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Can Causal Dynamical Triangulations Probe Factor-Ordering Issues?

Or, What Path-Integral Measure Does CDT Favor?

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Rachel Maitra
Utrecht University

CDT's measure comes from geometry

Regularized Euclidean path integral over causally well-behaved discrete geometries:

$$Z(G,\Lambda) = \int \mathcal{D}[g]\, e^{-S_E^{EH}[g]} \quad \rightarrow \quad Z(\kappa_0,\kappa_4,\Delta) = \sum_{\mathcal{T}} \frac{1}{C_{\mathcal{T}}} e^{-S_E^{\mathrm{Regge}}(\mathcal{T})}$$

$\mathcal{T}$ = causally well-behaved triangulation of spacetime ($\sim I\times S^3$)

$C_{\mathcal{T}}$ = symmetry factor (order of automorphism group of $\mathcal{T}$) = "discrete measure"

Δ = asymmetry parameter from relative scale of timelike and spacelike edges

κ_0, κ_4 = dimensionless coupling constants from G, Λ

Quantum fluctuations in volume

(Ambjorn, Görlich, Jurkiewicz, Loll 2008, arXiv:0807.4481v1)

Computationally measure covariance matrix for spatial volume:

Monte Carlo simulation $\rightarrow$ K independent histories

$N_3^{(k)}(i)$ = spatial volume of k^{th} history at timestep i

$$\overline{N}_3(i) \equiv \langle N_3(i) \rangle \cong \frac{1}{K} \sum_k N_3^{(k)}(i)$$

Compute eigenfunctions of $C(i,j)$.

$$C(i,j) \cong \frac{1}{K} \sum_k \left(N_3^{(k)}(i) - \overline{N}_3(i) \right) \left(N_3^{(k)}(j) - \overline{N}_3(j) \right)$$

Compare with semiclassical calculation

(Wick+conformal-rotated) Lagrangian for 3-volume in FRW cosmology:

$$L\left[V_3(s)\right] = \frac{c_1}{V_3(s)}\left(\frac{dV_3(s)}{ds}\right)^2 + c_2 V_3^{1/3}(s) - \lambda V_3(s)$$

Approximate path integral for correlator of volume fluctuation by expanding around the classical solution:

$$x(t) = V_3(t) - V_3^{cl}(t)$$

$$\left\langle x(t)\, x(t')\right\rangle \approx e^{-\frac{S\left[V_3^{cl}\right]}{\hbar}} \cdot \int \mathcal{D}x(s)\; x(t)\, x(t')\, e^{-\frac{1}{2\hbar}\iint ds\, ds'\; x(s) M(s,s') x(s')}$$

$$= e^{-\frac{S\left[V_3^{cl}\right]}{\hbar}} \cdot M^{-1}(t,t')$$

Compare eigenfunctions with computationally obtained operator *C*(*i,j*).

However, this calculation assumes a measure

Takes the combination

$$e^{-\frac{1}{2\hbar}\iint ds\,ds'\, x(s)M(s,s')x(s')}\mathcal{D}x(s)$$

to be the functional Gaussian measure.

But $\mathcal{D}x(s)$ is inherited from the full path integral measure $\mathcal{D}V_3(s)$, and should be nontrivial, since the effective Lagrangian

$$L\left[V_3(s)\right]=\frac{c_1}{V_3(s)}\left(\frac{dV_3(s)}{ds}\right)^2+c_2V_3^{1/3}(s)-\lambda V_3(s)$$

contains products involving $V_3(s)$ and $V_3{}'(s)$.

▶ Here is our opportunity.

How much does choice of measure matter?

