Relation between Ford's α-model and a model of Random tree growth by vertex splitting

S.Ö. Stefánsson, University of Iceland

8 June 2009

Zakopane, Poland

Outline

- Definition of the model
- Relation to the vertex splitting model
- Markovian self similarity
- Weak convergence of the finite volume measure
- Conclusions

Definition of the model

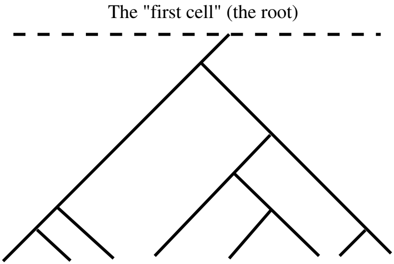
- A one parameter model of randomly growing rooted, planar, binary trees.
- Introduced by Daniel J. Ford in
    arXiv:math/0511246v1 [math.PR].
- Used to model phylogenetic trees

Definition of the model

- A one parameter model of randomly growing rooted, planar, binary trees.
- Introduced by Daniel J. Ford in
  arXiv:math/0511246v1 [math.PR].
- Used to model phylogenetic trees

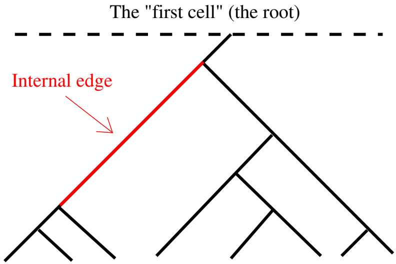
Definition of the model

- A one parameter model of randomly growing rooted, planar, binary trees.
- Introduced by Daniel J. Ford in
  arXiv:math/0511246v1 [math.PR].
- Used to model phylogenetic trees



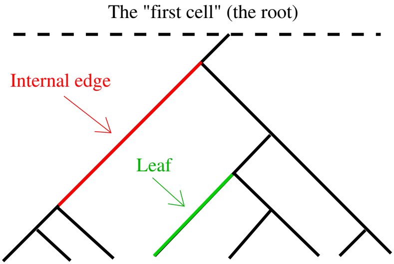
Definition of the model

- A one parameter model of randomly growing rooted, planar, binary trees.
- Introduced by Daniel J. Ford in
  arXiv:math/0511246v1 [math.PR].
- Used to model phylogenetic trees



Definition of the model

- A one parameter model of randomly growing rooted, planar, binary trees.
- Introduced by Daniel J. Ford in
  arXiv:math/0511246v1 [math.PR].
- Used to model phylogenetic trees

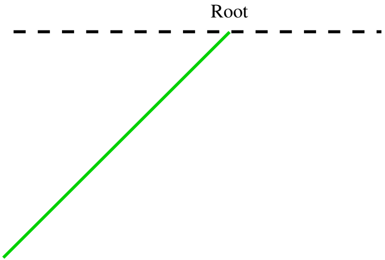


Definition of the model - Growth rules

Begin with a single rooted leaf. Attach (randomly) a new edge to

- an internal edge with weight α
- a leaf with weight $1 - \alpha$

where $0 \leq \alpha \leq 1$.



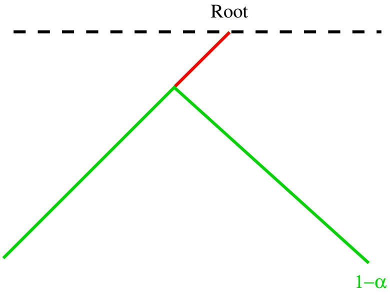
Weight $= 1$

Definition of the model - Growth rules

Begin with a single rooted leaf. Attach (randomly) a new edge to

- an internal edge with weight α
- a leaf with weight $1 - \alpha$

where $0 \leq \alpha \leq 1$.



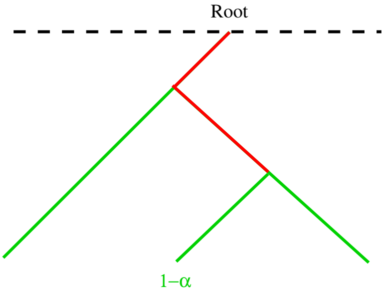
Weight $= \ 1 - \alpha$

Definition of the model - Growth rules

Begin with a single rooted leaf. Attach (randomly) a new edge to

- an internal edge with weight α
- a leaf with weight $1 - \alpha$

where $0 \leq \alpha \leq 1$.



