Infinite-N limit of the eigenvalue density of Wilson loops in 2D SU(N) YM

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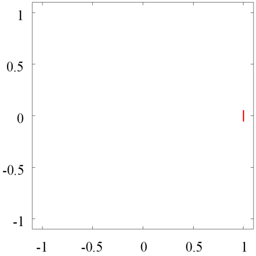
Motivation

- In two Euclidean dimensions: Dilated from a small size to a large one, Wilson loops in $SU(N)$ gauge theory exhibit an infinite-N phase transition (discovered by Durhuus and Olesen in 1981)

- Eigenvalue distribution of the untraced Wilson loop unitary matrix expands from a small arc to the entire unit circle
- Transition has universal properties (shared across dimensions and analog two-dimensional models)



- For finite N: integral representation for the resolvent
- In this talk: show that the known infinite-N result for the eigenvalue density can be obtained by a saddle point analysis

Outline

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Expectation values

- Probability density for Wilson loop matrix W is given by

$$\mathscr{P}_N(W,t) = \sum_r d_r \chi_r(W) e^{-\frac{t}{2N}C_2(r)}$$

with $t = \lambda\mathscr{A}$, standard 't Hooft coupling $\lambda = g^2 N$, Wilson loop encloses area $\mathscr{A}$
d_r, $C_2(r)$, $\chi_r(W)$: dimension, quadratic Casimir, character for irreducible representation r of $SU(N)$

- Averages over W at fixed t are given by

$$\langle \mathscr{O}(W) \rangle = \int dW \mathscr{P}_N(W,t)\mathscr{O}(W)$$

with Haar measure dW

Different densities

- True resolvent and eigenvalue density

$$G_N^{\text{true}}(z,t) = \frac{1}{N}\left\langle \text{Tr}\,\frac{1}{z-W}\right\rangle = \frac{1}{N}\frac{\partial}{\partial z}\langle \log\det(z-W)\rangle$$

$$\rho_N^{\text{true}}(\theta) = \text{Re}\left[2zG^{\text{true}}(z)-1\right] \qquad z=e^{i\theta+\epsilon},\ \epsilon\to 0^+$$

- Define in analogy

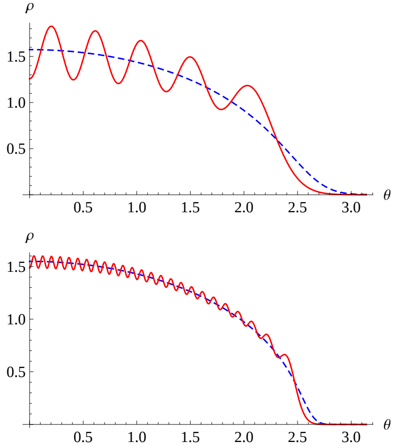
$$G_N^{\text{sym}}(z,T) = -\frac{1}{N}\frac{\partial}{\partial z}\log\langle\det\left(\frac{1}{z-W}\right)\rangle$$

$$\rho_N^{\text{sym}}(\theta) = \text{Re}\left[2zG_N^{\text{sym}}(z)-1\right]$$

- Densities have the same infinite-N limit (due to infinite-N factorization)

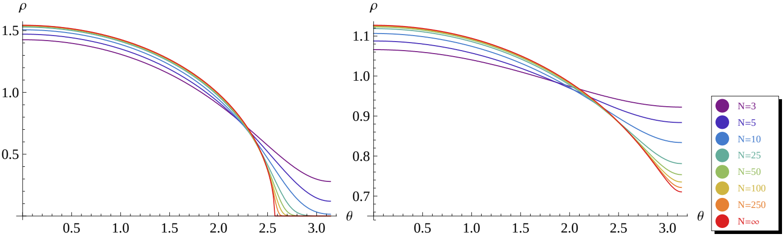
$$\lim_{N\to\infty}\rho_N^{\text{true}}(\theta) = \lim_{N\to\infty}\rho_N^{\text{sym}}(\theta) = \rho_\infty(\theta)$$

Plots



Plots of the densities $\rho_N^{\text{true}}(\theta, t)$ (red) and $\rho_N^{\text{sym}}(\theta, T)$ (blue) for $t = 2$ (left) and $t = 5$ (right), $N = 10$ (top), and $N = 50$ (bottom).