Rephrase question canonically in terms of factor ordering. Consider 2-parameter family (Steigl/Hinterleitner 2006):

$$H = \frac{1}{2}\left[-\frac{{p_a}^2}{a} - a + \frac{\Lambda}{3}a^3 \right]$$

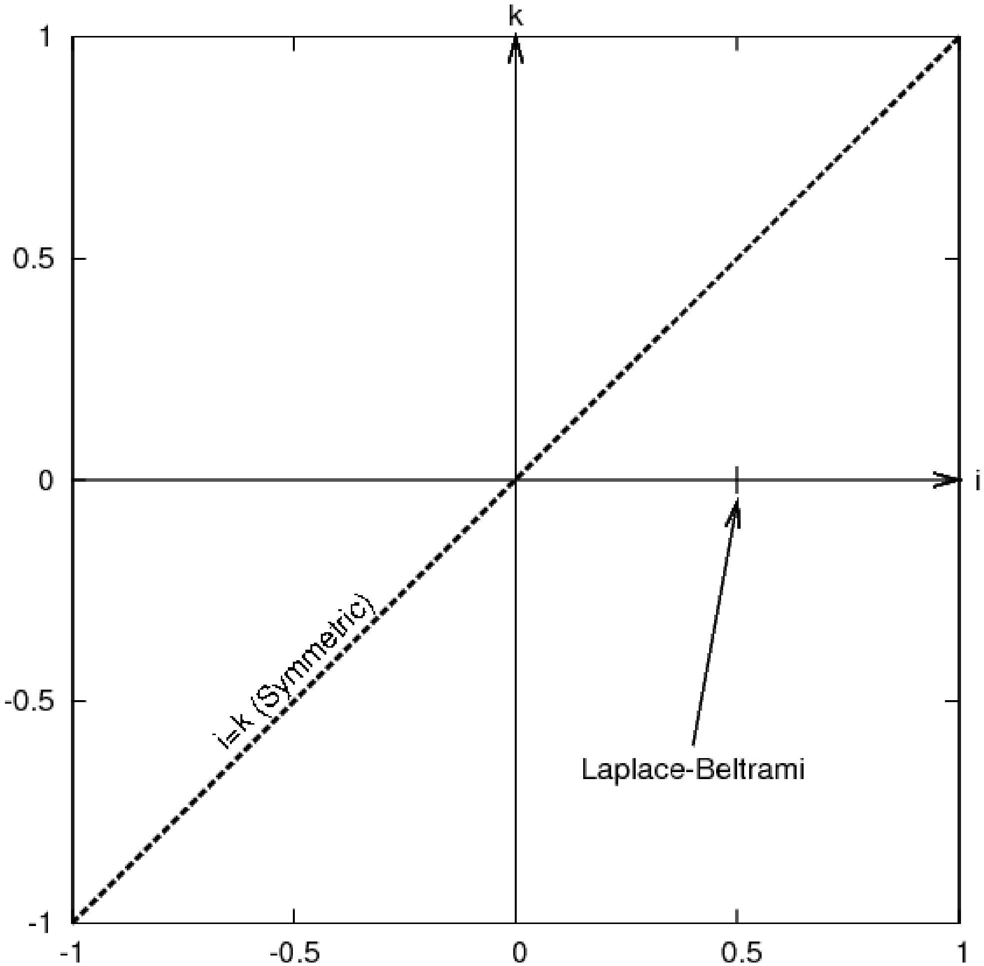
$$\xrightarrow{p_a \to -i\hbar\partial_a} \frac{1}{2}\left(\hbar^2 \frac{1}{a^i}\partial_a \frac{1}{a^j}\partial_a \frac{1}{a^k} - a + \frac{\Lambda}{3}a^3 \right), \; i + j + k = 1$$

With respect to the measure $a^{i\text{-}k}da$, $\hat{H}$ is self-adjoint.

$$\underbrace{\hbar^2 a^{-\frac{1}{2}(1-j)}\partial_a a^{-j}\partial_a a^{-\frac{1}{2}(1-j)}}_{Symmetric} \qquad \underbrace{\hbar^2 a^{-\frac{1}{2}}\partial_a a^{-\frac{1}{2}}\partial_a}_{Laplace-Beltrami,\; g=(a)}$$