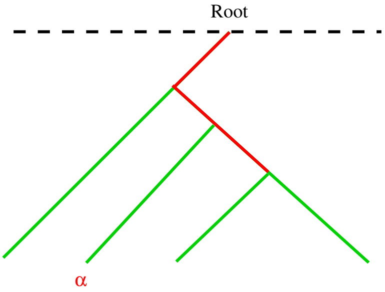
Weight $= (1 - \alpha) \cdot (1 - \alpha)$

Definition of the model - Growth rules

Begin with a single rooted leaf. Attach (randomly) a new edge to

- an internal edge with weight α
- a leaf with weight $1 - \alpha$

where $0 \leq \alpha \leq 1$.



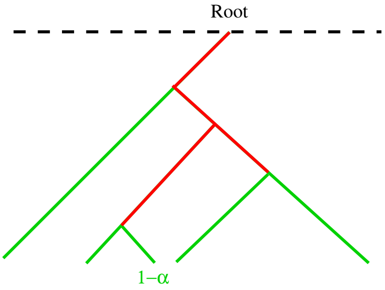
Weight $= (1 - \alpha) \cdot (1 - \alpha) \cdot \alpha$

Definition of the model - Growth rules

Begin with a single rooted leaf. Attach (randomly) a new edge to

- an internal edge with weight α
- a leaf with weight $1 - \alpha$

where $0 \leq \alpha \leq 1$.



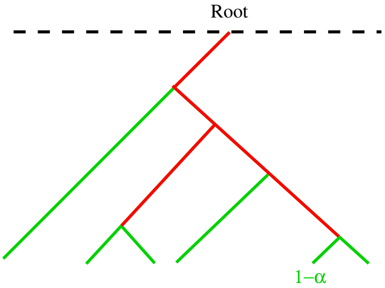
Weight $= (1 - \alpha) \cdot (1 - \alpha) \cdot \alpha \cdot (1 - \alpha)$

Definition of the model - Growth rules

Begin with a single rooted leaf. Attach (randomly) a new edge to

- an internal edge with weight α
- a leaf with weight $1-\alpha$

where $0 \leq \alpha \leq 1$.



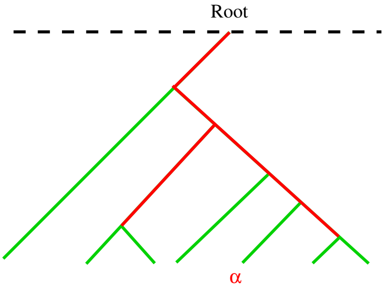
Weight $= (1-\alpha) \cdot (1-\alpha) \cdot \alpha \cdot (1-\alpha) \cdot (1-\alpha)$

Definition of the model - Growth rules

Begin with a single rooted leaf. Attach (randomly) a new edge to

- an internal edge with weight α
- a leaf with weight $1-\alpha$

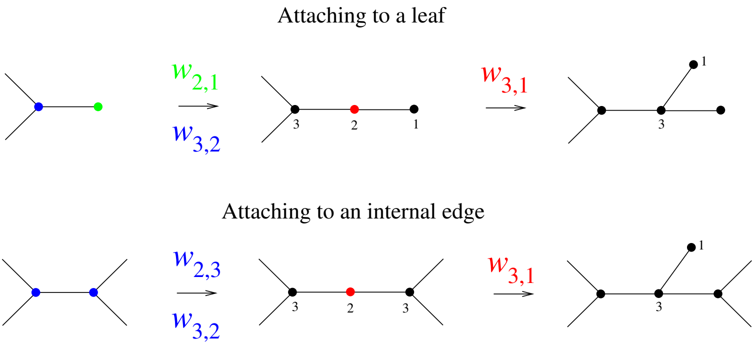
where $0 \leq \alpha \leq 1$.



