2 S_i}{\exp\left\{\frac{E_i-\mu_i}{T}\right\}+g_i}$$

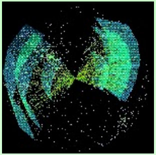
$$n_Q = \frac{1}{2\pi^2}\sum_{i=1}^{l}(2s_i+1)\int_0^\infty dp\,\frac{p^2 Q_i}{\exp\left\{\frac{E_i-\mu_i}{T}\right\}+g_i}$$

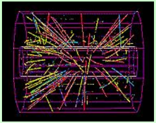
Supplemented by

-Van derWaals type interaction via excluded volume correction
- Finite volume corrections
- Width of all resonances included by integrating over BreitWigner distributions

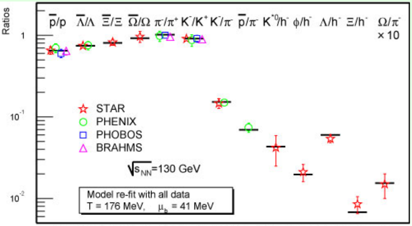
where

$$\mu_j = b_j\mu_b + s_j\mu_s + +q_j\mu_q$$

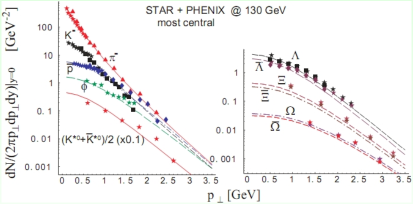




More particles (degrees of freedom) in the process: kinematics tends to dominate the behavior of the system

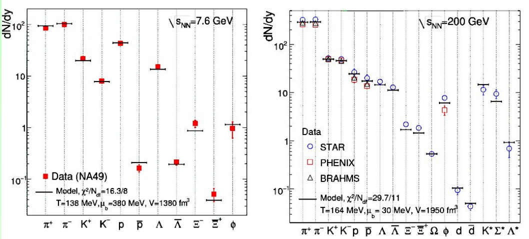


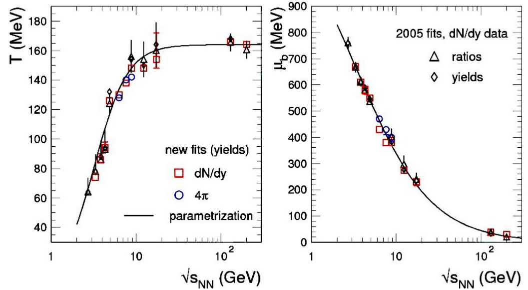
Braun-Munzinger et al., PLB 518 (2001) 41 D. Magestro (updated July 22, 2002)



We cannot solve pre-equilibrium HIC dynamics, we have no good description of hadronization processes. . . Nevertheless, the thermal statistical models work quite well.

Place for statistical physics





A.Andronic,P.BraunMunzinger,J.Stachel: Phys.Lett.B673,142(2009), ActaPhys.Polon.B40,1005(2009)

Direct variables

The chemical potential μ determines the average charge in the grand canonical ensemble

$$\langle Q \rangle = T \frac{\partial}{\partial \mu} \ln \mathcal{Z}^{GC} .$$

This allows to eliminate the chemical potential from further formulae for the grand canonical probabilities distributions

$$\frac{\mu}{T} = \operatorname{arcsinh} \frac{\langle Q \rangle}{2 V z_0} = \ln \frac{\langle Q \rangle + \sqrt{\langle Q \rangle^2 + 4 (V z_0)^2}}{2 V z_0} .$$

Probabilities in ensembles

To have N_- negative particles in the canonical ensemble

$$\mathcal{P}_Q^C(N_-, V) = \frac{(Vz_0)^{2N_-+Q}}{N_-!(N_- + Q)!} \frac{1}{I_Q(2Vz_0)} \, .$$

To have N_- negative particles in the grand canonical ensemble

$$\mathcal{P}_{\langle Q \rangle}^{GC}(N_-, V) =$$

$$\frac{1}{N_-!} \left[\frac{2(Vz_0)^2}{\langle Q \rangle + \sqrt{\langle Q \rangle^2 + 4(Vz_0)^2}} \right]^{N_-} \exp\left[-\frac{2(Vz_0)^2}{\langle Q \rangle + \sqrt{\langle Q \rangle^2 + 4(Vz_0)^2}} \right]$$