Plots



$\rho_N^{\text{sym}}(\theta, T)$ for $T = 2$ (left), $T = 5$ (right), and $N = 3, 5, 10, 25, 50, 100, 250$ together with $\rho_\infty(\theta, T)$.

Outline

Character expansion

- ρ^{true} can be obtained from the expectation value of

$$R(u,v,W) = \frac{\det(1+uW)}{\det(1-vW)} = \sum_{p=0}^{N}\sum_{q=0}^{\infty} u^p v^q \chi_p^A(W)\chi_q^S(W)$$

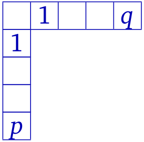
when we set $u = -v + \epsilon$ and expand to linear order in ϵ

$$R(-v+\epsilon, v, W) = 1 - \epsilon \,\mathrm{Tr}\,\frac{1}{v - W^\dagger}$$

- After decomposing the tensor product $p^A \otimes q^S$ into irreducible representations, we can compute the expectation value (character orthogonality)

$$\bar{R}(v) \equiv \left\langle \mathrm{Tr}\,\frac{1}{v - W^\dagger} \right\rangle = -\sum_{p=0}^{N-1}\sum_{q=0}^{\infty}(-1)^p v^{p+q} e^{-\frac{t}{2N}C(p,q)} d(p,q)$$

$C(p,q)$, $d(p,q)$: value of the quadratic Casimir operator and dimension of irreducible representation identified by Young diagram



$$C(p,q) = (p+q+1)\left(N - \frac{p+q+1}{N} + q - p\right)$$

$$d(p,q) = d^A(p)d^S(q)\frac{(N-p)(N+q)}{N}\frac{1}{p+q+1}$$

$$d^A(p) = \binom{N}{p}, \qquad d^S(q) = \binom{N+q-1}{q}$$

We can perform sums of the form

$$\sum_{p=0}^{N-1} z^p d^A(p)(N-p) = N(1+z)^{N-1}, \qquad \sum_{q=0}^{\infty} z^q d^S(q)(N+q) = \frac{N}{(1-z)^{N+1}}$$

Integral representation

- Write

$$\frac{1}{p+q+1} = \int_0^1 d\rho\, \rho^{p+q+1}$$

- Introduce Gaussian integrals to decouple the terms which are nonlinear in p and q in $\exp(-\frac{t}{2N}C(p,q))$
- Performing the (independent) sums over p, q then leads to

$$\bar{R}(v) = -\frac{N^2}{t} e^{-\frac{t}{2}} \int\int_{-\infty}^{\infty} \frac{dx\,dy}{2\pi} \int_0^1 d\rho\, e^{-\frac{N}{2t}(x^2+y^2)+\frac{1}{2t}(x+iy)^2-\frac{1}{2}(x-iy)} \times \frac{\left[1-v\rho e^{-x-t/2}\right]^{N-1}}{\left[1-v\rho e^{iy-t/2}\right]^{N+1}}$$

Outline

Integral representation for ρ^{sym}

- Similarly, $\psi(z) = \langle \det(z - W)^{(-1)} \rangle$ (which determines ρ^{sym}) has an integral representation (valid for $|z| > 1$)

$$\psi(z) = e^{\frac{NT}{8}} \sqrt{\frac{N}{2\pi T}} \int_{-\infty}^{\infty} dw \, e^{-\frac{N}{2T} w^2} \left(z e^{-i\frac{w}{2}} - e^{i\frac{w}{2}} \right)^{-N}$$

- We set $z = e^{i\theta + \epsilon}$ and take the limit $\epsilon \to 0^+$ at the end
- Integrand is $\exp(-N f(w))$ with

$$f(w) = \frac{w^2}{2T} + \log\left(z e^{-i\frac{w}{2}} - e^{i\frac{w}{2}} \right) .$$

- Singularities of the integrand are located on the line $\operatorname{Im} w = -\epsilon < 0$
- Integration path (along $\operatorname{Im} w_i = 0$) can be shifted upwards in the complex plane