Weight $= (1-\alpha) \cdot (1-\alpha) \cdot \alpha \cdot (1-\alpha) \cdot (1-\alpha) \cdot \alpha$

Relation to the vertex splitting model

The α–model is a limiting case of a tree growth model introduced by David et al. in arXiv:0811.3183v3 [cond-mat.stat-mech].
[Remember Thordur Jonsson's talk!]

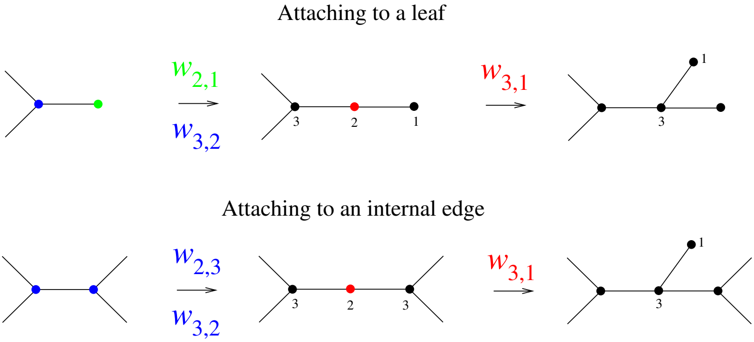


$w_{2,1} = 1 - 3\alpha/2$
$w_{2,3} = \alpha/2$
$w_{3,1} \to \infty$

$d_H = 1/\alpha$

Relation to the vertex splitting model

The α–model is a limiting case of a tree growth model introduced by David et al. in arXiv:0811.3183v3 [cond-mat.stat-mech].
[Remember Thordur Jonsson's talk!]



$$w_{2,1} = 1 - 3\alpha/2$$
$$w_{2,3} = \alpha/2$$
$$w_{3,1} \to \infty$$

$$d_H = 1/\alpha$$

Relation to the vertex splitting model

- Interesting relation since the α–model is simple.
- Gives insight into the more complicated vertex splitting model.
- What allows one to do calculations in the α–model is a property called Markovian self-similarity which is in general not present in the vertex splitting model.

Relation to the vertex splitting model

- Interesting relation since the α–model is simple.
- Gives insight into the more complicated vertex splitting model.
- What allows one to do calculations in the α–model is a property called Markovian self-similarity which is in general not present in the vertex splitting model.

Relation to the vertex splitting model

- Interesting relation since the α–model is simple.
- Gives insight into the more complicated vertex splitting model.
- What allows one to do calculations in the α–model is a property called Markovian self-similarity which is in general not present in the vertex splitting model.

Markovian self similarity

- Call the set of rooted, binary, planar trees on n leaves T_n.
- The α–model growth rules induce a probability distribution $p_{\alpha,n}$ on T_n. We write formally

$$P_{\alpha,n} = \sum_{\tau \in T_n} p_{\alpha,n}(\tau)\tau. \tag{1}$$

  and call $P_{\alpha,n}$ a random tree on n leaves with prob. dist. $p_{\alpha,n}$.
- Introduce an operation $*$ on trees, which joins them by the root $\longrightarrow$ compatible with the sum and scalar product in (1).

Markovian self similarity

- Call the set of rooted, binary, planar trees on n leaves T_n.
- The α–model growth rules induce a probability distribution $p_{\alpha,n}$ on T_n. We write formally

$$P_{\alpha,n} = \sum_{\tau \in T_n} p_{\alpha,n}(\tau)\tau. \quad (1)$$

and call $P_{\alpha,n}$ a random tree on n leaves with prob. dist. $p_{\alpha,n}$.

- Introduce an operation $*$ on trees, which joins them by the root $\longrightarrow$ compatible with the sum and scalar product in (1).

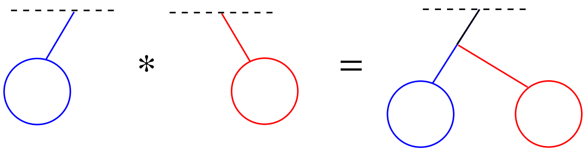
Markovian self similarity

- Call the set of rooted, binary, planar trees on n leaves T_n.
- The α–model growth rules induce a probability distribution $p_{\alpha,n}$ on T_n. We write formally

$$P_{\alpha,n} = \sum_{\tau \in T_n} p_{\alpha,n}(\tau)\tau. \qquad (1)$$

and call $P_{\alpha,n}$ a random tree on n leaves with prob. dist. $p_{\alpha,n}$.