Steigl-Hinterleitner orderings

$$a^{-i}\partial_a a^{-j}\partial_a a^{-k}, \quad i+j+k=1$$



$$\Delta_{LB} = -\left(dd^* + d^* d\right) = \frac{1}{\sqrt{|g|}}\partial_\alpha \sqrt{|g|}g^{\alpha\beta}\partial_\beta$$

$$g(a) = (a) \quad \Rightarrow \quad \Delta_{LB} = a^{-1/2}\partial_a a^{-1/2}\partial_a$$

Quadratic approximation detects asymmetry

Look at Wheeler-DeWitt equation:

$$\hbar^2 a^{-1} \partial_a^2 \psi(a) + \hbar^2 (-k + i - 1) a^{-2} \partial_a \psi(a)$$
$$+ \left[\hbar^2 k (-i + 2) a^{-3} - a + \frac{\Lambda}{3} a^3 \right] \psi(a) = 0$$

WKB-style series solution:

$$\psi(a) = \exp\left\{ -\frac{S(a)}{\hbar} \right\} \sum_{n=0}^{\infty} \hbar^n \varphi_n(a)$$

Yields

$$S(a) = \frac{1}{\Lambda}\left(1 - \frac{\Lambda}{3} a^2\right)^{\frac{3}{2}} - \frac{1}{\Lambda}, \qquad \varphi_0(a) = K \left| 1 - \frac{\Lambda}{3} a^2 \right|^{-\frac{1}{4}} \cdot a^{\frac{k-i}{2}}$$

Quadratic approximation detects asymmetries in factor ordering.

Correlators from propagators

Ordering *(i,j,k)* has Hilbert space of states $L^2(R^+, a^{i-k} da)$.

Using resolution of identity we can write any correlator in terms of propagators:

$$\left\langle a'', t'' \middle| f\left(\hat{a}(t_2)\right) f\left(\hat{a}(t_1)\right) \middle| a', t' \right\rangle_{ijk}$$

$$= \int_0^\infty a_2^{i-k} da_2\, f(a_2) \left\langle a'', t'' \middle| a_2, t_2 \right\rangle_{ijk} \left[\int_0^\infty a_1^{i-k} da_1\, f(a_1) \left\langle a_2, t_2 \middle| a_1, t_1 \right\rangle_{ijk} \left\langle a_1, t_1 \middle| a', t' \right\rangle_{ijk} \right]$$

Thus we focus on finding (Wick+conformal-rotated) path integral expressions for propagators of Euclidean Schrödinger equation for FRW cosmology with Steigl/Hinterleitner operator orderings.

Propagator (Wick- and conformal-rotated)

Want path integral expression for the propagator

$$\left(\frac{\partial}{\partial t}-\frac{\hbar^2}{2}a^{-i}\partial_a a^{-j}\partial_a a^{-k}-V(a)\right)K_{ijk}(a,t;a',t')=0,\quad a\neq a'$$

$$\lim_{t\searrow 0}K_{ijk}(a,t;a',t')=(a')^{k-i}\,\delta(a-a')$$

$$\boxed{V(a)=\frac{1}{2}\left(-a+\frac{\Lambda}{3}a^3\right)}$$

For the Laplace-Beltrami case, it is known (DeWitt (1957), Parker (1979)):