Technical details

Cleymans J., Redlich K., and Turko L. Phys. Rev. C **71** 047902 (2005)
Cleymans J., Redlich K., and Turko L. J. Phys. G **31** 1421 (2005)

The thermodynamic limit

The thermodynamic limit is understood as a limit $V \to \infty$ such that densities of the system remain constant.
The canonical ensemble

$$Q, N_- \to \infty; \ \frac{Q}{V} = q\,; \ \frac{N_-}{V} = n_-$$

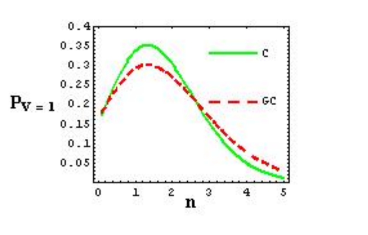
The grand canonical ensemble.

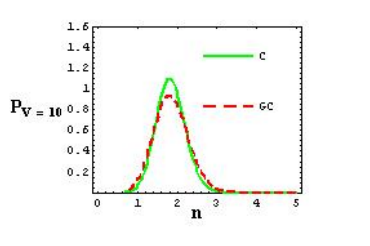
$$\langle Q \rangle, N_- \to \infty; \ \frac{\langle Q \rangle}{V} = \langle q \rangle\,; \ \frac{N_-}{V} = n_-$$

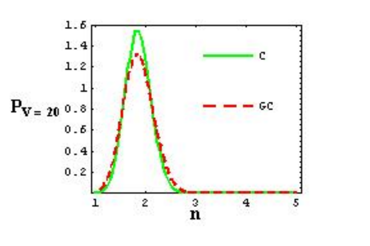
To formulate correctly the thermodynamic limit of quantities involving densities, one defines probabilities for densities

$$\begin{aligned} \mathbf{P}_q^C(n_-, V) &:= V\mathcal{P}_{Vq}^C(Vn_-, V)\,, \\ \mathbf{P}_{\langle q \rangle}^{GC}(q, V) &:= V\mathcal{P}_{V\langle q \rangle}^{GC}(Vq, V)\,. \end{aligned}$$

Volume dependence: probabilities







An average limiting density of (negatively) charged particles

$$\langle n_- \rangle_\infty = \left. \frac{\sqrt{q^2 + 4z_0^2} - q}{2} \right|_{q=\langle q \rangle}$$

Probability distribution of the canonical ensemble

$$\mathbf{P}_q^C(n_-; V) = \delta\left(n_- - \langle n_- \rangle_\infty\right) + \frac{1}{V} \frac{\langle n_- \rangle_\infty (q + \langle n_- \rangle_\infty)}{(2q + \langle n_- \rangle_\infty)^2}\, \delta'\left(n_- - \langle n_- \rangle_\infty\right) + \frac{1}{2V} \frac{\langle n_- \rangle_\infty (q + \langle n_- \rangle_\infty)}{2q + \langle n_- \rangle_\infty}\, \delta''\left(n_- - \langle n_- \rangle_\infty\right) + \mathcal{O}(1/V^2)\,,$$

Probability distribution of the grand canonical ensemble

$$\mathbf{P}_{\langle q \rangle}^{GC}(n_-, V) = \delta\left(n_- - \langle n_- \rangle_\infty\right) + \frac{\langle n_- \rangle_\infty}{2V}\, \delta''\left(n_- - \langle n_- \rangle_\infty\right) + \mathcal{O}(1/V^2)\,.$$

Moments in the thermodynamic limit

The canonical ensemble

$$\langle n_-^k \rangle^C \simeq \langle n_- \rangle_\infty^k - \frac{k}{V}\frac{q+\langle n_-\rangle_\infty}{(q+2\langle n_-\rangle_\infty)^2}\langle n_-\rangle_\infty^k + \frac{k(k-1)}{2V}\frac{q+\langle n_-\rangle_\infty}{q+2\langle n_-\rangle_\infty}\langle n_-\rangle_\infty^{k-1}.$$