- Introduce an operation $*$ on trees, which joins them by the root $\longrightarrow$ compatible with the sum and scalar product in (1).



Markovian self similarity

Proposition (Ford) *The random tree $P_{\alpha,n}$ satisfies the recursion*

$$P_{\alpha,n} = \sum_{n_1+n_2=n} q_\alpha(n_1,n_2) P_{\alpha,n_1} * P_{\alpha,n_2} \qquad (2)$$

with

$$q_\alpha(n_1,n_2) = \frac{n!}{\Gamma_\alpha(n)}\left(\frac{\alpha}{2}\frac{\Gamma_\alpha(n_1)}{n_1!}\frac{\Gamma_\alpha(n_2)}{n_2!} + (1-2\alpha)\frac{1}{n(n-1)}\frac{\Gamma_\alpha(n_1)}{(n_1-1)!}\frac{\Gamma_\alpha(n_2)}{(n_2-1)!}\right)$$

and

$$\Gamma_\alpha(n) = (n-1-\alpha)(n-2-\alpha)\cdots(2-\alpha)(1-\alpha), \quad \textit{and} \quad \Gamma_\alpha(1) = 1.$$

Definition *A random tree for which there exists a function q such that (2) holds is called Markovian self similar.*

Markovian self similarity

Proposition (Ford) *The random tree $P_{\alpha,n}$ satisfies the recursion*

$$P_{\alpha,n} = \sum_{n_1+n_2=n} q_\alpha(n_1,n_2) P_{\alpha,n_1} * P_{\alpha,n_2} \qquad (2)$$

with

$$q_\alpha(n_1,n_2) = \frac{n!}{\Gamma_\alpha(n)} \left(\frac{\alpha}{2} \frac{\Gamma_\alpha(n_1)}{n_1!} \frac{\Gamma_\alpha(n_2)}{n_2!} + (1-2\alpha) \frac{1}{n(n-1)} \frac{\Gamma_\alpha(n_1)}{(n_1-1)!} \frac{\Gamma_\alpha(n_2)}{(n_2-1)!} \right)$$

and

$$\Gamma_\alpha(n) = (n-1-\alpha)(n-2-\alpha)\cdots(2-\alpha)(1-\alpha), \quad \textit{and} \quad \Gamma_\alpha(1) = 1.$$

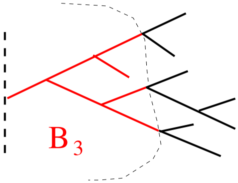
Definition *A random tree for which there exists a function q such that (2) holds is called Markovian self similar.*

Weak convergence of the finite volume measure

Let T be the set of all rooted planar binary trees.

Define a metric on T: $d(\tau,\tau') = \inf\left\{ \frac{1}{R+1} \;\middle|\; B_R(\tau) = B_R(\tau')\right\}$

$B_R(\tau)$ a subtree of τ spanned by vert. of graph dist. $\leq R$ from root.



Proposition *For $0 < \alpha \leq 1$ the probability measure $p_{\alpha,n}$ conv. weakly (in the topology generated by d) as $n \longrightarrow \infty$ to a measure p_α which is concentrated on trees with exactly one path to infinity to which finite trees are attached, i.i.d. by*

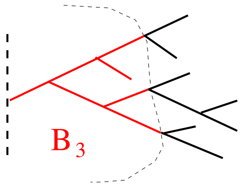
$$\mu(\tau) = \alpha \frac{\Gamma_\alpha(|\tau|)}{|\tau|!} p_{\alpha,|\tau|}(\tau), \qquad |\tau| = \sharp \text{ leaves in } \tau.$$

Weak convergence of the finite volume measure

Let T be the set of all rooted planar binary trees.

Define a metric on T: $d(\tau, \tau') = \inf \left\{ \frac{1}{R+1} \;\middle|\; B_R(\tau) = B_R(\tau') \right\}$

$B_R(\tau)$ a subtree of τ spanned by vert. of graph dist. $\leq R$ from root.