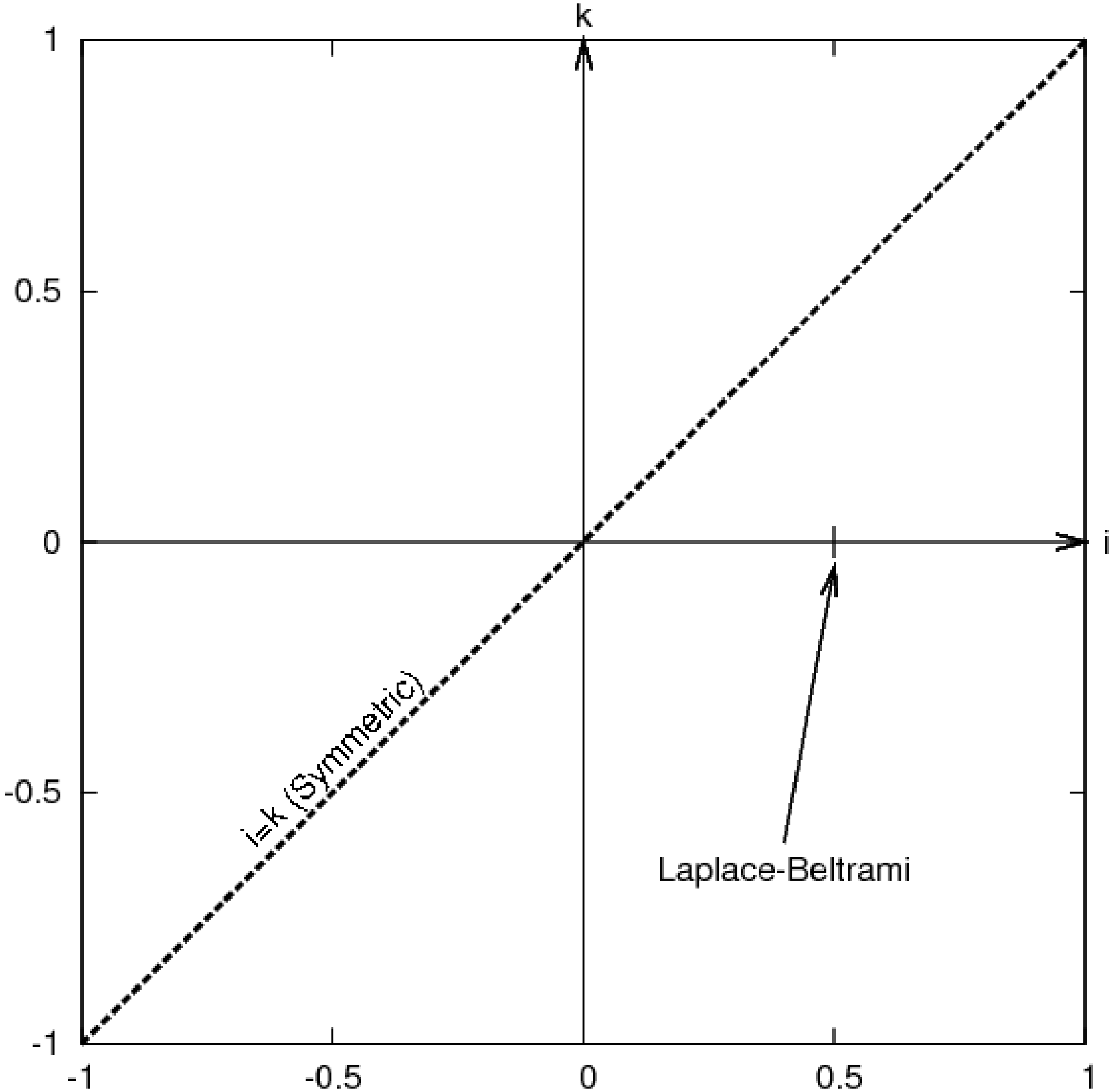
$$K_{LB}(a'',t'';a',t')=\int_{\mathcal{C}\{a',t''|a'',t'\}}\mathcal{D}\left[a^{\frac{1}{2}}(t)a(t)\right]\exp\left\{-\frac{1}{\hbar}\int_{t'}^{t''}dt\left[\frac{1}{2}a\dot{a}^2-V(a)\right]\right\}$$

$$\equiv\lim_{N\to\infty}\frac{1}{(2\pi\varepsilon\hbar)^{N/2}}\prod_{n=1}^{N-1}\int a_n^{\frac{1}{2}}da_n$$

$$\times\exp\left\{-\frac{1}{\hbar}\sum_{n=1}^{N}\left[\frac{1}{2\varepsilon}a_{n-1}(a_n-a_{n-1})^2-\varepsilon V(a_{n-1})\right]\right\}$$