The grand canonical ensemble

$$\langle n_-^k \rangle^{GC} \simeq \langle n_- \rangle_\infty^k + \frac{k(k-1)}{2V}\langle n_-\rangle_\infty^{k-1}.$$

Canonical suppression factor

Canonical suppression factor for densities

$$\frac{\langle n_-^k \rangle_q^C}{\langle n_-^k \rangle_{\langle q \rangle}^{GC}} \simeq 1 - \frac{1}{V} \frac{k(k+1)\langle q \rangle + 2k^2 \langle n_- \rangle_\infty}{2(2\langle n_- \rangle_\infty + \langle q \rangle)^2}$$

Canonical suppression factor for particles

$$\frac{\langle N_-^k \rangle_q^C}{\langle N_-^k \rangle_{\langle q \rangle}^{GC}} \simeq 1 - \frac{1}{V} \frac{k(k+1)\langle q \rangle + 2k^2 \langle n_- \rangle_\infty}{2(2\langle n_- \rangle_\infty + \langle q \rangle)^2}$$

Canonical suppression factor

Finite volume corrections

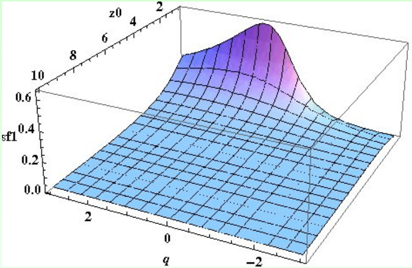


Figure: $1/V$ corrections to canonical suppression factors for the first moment

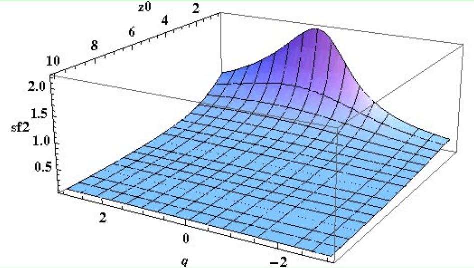


Figure: $1/V$ corrections to canonical suppression factors for the second moment

Abelian and nonabelian

Example

Perform and compare results of the statistical system: nucleons (n, p) and pions $(\pi^{\pm}, \pi^0)$ with an exact isospin $SU(2)$ and $U(1)_B$ symmetry.

- Abelian approach based on $U(1)_{I_3} \times U(1)_B$ symmetry. Abelian canonical partition function is given as

$$\mathcal{Z}^{(a)}_{B,I_3} = \mathrm{Tr}_{B,I_3}\, \mathrm{e}^{-\beta H}$$

with the trace-sum over all states with the given value I_3 of the third component of the isospin.

- Nonabelian approach based $SU(2) \times U(1)_B$ symmetry. Nonabelian canonical partition function is given as

$$\mathcal{Z}^{(na)}_{B,I} = \mathrm{Tr}_{B,I}\, \mathrm{e}^{-\beta H}$$

with the trace-sum over all states with the given value I of the total isospin.

General projective approach

A generating function is given as

$$\tilde{\mathcal{Z}}(g) = \mathrm{Tr}\{U(g)\,\mathrm{e}^{-\beta H}\} = \sum_{\Lambda} \frac{\chi_\Lambda(g)}{dim(\Lambda)}\, Z_\Lambda^{(na)}$$

$$Z_\Lambda^{(na)} = \mathrm{Tr}_\Lambda\, \mathrm{e}^{-\beta H} \ .$$

Then

$$Z_\Lambda^{(na)} = dim(\Lambda) \int d\mu(g)\, \chi_\Lambda(g) \tilde{\mathcal{Z}}(g)\,.$$

Technical details

Redlich K., and T.L.: Z. Phys. **C 5** (1980) 201

T.L.: Phys. Lett. **B 104** (1981) 153

$SU(2)$ case

One can compare analytically abelian and nonabelian approach
Characters if irreducible representation are given as

$$\chi_J(\gamma) = \frac{\sin\left(J + \frac{1}{2}\right)\gamma}{\sin\frac{\gamma}{2}} = \sum_{j_3=-J}^{J} e^{ij_3\gamma}$$

with the measure

$$d\mu(\gamma) = \sin^2\frac{\gamma}{2}\, d\gamma = \frac{1-\cos\gamma}{2}\, d\gamma$$

and the integration domain $\{0, 2\pi\}$.

