Quantum Anomalies in Noncommutative Geometry

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Outline

1. "Standard" anomalies
   - What are anomalies?
   - Approaches to anomalies
2. Noncommutative spaces
   - The axioms
   - Example
3. Anomalies on spectral triples
   - Some results
   - Geometrical approach
   - Noncommutative 2-torus
4. Conclusions

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Heuristic introduction

Classical FT	Quantum FT
imposed symmetries	broken symmetries
$\mathcal{L}[A_\mu] = \mathcal{L}[A_\mu^g]$	$Z[A_\mu] \neq Z[A_\mu^g]$
$\downarrow$	$\uparrow$
conserved currents	quantum anomalies
$D_\mu j^\mu = 0$	$D_\mu j^\mu = \mathbb{A}$

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QFT methods

- Feynman diagrams

- Path-integrals - the Fujikawa method
  - The non-invariance of the path-integral measure!

$$J[\beta, A_\mu] = e^{-\int_M \beta \mathbf{A}}$$

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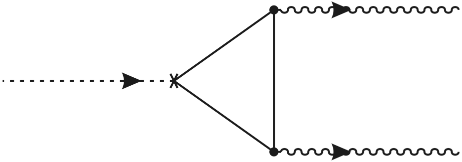

Figure: Triangle diagram appearing in the famous ABJ anomaly computation.

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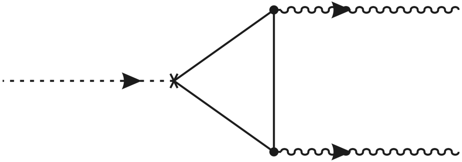

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Geometry of anomalies

- BRST cohomology
  - BRST (non)invariance $s\mathcal{L} = 0$, $sW[A_\mu] = \mathbf{A}$
  - Nilpotency $s^2 = 0 \;\Rightarrow\; s \int_M \nu^a \mathbf{A}^a_{\text{consistent}} = 0$
    - Wess-Zumino consistency conditions
    - Chain of descent equations $\Longrightarrow \mathbf{A}^{2n}_{\text{consistent}} \longleftrightarrow \mathbf{A}^{2n+2}_{\text{singlet}}$
- Index theory
  - Singlet anomaly $\longleftrightarrow$ AS index theorem
  - Non-abelian anomaly
    - AS index theorem for families of elliptic operators
    - AS index theorem in $2n+2$ dimensions (ACG)
      The winding number of the fermionic determinant

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The axioms of noncommutative geometry

$(\mathcal{A}, \mathcal{H}, \mathcal{D})$ - spectral triple

- $\mathcal{A}$ - dense $*$-subalgebra of a C^*-algebra, unital
- $\mathcal{H}$ - Hilbert space; by GNS construction
  $\exists$ a faithful representation $\rho(\mathcal{A}) \simeq \mathcal{B}(\mathcal{H})$
- $\mathcal{D}$ - the Dirac operator - selfadjoint, unbounded
  - $\mathcal{D}^{-1}$ - compact
  - $[\mathcal{D}, \rho(a)] \in \mathcal{B}(\mathcal{H})$ for all $a \in A$
- $\gamma \in \mathcal{B}(\mathcal{H})$ - chirality - selfadjoint, unitary

$$\gamma^2 = \mathrm{id}, \qquad \forall_{a \in \mathcal{A}}\, \gamma a = a\gamma, \qquad \mathcal{D}\gamma = -\gamma\mathcal{D}$$

The $\mathbb{Z}_2$-grading of the Hilbert space $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$

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A commutative spectral triple

Example 1

Let M be compact Riemannian spin manifold, then

- $\mathcal{A} = C^{\infty}(M)$ - dense in $C^{0}(M)$
  - unital if M is compact
- $\mathcal{H} = L^{2}(S(M))$
  - representation: $\rho(f)\xi = f(x) \cdot \xi(x)$
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Some results

- Moyal plane $\mathbb{R}^4_\star$
  (J.M. Gracia-Bondía, C.P. Martín, *Phys. Lett. B* **479** (2000) 321)

$$\mathtt{A} = \alpha \ \mathrm{Tr}\{T^a, T^b\}T^c + \beta \ \mathrm{Tr}[T^a, T^b]T^c$$

- Geometric approach
  (D. Perrot, *Contemp. Math.* 434 (2007) 125.)
  - The role of projections in $\mathcal{A}$.

