Exact solutions in $D = 2$, supersymmetric Yang-Mills quantum mechanics with $SU(3)$ gauge group and higher

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Outline of the talk

- Motivations
- $D = 2$, supersymmetric Yang-Mills quantum mechanics
- Numerical algorithm and numerical results
- Exact solutions
- Further perspectives

Motivations

Supersymmetric Yang-Mills Quantum Mechanics

- dimensional reduction of $\mathcal{N} = 1$, D dimensional Yang-Mills quantum field theory to one point in space (Halpern, Claudson)
- generalization of supersymmetric quantum mechanics (Witten, Cooper)
- Hamiltonian formulation, fermions and bosons are treated on equal footing
- the physical Hilbert space is composed of singlet states, as well as, all relevant operators are invariant

Numerical method

- gauge invariant cut-off (Wosiek)
- fermions can be introduced without difficulties
- rotational symmetry is preserved

Earlier analytic developments

- Claudson-Halpern solutions for $SU(2)$
- Samuel solutions for $SU(N)$
- Trzetrzelewski solutions for $SU(N)$

Model

System is described by a bosonic variable ϕ_A and a complex fermion λ_A, where A labels the generators of the gauge group.

$$G_A = f_{ABC}\left(\phi_B \pi_C - i\bar{\lambda}_B \lambda_C\right),$$

$$Q = \lambda_A \pi_A, \qquad \bar{Q} = \bar{\lambda}_A \pi_A,$$

$$\{Q, \bar{Q}\} = \pi_A \pi_A = 2H - 2g\phi_A G_A.$$

Thus, on physical states,

$$H = \frac{1}{2}\pi_A \pi_A.$$

Method - construction of the Fock basis

We construct the Fock basis recursively,

- define the set of elementary, gauge invariant creation operators

$SU(2)$	$SU(3)$	$SU(4)$	$\dots$
$(a^\dagger a^\dagger)$	$(a^\dagger a^\dagger)$ $(a^\dagger a^\dagger a^\dagger)$	$(a^\dagger a^\dagger)$ $(a^\dagger a^\dagger a^\dagger)$ $(a^\dagger a^\dagger a^\dagger a^\dagger)$	

Table: Elementary bosonic bricks for $SU(2)$,$SU(3)$ and $SU(4)$.

Thus, a general state with n_B quanta for some given N, can be written as

$$|s_{n_B,0}\rangle_N = \sum_{\left\{\sum_{j=2}^{N} jk_j = n_B\right\}} \gamma_{k_2,\dots,k_N} (a^{\dagger 2})^{k_2}(a^{\dagger 3})^{k_3}\dots(a^{\dagger N})^{k_N}|0\rangle.$$

Method - construction of the Fock basis

$F=1$	$F=2$	$F=3$
$(f^\dagger a^\dagger)$	$(f^\dagger f^\dagger a^\dagger)$	$(f^\dagger f^\dagger f^\dagger)$

Table: $SU(2)$ fermionic bricks.

$F=1$	$F=2$	$F=3$	$F=4$
$(f^\dagger a^\dagger)$	$(f^\dagger f^\dagger a^\dagger)$	$(f^\dagger f^\dagger f^\dagger)$	$(f^\dagger f^\dagger f^\dagger f^\dagger a^\dagger)$
$(f^\dagger a^\dagger a^\dagger)$	$(f^\dagger f^\dagger a^\dagger a^\dagger)$	$(f^\dagger f^\dagger f^\dagger a^\dagger)$	$(f^\dagger a^\dagger)(f^\dagger f^\dagger f^\dagger)$
	$(f^\dagger a^\dagger a^\dagger f^\dagger a^\dagger)$	$(f^\dagger f^\dagger f^\dagger a^\dagger a^\dagger)$	$(f^\dagger f^\dagger f^\dagger f^\dagger a^\dagger a^\dagger)$
	$(f^\dagger a^\dagger)(f^\dagger a^\dagger a^\dagger)$	$(f^\dagger a^\dagger)(f^\dagger f^\dagger a^\dagger)$	$(f^\dagger a^\dagger a^\dagger)(f^\dagger f^\dagger f^\dagger)$
		$(f^\dagger a^\dagger f^\dagger f^\dagger a^\dagger a^\dagger)$	$(f^\dagger a^\dagger)(a^\dagger f^\dagger f^\dagger f^\dagger)$
		$(f^\dagger a^\dagger)(f^\dagger f^\dagger a^\dagger a^\dagger)$	$(f^\dagger f^\dagger a^\dagger)(f^\dagger f^\dagger a^\dagger)$
		$(f^\dagger a^\dagger a^\dagger)(f^\dagger f^\dagger a^\dagger)$	$(f^\dagger a^\dagger a^\dagger)(f^\dagger f^\dagger f^\dagger a^\dagger)$
		$(f^\dagger a^\dagger a^\dagger)(f^\dagger f^\dagger a^\dagger a^\dagger)$	$(f^\dagger f^\dagger a^\dagger)(f^\dagger f^\dagger a^\dagger a^\dagger)$
			$(f^\dagger a^\dagger)(f^\dagger a^\dagger a^\dagger)(f^\dagger f^\dagger a^\dagger)$
			$(f^\dagger f^\dagger a^\dagger)(f^\dagger a^\dagger f^\dagger a^\dagger a^\dagger)$

