Symmetry Reduction and Exact Solutions in Twisted Noncommutative Gravity

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49. Cracow School of Theoretical Physics 2009

Non-perturbative Gravity and Quantum Chromodynamics

Zakopane, Poland,
May 31- June 10, 2009

Why Noncommutative Geometry?

Basics of Noncommutative Geometry

NC Symmetry Reduction

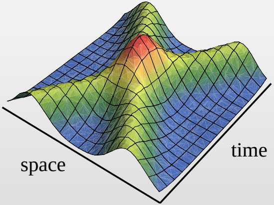
Dynamics of Symmetry Reduced Sectors

Field Fluctuations on NC Backgrounds

Why Noncommutative Geometry?

Einstein gravity

- based on smooth manifolds
  i. e. spacetime made out of points
- ↯ points not physical (black holes!)
- ∴ wrong category for short distance gravity



Quantum gravity

- motivation: “get rid of points”
- approaches: Strings, Loop Quantum Gravity, CDT, . . .
- hardest problem: making contact with the “real world”

“Almost quantum” gravities

- intermediate step incorporating most important quantum effects
- ideas: infrared expansion, noncommutative geometry, . . .
- ☺ NC Geometry without points and spacetime uncertainties built in

Basics of Noncommutative Geometry

- wanted: coordinate operators $[\hat{x}^\mu, \hat{x}^\nu] = i\Theta^{\mu\nu}(\hat{x})$

$\Rightarrow$ spacetime uncertainty relations $\Delta x^\mu \Delta x^\nu \neq 0$

- equivalently: use $\star$-products $f(x) \star g(x) \neq g(x) \star f(x)$
- examples:
  - Moyal-Weyl product:

$$f \star g = f e^{\frac{i\lambda}{2} \overleftarrow{\partial}_\mu \Theta^{\mu\nu} \overrightarrow{\partial}_\nu} g$$

  - Reshetikhin-Jambor-Sykora (RJS) product:

$$f \star g = f e^{\frac{i\lambda}{2} \overleftarrow{X}_\alpha \Theta^{\alpha\beta} \overrightarrow{X}_\beta} g \,, \quad [X_\alpha, X_\beta] = 0$$

- **NB:** RJS and Moyal-Weyl products are obtained from twists

$$\mathcal{F} = \exp\Big(-\frac{i\lambda}{2}\Theta^{\alpha\beta} X_\alpha \otimes X_\beta\Big) \in U\Xi \otimes U\Xi$$

[Wess group, Madore, ...]

- classical spaces $\leftrightsquigarrow$ classical symmetries (Lie groups/algebras)
  - euclidean space $\leftrightsquigarrow$ euclidean group $SO(3) \ltimes \mathbb{R}^3$
  - Minkowski space $\leftrightsquigarrow$ Poincaré group $SO(3,1) \ltimes \mathbb{R}^4$
- NC spaces $\leftrightsquigarrow$ "quantum symmetries" (quantum groups/Hopf algebras)
  - q-euclidean space $\leftrightsquigarrow$ q-euclidean Hopf algebra
  - Moyal-plane $\leftrightsquigarrow$ θ-Poincaré Hopf algebra
- general feature:

  noncommutative spacetime $\leftrightsquigarrow$ noncocommutative Hopf algebra

  commutative spacetime $\leftrightsquigarrow$ Lie algebra $\rightarrow$ cocommutative HA
- Basic idea of (twisted) NC gravity:

  Einstein gravity $\leftrightsquigarrow$ diffeomorphism Lie algebra Ξ

  NC Einstein gravity $\leftrightsquigarrow$ deformed diffeomorphism Hopf algebra

$$(\Xi, [\ ,\ ]) \xrightarrow{\text{construct}} (U\Xi, \cdot, \Delta, S, \epsilon) \xrightarrow{\mathcal{F}} (U\Xi, \cdot, \Delta_{\mathcal{F}}, S_{\mathcal{F}}, \epsilon)$$

[Wess group]