Steigl-Hinterleitner orderings

$$a^{-i}\partial_a a^{-j}\partial_a a^{-k}, \quad i+j+k=1$$



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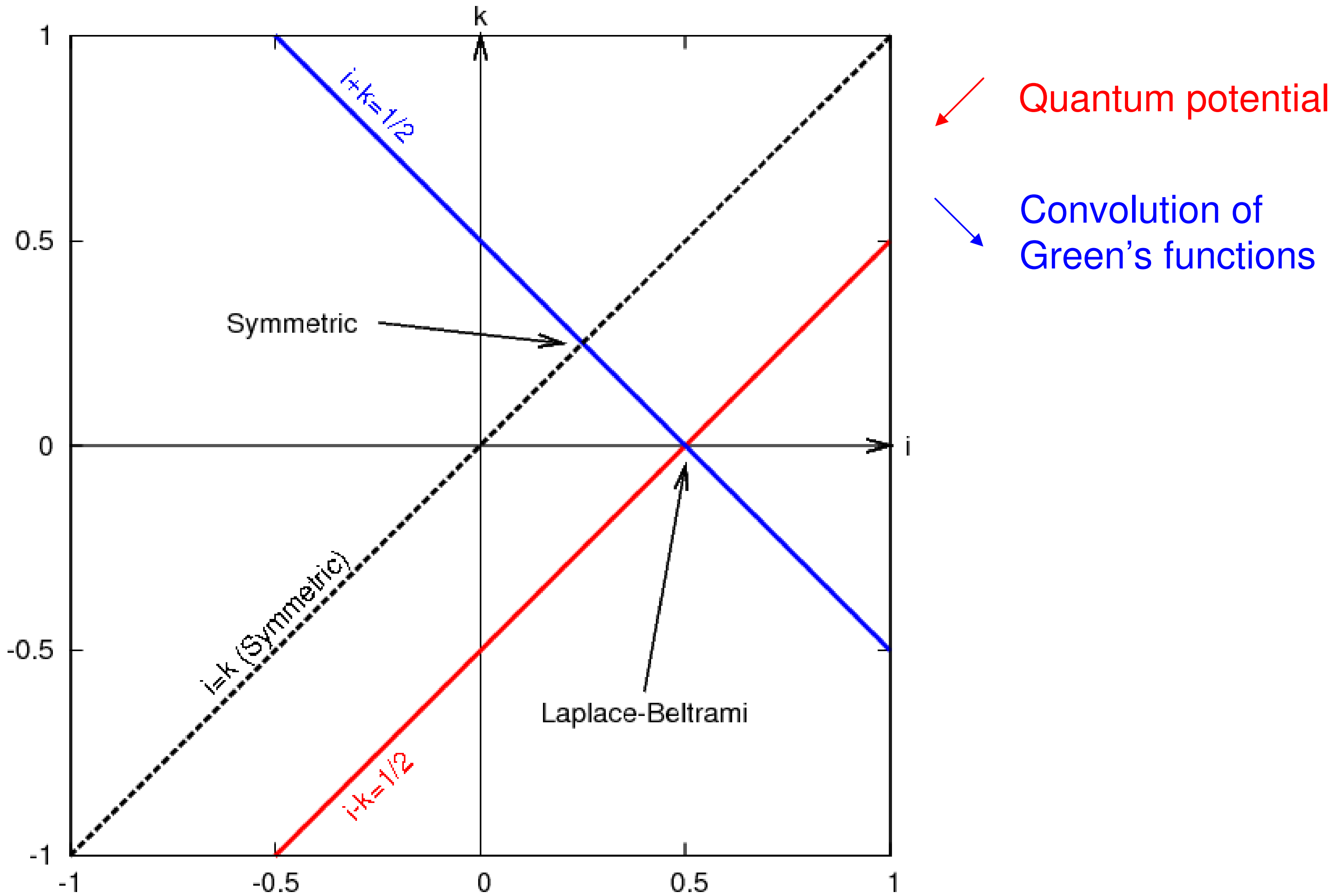