Saddle points

- Saddle point equation $f'(w_0) = 0$ can be written as

$$e^{-TU(\theta,T)} \frac{U(\theta,T)+1/2}{U(\theta,T)-1/2} = e^{\epsilon+i\theta} \, .$$

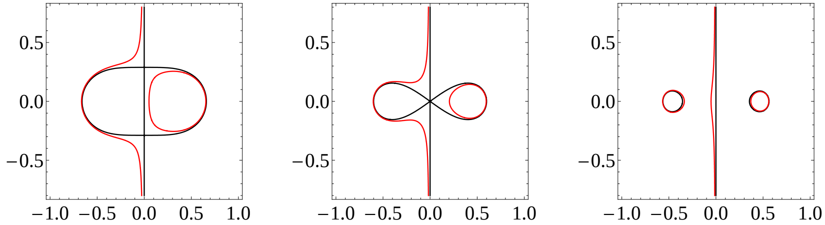
  with $w_0 = iTU(\theta,T) = iT(U_r(\theta,T) + iU_i(\theta,T))$

- Taking the absolute value leads to

$$U_i^2 = U_r \coth(TU_r + \epsilon) - U_r^2 - \frac{1}{4}$$

- This equation describes one or more curves in the complex U plane on which the saddle points have to lie
- For given value of θ, saddles are isolated points on these curves

Curves of solutions

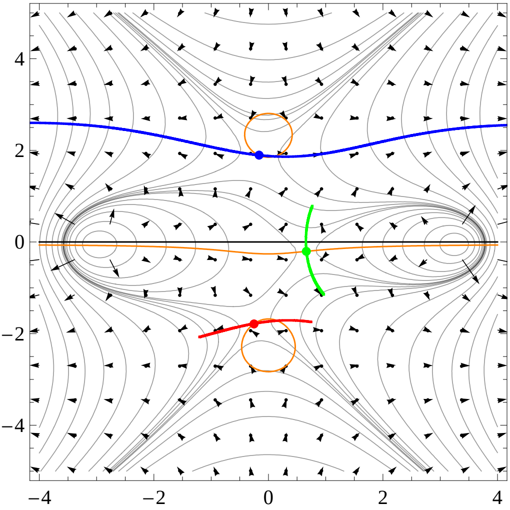


Curves in the complex-U plane for $T = 3$ (left), $T = 4$ (middle), and $T = 5$ (right). Red curves: small $\epsilon > 0$; black curves: $\epsilon = 0$

- For given θ: always one (and only one) saddle point on the closed curve encircling $U = \frac{1}{2}$
- Integration contour (along imaginary U axis) can be smoothly deformed to go through this saddle along a path of steepest descent (no singularities are crossed)

Deformation of the integration contour

Saddle points and corresponding paths of steepest descent in complex w-plane (arrows: direction of increasing $\operatorname{Re} f(w)$)



Saddle point result

- Limit $\epsilon \to 0$ can be taken once the integration contour has been deformed to go through the saddle point
- Parametrize the contour in the vicinity of the saddle point by $w = w_0 + xe^{i\beta}$ (β: angle which the path of steepest descent makes with the real w axis)
- Expanding the exponent to quadratic order in x and integrating over x leads to

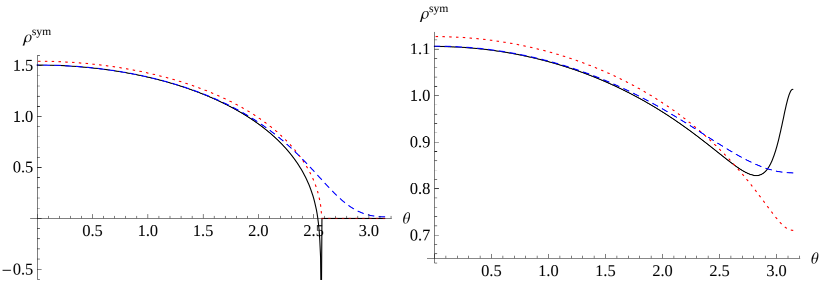
$$\psi(z) = e^{\frac{NT}{8}} \sqrt{\frac{N}{2\pi T}} e^{-Nf(w_0)} \sqrt{\frac{2\pi}{Nf''(w_0)}} + \mathcal{O}(1/N)$$