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Perrot's procedure

Assumption: $\mathfrak{p}$-summability *i.e.* $\mathrm{Tr}(\mathcal{D}^{-\mathfrak{p}}) < \infty$, r - biggest integer $< \mathfrak{p}$

- The action of the "classical" theory

$$S(\psi, \bar{\psi}) = \langle \bar{\psi}, (Q + A)\psi \rangle , \qquad Q = \mathcal{D} + \gamma m, \qquad A \in \mathcal{B}(\mathcal{H})$$

- Path-integral quantization

$$Z(A) = \int \mathrm{d}\psi \mathrm{d}\bar{\psi} \; \mathrm{e}^{-S(\psi, \bar{\psi}, A)} = \det(1 + Q^{-1}A)$$

$$W(A) = \mathrm{Tr}\, \log(1 + Q^{-1}A) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \mathrm{Tr}(Q^{-1}A)^n$$

- Regularization - trace extension $\tau(T) = \operatorname*{Res}_{z=0} \frac{1}{z} \; \mathrm{Tr}(T|Q|^{-2z})$

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$$W(A) = \mathrm{Tr}\ \log(1 + Q^{-1}A) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \mathrm{Tr}(Q^{-1}A)^n$$

- Regularization - trace extension $\tau(T) = \operatorname*{Res}_{z=0} \frac{1}{z}\ \mathrm{Tr}(T|Q|^{-2z})$

Perrot's procedure

Assumption: $\mathfrak{p}$-summability *i.e.* $\mathrm{Tr}(\mathcal{D}^{-\mathfrak{p}}) < \infty$, r - biggest integer $< \mathfrak{p}$

- The action of the "classical" theory

$$S(\psi, \bar{\psi}) = \left\langle \bar{\psi}, (Q + A)\psi \right\rangle, \qquad Q = \mathcal{D} + \gamma m, \qquad A \in \mathcal{B}(\mathcal{H})$$

- Path-integral quantization

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Perrot's procedure

- BRST operator - de Rham differential on S^1
$$s\,:\,C^\infty(S^1)\otimes\mathcal{A}\to\Omega^1(S^1)\otimes\mathcal{A}.$$
- Idempotent loops
$g=1+(\beta-1)p,\quad C^\infty(S^1)\ni\beta=\mathrm{e}^{2\pi\,\mathrm{i}\,t},\quad\mathcal{A}\ni p=p^2$
- Note: $(\mathcal{A},\mathcal{H},\mathcal{D})\longleftrightarrow(M_N(\mathcal{A}),\mathcal{H}\otimes\mathbb{C}^N,\mathcal{D}\otimes\mathbf{1})$
- Mauer-Cartan form (ghost field) $\omega=g^{-1}sg=2\pi\,\mathrm{i}\,p\,\mathrm{d}t$
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The anomaly $\mathbb{A}(\omega, A) := s\, W_\tau(A) =$

$$\sum_{n=1}^{r} \frac{(-1)^{n+1}}{n} s\, \tau(Q^{-1}A)^n + (-1)^r \operatorname{Tr}\left((\omega_+ - Q^{-1}\omega_- Q)(Q^{-1}A)^r\right),$$

Theorem (Perrot)

1. $$\langle [\mathcal{D}], [p] \rangle = \frac{1}{2\pi\, \mathrm{i}} \oint_{S^1} \mathbb{A}(\omega, A) \in \mathbb{Z},$$