Table: $SU(3)$ fermionic bricks.

Method - cut-off

We must introduce a cut-off: N_{cut}

- limit the maximal number of bosonic quanta
- finite number of fermionic quanta (Pauli exclusion principle)
- gauge symmetry as well as rotational symmetry are preserved

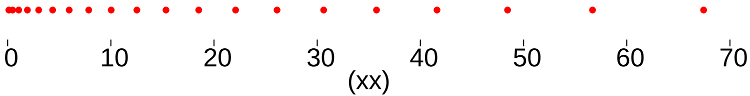

Figure: Eigenvalues of the (x^2) operator for $SU(2)$.

Method - recursive relations

We relate an expectation value of an operator O to its expectation values in sectors with lower number of quanta.

$$\begin{aligned}\langle s_{n'_B,0}|O(n_B^O,0)|s_{n_B,0}\rangle &= \Big(\langle s_{n'_B,0}|\big[O(n_B^O,0),C(p,0,\alpha)\big]|s_{n_B-p,0}\rangle \\ &+ \langle s_{n'_B,0}|C(p,0,\alpha)O(n_B^O,0)|s_{n_B-p,0}\rangle\Big)\cdot R(n_B,0)\end{aligned}$$

Numerical results

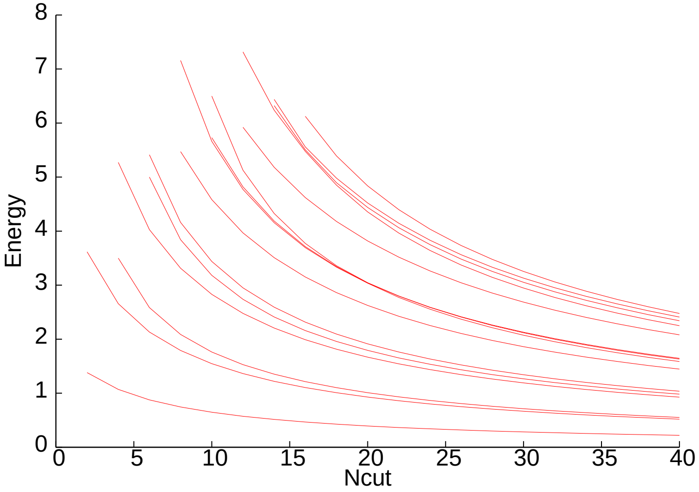

Figure: Dependence of the eigenenergies on the cut-off for the $SU(3)$ model in the $F = 0$ sector.

Numerical results

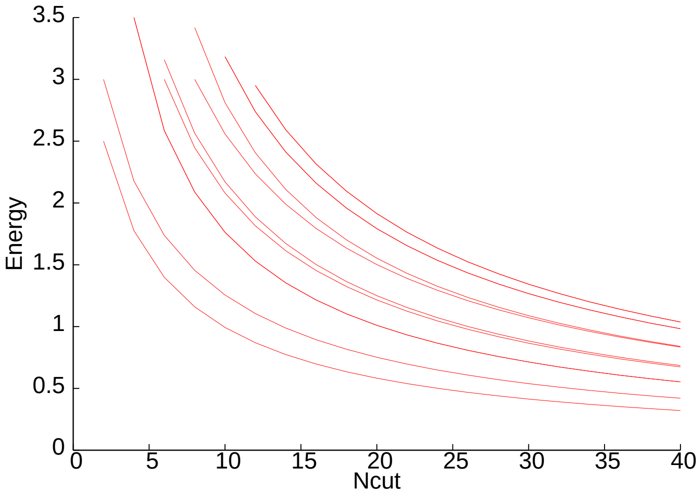

Figure: Dependence of the eigenenergies on the cut-off for the $SU(3)$ model in the $F = 2$ sector.