- The density $\rho^{\text{sym}} = -2\,\text{Re}\left(1/2 + 1/Nz\partial_z \ln \psi\right)$ is given by

$$\rho_N^{\text{sym}}(\theta, T) = 2\,\text{Re}\left[U\left(1 + \frac{1}{N}\frac{T(1/4 - U^2)}{[1 - T(1/4 - U^2)]^2}\right)\right] + \mathcal{O}(1/N^2)$$

- Infinite-N result is $\rho^{\text{sym}} = 2\,\text{Re}[U(\theta, T)]$
- Next order term diverges if denominator $1 - T(1/4 - U^2) = 0$ (this corresponds to $f''(w_0) = 0$)
- This happens for $T \leq 4$ at the transition point θ_c (from zero to non-zero ρ_∞)
- For $T \leq 4$ and $|\theta| > \theta_c$, leading order and $1/N$ term are both zero
- In this interval ρ^{sym} approaches zero by corrections that are exponentially suppressed in N
- This saddle point analysis is not the right tool to compute finite N effects in this region

Examples for the $1/N$ corrections to $\rho_\infty(\theta, T)$ for $N = 10$, $T = 2$ (left), and $T = 5$ (right)



blue: exact result for $\rho_N^{\text{sym}}(\theta, T)$
red: infinite-N result (blue dashed curve)
black: asymptotic expansion of $\rho_N^{\text{sym}}(\theta, T)$ up to order $\mathcal{O}(1/N)$

Outline

True eigenvalue density ρ^{true}

- Infinite-N limit of ρ^{true} is obtained by saddle point approximation of

$$\bar{R}(v) = -\frac{N^2}{t} e^{-\frac{t}{2}} \int\!\!\int_{-\infty}^{\infty} \frac{dx dy}{2\pi} \int_0^1 d\rho\, e^{-\frac{N}{2t}\left(x^2+y^2\right)+\frac{1}{2t}(x+iy)^2-\frac{1}{2}(x-iy)} \\ \times e^{(N-1)\log\left(1-v\rho e^{-x-t/2}\right)-(N+1)\log\left(1-v\rho e^{iy-t/2}\right)}.$$

- valid for $|v| < 1$; $v = e^{i\theta-\epsilon}$, $\epsilon \to 0^+$
- Approximate integrals over x and y, integrate over ρ at the end
- Integrals decouple at leading order and can be approximated independently
- coefficients of $-N$ in the exponent are

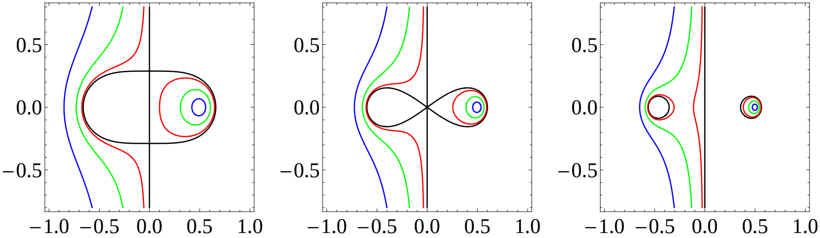
$$\bar{f}(y) = \frac{1}{2t}y^2 + \log\left[1 - v\rho e^{iy-\frac{t}{2}}\right]$$
$$\tilde{f}(x) = \frac{1}{2t}x^2 - \log\left[1 - v\rho e^{-x-t/2}\right] = -\bar{f}(ix)$$

Relation to integral for ρ^{sym}

- Substituting $y = w - it/2$ leads to the integral for ψ (with $z \to 1/(\nu\rho)$), integration over w is along line from $-\infty + it/2$ to $+\infty + it/2$
- No singularities between this line and the real w axis
- Saddle point equation reads ($y_s = it(U - 1/2)$)

$$e^{-tU} \frac{U + 1/2}{U - 1/2} = \frac{1}{\nu\rho},$$

- Difference to previous analysis: $0 \leq |\nu\rho| < 1$
- Relevant saddle point is located on a closed curve around $U = 1/2$ (corresponds to $y = 0$)
- Curve shrinks for decreasing ρ
- Integral can be approximated by one single saddle point, $y_0(\theta, t\rho)$