2. $\mathbb{A}(\omega, A)$ is cohomologous, as a one-form, to the sum of residues

$$\widetilde{\mathbb{A}}(\omega, A) := \operatorname*{Res}_{z=0} \frac{1}{z} \operatorname{Tr}\left(\gamma\omega Q^{-2z}\right) + \sum_{n\geq 1} \sum_{k\in\mathbb{N}^n} (-1)^{n+k} c(k) \times$$

$$\times \operatorname*{Res}_{z=0} \frac{\Gamma(z+n+k)}{\Gamma(z+1)} \operatorname{Tr}\left(\tfrac{1+\gamma}{2} [\omega, Q] A^{(k_1)} Q A^{(k_2)} Q \ldots A^{(k_n)} Q^{-2(z+n+k)}\right).$$

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Results

- The anomaly result $\widetilde{\mathbf{A}}(\omega, A) = \mathbf{A}(\omega, A) =$

$$4\pi\,\mathrm{i}\,\mathrm{d}t \operatorname*{Res}_{z=0} \mathrm{tr}\left((\beta - 1)[\mathcal{D}_-, p][\mathcal{D}_+, p] + (2 - \beta - \beta^{-1})[\mathcal{D}_-, p]p[\mathcal{D}_+, p]\right) Q^{-2z-2}$$

- The index result

$$\frac{1}{2\pi\,\mathrm{i}} \oint_{S^1} \mathbf{A}(\omega, A) = 2\operatorname*{Res}_{z=0} \mathrm{tr}\left(- [\mathcal{D}_-, p][\mathcal{D}_+, p] + 2[\mathcal{D}_-, p]p[\mathcal{D}_+, p]\right) Q^{-2z-2}$$

- Agreement with Connes-Moscovici

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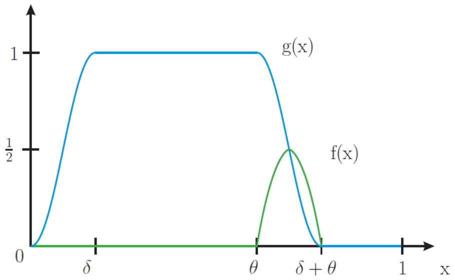
Projections

$$VU = \mathrm{e}^{2\pi\,\mathrm{i}\,\theta}\,UV, \qquad f(V) = \sum_{n\in\mathbb{Z}} f_n\,V^n$$

- Power-Rieffel projection $p = f(V)U + U^{-1}\bar{f}(V^*) + g(V)$

$$\sigma(p) = \theta$$

$$\langle[\mathcal{D}],[p]\rangle = -1$$



Projections

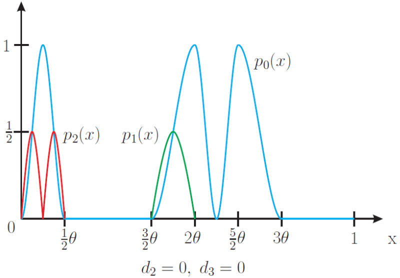
$$VU = \mathrm{e}^{2\pi\,\mathrm{i}\,\theta}\, UV, \qquad f(V) = \sum_{n\in\mathbb{Z}} f_n\, V^n$$

- Second order projection

$$p^{(2)} = p_2(V)U^{-2} + p_1(V)U^{-1} + p_0(V) + U\bar{p}_1(V^*) + U^2\bar{p}_2(V^*)$$

$\langle[\mathcal{D}], [p^{(2)}]\rangle$	$d_2 = 0$	$d_2 = 1$
$d_3 = 0$	2	4
$d_3 = 1$	-4	-2

$\sigma(p^{(2)})$	$d_2 = 0$	$d_2 = 1$
$d_3 = 0$	θ	2θ
$d_3 = 1$	$1-2\theta$	$1-\theta$



Conclusions

- Anomaly cancellation condition
  - Classically: $\mathrm{Tr}\{T^a, T^b\}T^c$
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  - Generally: ?
- The geometry of anomalies
  - The classical correspondence not clear
  - The role of projections

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