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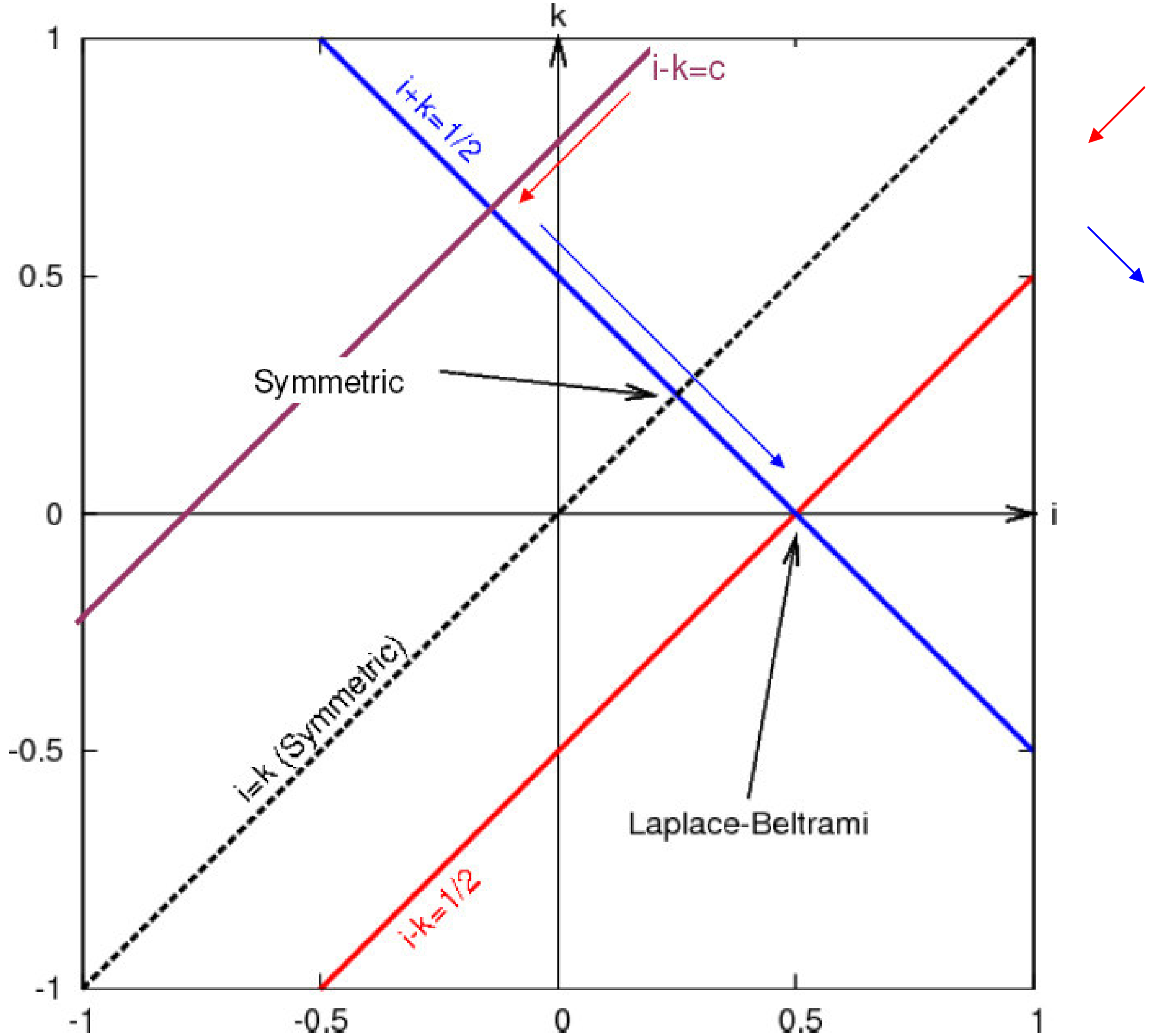