Projections

A generating function is given as

$$\tilde{\mathcal{Z}} = \mathrm{Tr}\{U(g)\,\mathrm{e}^{-\beta H}\} = \sum_{J=0}^{\infty} \frac{\chi_J(\gamma)}{2J+1}\, Z_J^{(na)}\,; \qquad Z_J^{(na)} = \mathrm{Tr}_J\,\mathrm{e}^{-\beta H}\ .$$

So we have

$$Z_J^{(na)} = \frac{2J+1}{\pi} \int_0^{2\pi} d\gamma\, \chi_J(\gamma) \tilde{\mathcal{Z}}(\gamma) \sin^2\frac{\gamma}{2}\,.$$

The abelian canonical partition function

$$Z_{j_3}^{(a)} = \mathrm{Tr}_{j_3}\,\mathrm{e}^{-\beta H} = \frac{1}{2\pi}\int_0^{2\pi} d\gamma\, \tilde{\mathcal{Z}}(\gamma)\,\mathrm{e}^{-ij_3\gamma} = \frac{1}{2\pi}\int_0^{2\pi} d\gamma\, \tilde{\mathcal{Z}}(\gamma)\,\cos j_3\gamma\,.$$

Projections from trigonometry

For the abelian canonical partition function $Z^{(a)}_{j_3}$

$$Z^{(a)}_{j_3} = \mathrm{Tr}_{j_3}\, \mathrm{e}^{-\beta H} = \frac{1}{2\pi}\int_0^{2\pi} d\gamma\, \tilde{\mathcal{Z}}(\gamma)\, \mathrm{e}^{-ij_3\gamma} = \frac{1}{2\pi}\int_0^{2\pi} d\gamma\, \tilde{\mathcal{Z}}(\gamma)\, \cos j_3\gamma\, .$$

But

$$\chi_J(\gamma)\, \sin^2\frac{\gamma}{2} = \frac{\sin\left(J+\frac{1}{2}\right)\gamma}{\sin\frac{\gamma}{2}}\sin^2\frac{\gamma}{2} = \frac{1}{2}\left(\cos J\gamma - \cos\left(J+1\right)\gamma\right)\, .$$

This allows to express a nonabelian $SU(2)$ partition function by means of abelian partition functions

$$Z^{(na)}_J = (2J+1)\left(Z^{(a)}_J - Z^{(a)}_{J+1}\right)\, .$$

Canonical functions

For the pion gas

$$Z_{I_3} = \exp\left[\lambda_0 Z_\pi^{(1)}\right] \left(\frac{\lambda_+}{\lambda_-}\right)^{I_3/2} I_{I_3}(2Z_\pi^{(1)}\sqrt{\lambda_+\lambda_-}\,) \qquad (1)$$

For the $\pi - N$ system

$$Z_{B,I_3} = \exp\left[\lambda_0 Z_\pi^{(1)}\right] \sum_{n=-\infty}^{\infty} \left(\frac{\lambda_p}{\lambda_{\bar{p}}}\right)^{n/2} \left(\frac{\lambda_n}{\lambda_{\bar{n}}}\right)^{(B-n)/2} \left(\frac{\lambda_+}{\lambda_-}\right)^{(B/2+I_3-n)/2}$$
$$\times\, I_n(2Z_N^{(1)}\sqrt{\lambda_p\lambda_{\bar{p}}}\,) I_{B-n}(2Z_N^{(1)}\sqrt{\lambda_n\lambda_{\bar{n}}}\,) I_{B/2+I_3-n}(2Z_\pi^{(1)}\sqrt{\lambda_+\lambda_-}\,)$$

Conclusions

- In the thermodynamic limit relevant probabilities are density distributions.
- Density probability distributions obtained from different statistical ensembles have the same thermodynamical limit.
- Finite volume effect more relevant for higher moments.
- Canonical suppression factor for particles depends on densities.
- Canonical ensembles based on the nonabelian symmetries are different from ensembles based on the direct product of abelian subgroups.
- Quantitative results are also different.
- There is a hope to calculate canonical "nonabelian" partition function without using poorly defined oscilating integrals - also for higher internal symmetries, beyond $SU(2)$.
- . . . work in progress.