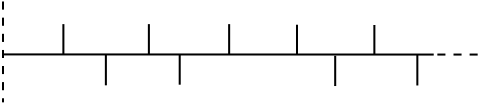


Proposition *For $0 < \alpha \leq 1$ the probability measure $p_{\alpha,n}$ conv. weakly (in the topology generated by d) as $n \longrightarrow \infty$ to a measure p_α which is concentrated on trees with exactly one path to infinity to which finite trees are attached, i.i.d. by*

$$\mu(\tau) = \alpha \frac{\Gamma_\alpha(|\tau|)}{|\tau|!} p_{\alpha,|\tau|}(\tau), \qquad |\tau| = \sharp \text{ leaves in } \tau.$$

Special cases

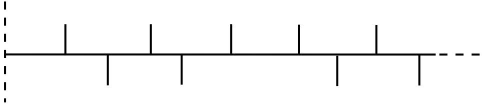
$\alpha = 1:$ a comb with single leaf teeth $\longrightarrow d_s = d_H = 1.$



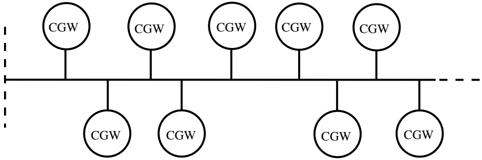
$\alpha = 1/2:$ generic tree $\longrightarrow d_s = 4/3, \; d_H = 2.$

Special cases

$\alpha = 1:$ a comb with single leaf teeth $\longrightarrow d_s = d_H = 1.$



$\alpha = 1/2:$ generic tree $\longrightarrow d_s = 4/3, \quad d_H = 2.$



Outline of proof of convergence:

The metric space (T, d) is compact so it is sufficient to prove for all $R \geq 1$ the convergence of the probability

$$p_{\alpha,n}(\{\tau \in T \mid B_R(\tau) = \tau_0\}) =: p_{\alpha,n}^{(R)}(\tau_0)$$

as $n \longrightarrow \infty$ for any tree τ_0 of height R
[Remember Bergfinnur Durhuus' talk!].

Markovian self similarity allows us to prove this by induction on R.

- It clearly holds for $R = 1$.
- Assume it holds for some R.
- Take a tree τ_0 of height $R + 1$. It can be written as $\tau_0 = \tau_1 * \tau_2$ where τ_1 and τ_2 have height $\leq R$.

Outline of proof of convergence:

The metric space (T, d) is compact so it is sufficient to prove for all $R \geq 1$ the convergence of the probability

$$p_{\alpha,n}(\{\tau \in T \mid B_R(\tau) = \tau_0\}) =: p_{\alpha,n}^{(R)}(\tau_0)$$

as $n \longrightarrow \infty$ for any tree τ_0 of height R
[Remember Bergfinnur Durhuus' talk!].

Markovian self similarity allows us to prove this by induction on R.

- It clearly holds for $R = 1$.
- Assume it holds for some R.
- Take a tree τ_0 of height $R + 1$. It can be written as $\tau_0 = \tau_1 * \tau_2$ where τ_1 and τ_2 have height $\leq R$.

Outline of proof of convergence:

The metric space (T, d) is compact so it is sufficient to prove for all $R \geq 1$ the convergence of the probability

$$p_{\alpha,n}(\{\tau \in T \mid B_R(\tau) = \tau_0\}) =: p^{(R)}_{\alpha,n}(\tau_0)$$

as $n \longrightarrow \infty$ for any tree τ_0 of height R
[Remember Bergfinnur Durhuus' talk!].

Markovian self similarity allows us to prove this by induction on R.

- It clearly holds for $R = 1$.
- Assume it holds for some R.
- Take a tree τ_0 of height $R + 1$. It can be written as $\tau_0 = \tau_1 * \tau_2$ where τ_1 and τ_2 have height $\leq R$.

Outline of proof of convergence:

The metric space (T, d) is compact so it is sufficient to prove for all $R \geq 1$ the convergence of the probability

$$p_{\alpha,n}(\{\tau \in T \mid B_R(\tau) = \tau_0\}) =: p_{\alpha,n}^{(R)}(\tau_0)$$

as $n \longrightarrow \infty$ for any tree τ_0 of height R
[Remember Bergfinnur Durhuus' talk!].