Curves in the complex-U plane for $t = 3$, $t = 4$, and $t = 5$ (right)
black: $\rho = 1$, red: $\rho = 0.9$, green: $\rho = 0.6$, blue: $\rho = 0.3$

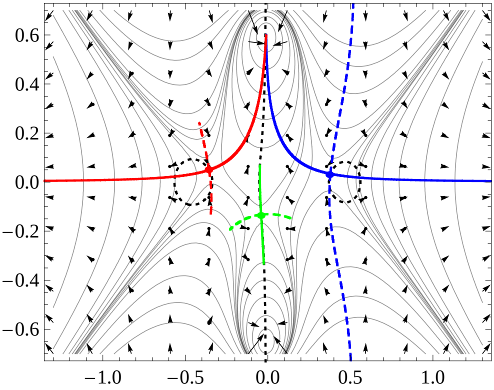
Integral over x

- Due to $\tilde{f}(x) = -\bar{f}(ix)$, saddle points of x and y integrals are related by rotation of $\pi/2$ in the complex plane, $x_s = -iy_s$
- Relation to U is $x_s = t(U - 1/2)$ (integration is now along the real U axis)
- Directions of steepest descent through y_s and x_s are identical (no rotation)

$$\tilde{f}''(x_s) = \frac{1}{t} + \frac{x_s}{t}\left(1 + \frac{x_s}{t}\right) = \bar{f}''(y_s = ix_s)$$

- Integration contour can always be deformed to go through the (single) saddle-point in the right half-plane (on the curve around $U = 1/2$)
- Depending on ρ, v, and t: either one or no additional saddle point on the contour(s) in the left half-plane through which we can also go in the direction of steepest descent
- But: contribution of additional saddle point (if there is one) is exponential suppressed
- Relevant saddle point is $x_0 = -iy_0$

Contours of steepest descent



Example for $t = 5$, $\rho = 0.95$, $\theta = 3.0$
dashed black: curves on which all saddle points (for $t = 5$ and $\rho = 0.95$) have to lie; dashed blue: integration path for y integral; solid red-blue: integration path for x integral;

ρ integral

- Combining saddle point approximations for x and y integrals gives ($x_0 = x_0(\theta, t, \rho)$)

$$\bar{R}(v) = -\frac{N}{t} e^{-\frac{t}{2}} \int_0^1 d\rho \, \frac{(t + x_0)^2}{t + x_0\,(t + x_0)} \, e^{-x_0}$$

- Differentiating the saddle point equation with respect to ρ leads to

$$\frac{\partial x_0}{\partial \rho} = v e^{-x_0 - t/2} \frac{(t + x_0)^2}{t + x_0\,(t + x_0)}$$

and

$$\bar{R}(v) = -\frac{N}{tv} \int_0^1 d\rho \, \frac{\partial x_0}{\partial \rho} = -\frac{N}{tv} \left[x_0(\theta, t, \rho = 1) - x_0(\theta, t, \rho = 0) \right]$$

- For the eigenvalue density

$$\lim_{N\to\infty} \rho^{\text{true}}(\theta, t) = 2\,\mathrm{Re}\, U(\theta, t, \rho = 1) = \lim_{N\to\infty} \rho^{\text{sym}}(\theta, t)$$

Outline

Summary

- For $SU(N)$ YM in 2 Euclidean dimensions: Probability distribution of Wilson loop given by sum over all irreducible representations (only Casimir, dimension, character enter)
- Definition of different density functions which have the same infinite-N limit
- For fintite N: exact integral representations
- Infinite N results can be obtained by saddle point approximations in leading order
- Next order terms give reasonable results in the interval where $\rho_\infty > 0$ (power corrections in $1/N$)
- More work is needed to get finite N effects in the interval where $\rho_\infty = 0$ (corrections are exponentially suppressed in N)