✓ construction of cov. derivatives and curvature on NC manifolds

basic idea: deform everything using the twist ⇒ deformed covariant theory

⇒ NC Einstein equations:

$$\mathrm{Ric}_{ab} - \frac{1}{2} g_{ab} \star \mathfrak{R} = 8\pi G\, T_{ab}$$

- **NB:**
  - ▷ nonlocal and nonlinear equations of motion → i.g. complicated
  - ▷ ambiguities in defining Einstein equations ☹
- wanted: solutions of NC Einstein equations

NC Symmetry Reduction:

a first step towards solutions

Classical symmetry reduction:

- isometries $\hat{=}$ symmetry Lie algebra $\mathfrak{g}$
- represent $\mathfrak{g}$ in terms of vector fields Ξ
- demand $\mathcal{L}_\mathfrak{g}(\tau) = \{0\}$ for all symmetric tensor fields

NC symmetry reduction: [Th. Ohl, AS: JHEP 0901:084,2009]

- isometries $\hat{=}$ symmetry Lie algebra $\mathfrak{g}$
- represent $\mathfrak{g}$ in terms of vector fields Ξ
- demand $\mathcal{L}_\mathfrak{g}(\tau) = \{0\}$ for all symmetric tensor fields

\+ consistency condition: $\mathcal{L}_\mathfrak{g}(\tau \star \tau') = \{0\}$, if $\mathcal{L}_\mathfrak{g}(\tau) = \mathcal{L}_\mathfrak{g}(\tau') = \{0\}$!

- **NB:**
  - CC from nontrivial coproduct $\Delta_\mathcal{F}$ in Hopf algebra
  - restrictions among twist $\mathcal{F}$ and symmetry Lie algebra $\mathfrak{g}$
  - for RJS twists $[X_\alpha, \mathfrak{g}] \subseteq \mathfrak{g}$, $\forall_\alpha$ $\rightarrow$ classification! ☺

✓ FRW models, Schwarzschild black holes (& black branes, AdS, . . .)

1. favorite model: $\left[\hat{t}, \hat{x}^i\right] = i\lambda \widehat{X(t)x^i}$
   - ▷ isotropic but nonhomogeneous model (interesting for CMB)
   - ▷ $X(t)$ can be used to tune away NC effects for large t
   - ▷ NC can drive gravity (see below!)
   - ☹ lies in the model class we understand less
2. next-to-favorite model: $\left[\widehat{\exp i\phi}, \hat{t}\right] = \lambda \widehat{\exp i\phi}$
   - ▷ discrete time spectrum $\sigma(\hat{t}) = \lambda(\mathbb{Z} + \delta)$
   - → singularity avoidance in cosmology!?!
   - ☺ we understand background dynamics (see below!)
   - ▷ nonisotropic model: maybe problems with CMB
3. less favored models: e. g. $\left[\hat{x}^i, \hat{x}^j\right] = i\lambda^{ij}\hat{1}$
   - ☹ NC scale growing with time
   - ☺ backgrounds and (Q)FT (see below!)
   - → nice playground for mathematical aspects (e. g. interacting fields)

1. isotropic model:

$$\left[\hat{t}, \hat{r}\right] = i\lambda \widehat{f(r)}$$

2. discrete time model:

$$\left[\widehat{\exp i\phi}, \hat{t}\right] = \lambda \widehat{\exp i\phi}$$

3. discrete radius model:

$$[\exp i\phi \stackrel{\star}{,} r] = -2 \sinh\left(\frac{\lambda}{2} f(r)\partial_r\right) r \cdot \exp i\phi$$

- BH models solve NC Einstein equations using undeformed metric!

Dynamics of Symmetry Reduced Sectors:

general properties and explicit solutions

Proposition (Th. Ohl, AS: to appear)

Let $\mathcal{F} = \exp\left(-\frac{i\lambda}{2}\Theta^{\alpha\beta}X_\alpha \otimes X_\beta\right)$ be a $\mathfrak{g}$-compatible RJS twist. Then the symmetry reduced Riemannian geometry is undeformed if one $X_\alpha \in \mathfrak{g}$, for all pairs of vector fields connected by $\Theta^{\alpha\beta}$.