Quantum potential

Convolution of Green's functions

▶ Relate any propagator to that for the Laplace-Beltrami ordering, using a combination of a quantum potential and convolution of Green's functions

$i + k = 1/2$

$$\hat{H} = \frac{\hbar^2}{2}\hat{a}^{k-1/2}\partial_a \hat{a}^{-1/2}\partial_a \hat{a}^{-k} + V(\hat{a})$$

Need propagator satisfying

$$\left(\frac{\partial}{\partial t} - \hat{H}\right) K_{ijk}\left(a'', t''; a', t'\right) = 0, \quad a'' \neq a'; \qquad \lim_{t \searrow 0} K_{ijk}\left(a'', t''; a', t'\right) = \left(a'\right)^{2k-1/2} \delta\left(a'' - a'\right)$$

$$\left(\frac{\partial}{\partial t} - \hat{H}\right) = a^k \left(\frac{\partial}{\partial t} - \hat{H}_{LB}\right) a^{-k}, \qquad \hat{H}_{LB} = \frac{\hbar^2}{2}\hat{a}^{-1/2}\partial_a \hat{a}^{-1/2}\partial_a + V(\hat{a})$$

By convolution of Green's functions and conversion to correct measure

$$K_{ijk}\left(a'', t''; a', t'\right) = \left(a'\right)^{2k-1/2}\left(a'\right)^{1/2}\left[\left(a''\right)^k K_{LB}\left(a'', t''; a', t'\right)\left(a'\right)^{-k}\right]$$

$$= \left(a' a''\right)^k K_{LB}\left(a'', t''; a', t'\right)$$

$i - k = c$

Rewrite Hamiltonian in terms of an effective kinetic term satisfying *i+k=1/2,* and a quantum potential:

$$\hat{H} = \frac{\hbar^2}{2} a^{-k-c}\partial_a a^{2k+c-1}\partial_a a^{-k} + V(a)$$

$$= \frac{\hbar^2}{2}\left[a^{-1}\partial_a^2 - (1-c)a^{-2}\partial_a + k(2-c-k)a^{-3}\right] + V(a)$$

$$= \underbrace{\frac{\hbar^2}{2}\left[a^{-1/2(1/2+c)}\partial_a a^{-1/2}\partial_a a^{-1/2(1/2-c)}\right]}_{\text{new effective kinetic term}} +$$

$$\boxed{k' = \frac{1}{2}\left(\frac{1}{2} - c\right) = \frac{1}{4} + \frac{k-i}{2}}$$

$$\underbrace{\left[\overbrace{\frac{\hbar^2}{2}\left(k(2-c-k) - \tfrac{1}{4}\left(\tfrac{1}{2}-c\right)\left(\tfrac{7}{2}-c\right)\right)a^{-3}}^{\text{"quantum potential"}} + V(a)\right]}_{V_{QP}(a)}$$

Path integral expression for propagator

- Rewrite Hamiltonian in terms of an effective kinetic term satisfying *i+k=1/2* and a quantum potential

- Relate the relating the resulting propagator to that for the Laplace-Beltrami ordering using convolution of Green's functions

$$K_{ijk}\left(a'',t'';a',t'\right)=$$

$$\left(a'a''\right)^{1/4+\frac{k-i}{2}}\int_{\mathcal{C}\{a',t''|a'',t'\}}\mathcal{D}\left[a^{\frac{1}{2}}(t)\,a(t)\right]\exp\left\{-\frac{1}{\hbar}\int_{t'}^{t''}dt\left[\frac{1}{2}a\dot{a}^2-\left(V(a)+\frac{\hbar^2}{2}f(k,c)\,a^{-3}\right)\right]\right\}$$

$$\equiv\lim_{N\to\infty}\frac{\left(a'a''\right)^{1/4+\frac{k-i}{2}}}{\left(2\pi\varepsilon\hbar\right)^{N/2}}\prod_{n=1}^{N-1}\int a_n^{\frac{1}{2}}da_n$$