Markovian self similarity allows us to prove this by induction on R.

- It clearly holds for $R = 1$.
- Assume it holds for some R.
- Take a tree τ_0 of height $R + 1$. It can be written as $\tau_0 = \tau_1 * \tau_2$ where τ_1 and τ_2 have height $\leq R$.

Outline of proof of convergence:

The metric space (T, d) is compact so it is sufficient to prove for all $R \geq 1$ the convergence of the probability

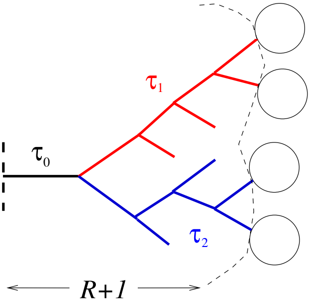
$$p_{\alpha,n}(\{\tau \in T \mid B_R(\tau) = \tau_0\}) =: p_{\alpha,n}^{(R)}(\tau_0)$$

as $n \longrightarrow \infty$ for any tree τ_0 of height R
[Remember Bergfinnur Durhuus' talk!].

Markovian self similarity allows us to prove this by induction on R.

- It clearly holds for $R = 1$.
- Assume it holds for some R.
- Take a tree τ_0 of height $R + 1$. It can be written as $\tau_0 = \tau_1 * \tau_2$ where τ_1 and τ_2 have height $\leq R$.

Outline of proof of convergence:

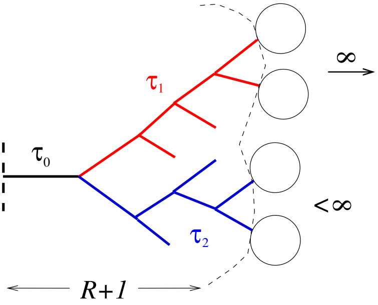


$$p^{(R+1)}_{\alpha,n}(\tau_0) = \sum_{n_1+n_2=n} q_\alpha(n_1,n_2)\, p^{(R)}_{\alpha,n_1}(\tau_1)\, p^{(R)}_{\alpha,n_2}(\tau_2)$$

$$q_\alpha(n_1,n_2) = \frac{n!}{\Gamma_\alpha(n)}\left(\frac{\alpha}{2}\frac{\Gamma_\alpha(n_1)}{n_1!}\frac{\Gamma_\alpha(n_2)}{n_2!} + (1-2\alpha)\frac{1}{n(n-1)}\frac{\Gamma_\alpha(n_1)}{(n_1-1)!}\frac{\Gamma_\alpha(n_2)}{(n_2-1)!}\right)$$

All except finite (but arbitrarily large) mass goes to either τ_1 or τ_2
$\longrightarrow$ convergence follows from ind. hyp.

Outline of proof of convergence:



$$p^{(R+1)}_{\alpha,\infty}(\tau_0) = p^{(R)}_{\alpha,\infty}(\tau_1) \sum_{n_2} q_\alpha(\infty, n_2) p^{(R)}_{\alpha,n_2}(\tau_2) + (\tau_1 \leftrightarrow \tau_2)$$

$$q_\alpha(\infty, n_2) = \frac{n!}{\Gamma_\alpha(n)} \left(\frac{\alpha}{2} \frac{\Gamma_\alpha(n_1)}{n_1!} \frac{\Gamma_\alpha(n_2)}{n_2!} + (1-2\alpha) \frac{1}{n(n-1)} \frac{\Gamma_\alpha(n_1)}{(n_1-1)!} \frac{\Gamma_\alpha(n_2)}{(n_2-1)!} \right)$$

All except finite (but arbitrarily large) mass goes to either τ_1 or τ_2
$\longrightarrow$ convergence follows from ind. hyp.

Conclusions

- We have proven convergence of the finite volume measure generated by the growth rules of the α–model for $0 < \alpha \leq 1$ and characterized the limiting measure.
- Possibility of a better understanding of the vertex splitting model.
- Work in progress: What are the dimensions d_s and d_H of the infinite α–trees? Is it true that $d_H = 1/\alpha$? At least for $\alpha = 1$ and $\alpha = 1/2$.
- Conjecture: $d_s = \dfrac{2}{1+\alpha}$.
- $\alpha = 0$?