- most FRW and black hole models are exactly solvable ☺
- **NB:** this **does not** mean our models are trivial!
- $\exists$ examples with NC corrections to backgrounds
- e. g. $\left[\hat{t}, \hat{x}^i\right] = i\lambda\hat{x}^i \;\Rightarrow$ NC Friedmann equations:

$$3\frac{\dot{A}(t-i\lambda)}{A(t-i\lambda)}\frac{\dot{A}(t+i\lambda)}{A(t+i\lambda)} + \frac{3}{2}\frac{\dot{N}(t)}{N(t)}\left(\frac{\dot{A}(t-i\lambda)}{A(t-i\lambda)} - \frac{\dot{A}(t+i\lambda)}{A(t+i\lambda)}\right) + \frac{3}{2}\left(\frac{\ddot{A}(t+i\lambda)}{A(t+i\lambda)} - \frac{\ddot{A}(t-i\lambda)}{A(t-i\lambda)}\right) = \rho(t)$$

$$-\frac{A(t)\dot{A}(t)\dot{A}(t-2i\lambda)}{A(t-2i\lambda)N(t-i\lambda)^2} + \frac{A(t)\dot{A}(t)\dot{N}(t-i\lambda)}{2N(t-i\lambda)^3} + \frac{3A(t)^2\dot{A}(t-2i\lambda)\dot{N}(t-i\lambda)}{2A(t-2i\lambda)N(t-i\lambda)^3} - \frac{A(t)\ddot{A}(t)}{2N(t-i\lambda)^2} - \frac{3A(t)^2\ddot{A}(t-2i\lambda)}{2A(t-2i\lambda)N(t-i\lambda)^2} = p(t)$$

- i. g. extremely complicated ☹,
  ... but de Sitter space + cosmological constant solves it ☺

Field Fluctuations on NC Backgrounds:

a first step towards physics

- fixed Riemannian manifold $(\mathcal{M}, g)$ with isometries $\mathfrak{g}$
- Definition: **Killing twist** $\mathcal{F} \in U\mathfrak{g} \otimes U\mathfrak{g} \subseteq U\Xi \otimes U\Xi$
  - ☑ actions: $S_\Phi^\star = -\frac{1}{2}(d\Phi, d\Phi)_\star - \frac{m^2}{2}(\Phi, \Phi)_\star - \sum_{k=3}^{N} \lambda_k (1, \Phi^{\star k})_\star$
  - ☑ phasespace with deformed Peierls brackets (for free scalars)
  - ☑ Fock Space Quantization of deformed Peierls algebras
  - ☑ first ideas and calculations for interacting scalar fields
- **Non-Killing twists:** (as required for cosmology)
  - ☑ wave equations: $\square^\star \Phi + F[\Phi] = 0$

  e. g. Free scalar field on NC de Sitter space:

$$\ddot{\Phi}(x) + 3H\dot{\Phi}(x) - e^{-2Ht}\Delta\tilde{\mathbf{\Phi}}(\mathbf{x}) + M^2\Phi(x) = 0 \,, \quad \text{where}$$

  1. $\tilde{\mathbf{\Phi}}(\mathbf{x}) = \exp\big(i\lambda(\partial_t - Hr\partial_r)\big)\Phi(x)$ for $\big[\hat{t}, \hat{x}^i\big] = i\lambda\hat{x}^i$
  2. $\tilde{\mathbf{\Phi}}(\mathbf{x}) = \exp\big(i\lambda H\partial_\phi\big)\Phi(x)$ for $\Big[\hat{t}, \widehat{\exp i\phi}\Big] = \lambda\widehat{\exp i\phi}$

  - ☑ (at least perturbative) solutions for free fields
  - ☐ actions, deformed Peierls algebras, quantization, physics, …

Conclusions and Outlook

- NCG is interesting step between classical and quantum gravity
- we found approach to NC symmetry reduction

$\rightarrow$ cosmological, black hole (& black brane, AdS, ...) solutions

- distinct NC effects depending on model, e. g.
  - discrete time spectra in cosmology
  - discrete radius spectra for black holes
  - lattice structure of position eigenvalues for black branes
- $\exists$ "realistic" models worth for cosmological studies
- free QFT on curved Killing RJS backgrounds

... still many open questions and undone calculations remain:

  - cosmological powerspectra and CMB predictions
  - (Q)FT on curved non-Killing RJS backgrounds
  - ¿¿ Deformed AdS spaces for particle physics ?? (with C. Uhlemann)