$$\boxed{f(a)=\left(k(2-c-k)-\tfrac{1}{4}\left(\tfrac{1}{2}-c\right)\left(\tfrac{7}{2}-c\right)\right)}$$

$$\times\exp\left\{-\frac{1}{\hbar}\sum_{n=1}^{N}\left[\frac{a_{n-1}\left(a_n-a_{n-1}\right)^2}{2\varepsilon}-\varepsilon\left(V\left(a_{n-1}\right)+\frac{\hbar^2}{2}f(k,c)\,a_{n-1}{}^{-3}\right)\right]\right\}$$

Note resemblance to WKB solution

From WKB series approximation to solution for Wheeler-DeWitt equation, we obtained

$$\psi(a) = \exp\left\{-\frac{S(a)}{\hbar}\right\} \sum_{n=0}^{\infty} \hbar^n \varphi_n(a)$$

$$S(a) = \frac{1}{\Lambda}\left(1 - \frac{\Lambda}{3}a^2\right)^{\frac{3}{2}} - \frac{1}{\Lambda}, \qquad \varphi_0(a) = K a^{\frac{k-i}{2}} \left|1 - \frac{\Lambda}{3}a^2\right|^{-\frac{1}{4}}$$

Order-dependent prefactor mirrors that in path integral.

$$(a'a'')^{1/4+\frac{k-i}{2}} \int_{\mathcal{C}\{a',t''|a'',t'\}} \mathcal{D}\left[a^{\frac{1}{2}}(t)a(t)\right] \exp\left\{-\frac{1}{\hbar}\int_{t'}^{t''} dt \left[\frac{1}{2}a\dot{a}^2 - \left(V(a) + \frac{\hbar^2}{2} f(k,c) a^{-3}\right)\right]\right\}$$

This confirms that the path integral will detect asymmetry in operator ordering.

Propagators back into correlators

Recall we disassembled the correlator into propagators:

$$\left\langle a'',t''\right| f\left(\hat{a}\left(t_2\right)\right) f\left(\hat{a}\left(t_1\right)\right)\left|a',t'\right\rangle_{ijk}$$

$$= \int_0^\infty a_2^{i-k} da_2\, f\left(a_2\right)\left\langle a'',t''\middle|a_2,t_2\right\rangle_{ijk} \left[\int_0^\infty a_1^{i-k} da_1\, f\left(a_1\right)\left\langle a_2,t_2\middle|a_1,t_1\right\rangle_{ijk} \left\langle a_1,t_1\middle|a',t'\right\rangle_{ijk}\right]$$

But

$$\left\langle a'',t''\middle|a_2,t_2\right\rangle_{ijk} = \left(a''a_2\right)^{\frac{1}{4}+\frac{k-i}{2}} \left\langle a'',t''\middle|a_2,t_2\right\rangle_{LB,QP}$$

yielding

$$\boxed{\begin{aligned}&\left\langle a'',t''\right| f\left(\hat{a}\left(t_2\right)\right) f\left(\hat{a}\left(t_1\right)\right)\left|a',t'\right\rangle_{ijk}\\ &\qquad = \left(a''a_2\right)^{\frac{1}{4}+\frac{k-i}{2}} \left\langle a'',t''\right| f\left(\hat{a}\left(t_2\right)\right) f\left(\hat{a}\left(t_1\right)\right)\left|a',t'\right\rangle_{LB,QP}\end{aligned}}$$

Future

- Calculate correlators to quadratic approximation
- Bridge the gap to CDT, where volume is the more realistic degree of freedom than scale parameter
- Directly obtain path integral measure for Steigl/Hinterleiter orderings a la DeWitt's method of relating Laplace-Beltrami to Weyl ordering
- Other families of operator orderings? Other minisuperspace models (e.g. Bianchi IX)?

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THANK YOU FOR YOUR ATTENTION.