Exact solutions - $SU(2)$

The Hamiltonian reads

$$H = (a^\dagger a) + \frac{3}{2} - \frac{1}{2}\Big((a^\dagger a^\dagger) + (aa)\Big).$$

The eigenequation is

$$H|E\rangle = E|E\rangle.$$

We expand $|E\rangle$ in the Fock basis

$$|E\rangle = \sum_{j=0}^{\infty} a_j(E)(a^\dagger a^\dagger)^j|0\rangle.$$

a_j must obey the recursion relation

$$a_j(E) - \big(2j + \frac{7}{2} - 4E\big)a_{j+1}(E) + (j+2)\big(j + \frac{5}{2}\big)a_{j+2}(E) \quad = \quad 0,$$

which is solved by

$$a_j(E) \quad = \quad \sqrt{\frac{j!}{\Gamma(j+\frac{3}{2})}} L_j^{\frac{1}{2}}(E).$$

Exact solutions - $SU(2)$

$$a_j(E) - \left(2j + \frac{7}{2} - 4E\right) a_{j+1}(E) + (j+2)\left(j + \frac{5}{2}\right) a_{j+2}(E) = 0$$

$$|E> = \sum_{j=0}^{\infty} \sqrt{\frac{j!}{\Gamma(j+\frac{3}{2})}} L_j^{\frac{1}{2}}(E) (a^\dagger a^\dagger)^j |0>$$

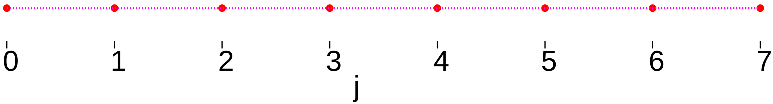

Figure: Schematic structure of solutions for the $SU(2)$ model.

Exact solutions - $SU(2)$

The bosonic eigensolution in the position representation can be written as

$$\begin{aligned}\langle R|E\rangle &= \sum_{j=0}^{\infty}\langle R|j\rangle\langle j|E\rangle = \sum_{j=0}^{\infty} a_j(E)\langle R|(a^\dagger a^\dagger)^j|0\rangle \\ &= \mathcal{N} f(E) e^{-\frac{R}{2}} \sum_{j=0}^{\infty} \frac{(-1)^j j!}{\Gamma(j+\frac{3}{2})} L_j^{\frac{1}{2}}(E) L_j^{\frac{1}{2}}(R).\end{aligned}$$

Setting $f(E) = e^{-\frac{E}{2}}$, as well as, $R = r^2$ and $E = k^2$, we get,

$$\langle R|E\rangle = \mathcal{N}\frac{\sin(kr)}{kr},$$

which is, up to a multiplicative factor, the Claudson-Halpern solution of the $SU(2)$ model.

Exact solutions - $SU(3)$

The Hamiltonian reads

$$H = (a^\dagger a) + 4 - \frac{1}{2}\Big((a^\dagger a^\dagger) + (aa)\Big).$$

The eigenequation is

$$H|E\rangle = E|E\rangle.$$

We expand $|E\rangle$ in the Fock basis

$$|E\rangle = \sum_{j,k=0}^{\infty} a_{j,k}(E)(a^\dagger a^\dagger)^j (a^\dagger a^\dagger a^\dagger)^k |0\rangle.$$

Degeneracy of basis states: $n_B = 2j + 3k$.

$a_{j,k}(E)$ must obey the recursion relation

$$\begin{aligned} a_{j-1,k}(E) - \big(2j + 3k + 4 - 4E\big)a_{j,k}(E) \quad &+ \quad (j+1)(j+3k+4)a_{j+1,k}(E) \\ &+ \quad 3(k+2)^2 a_{j-2,k+2}(E) = 0. \end{aligned}$$

Exact solutions - $SU(3)$

$$a_{j-1,k}(E) - \big(2j + 3k + 4 - 4E\big) a_{j,k}(E) + (j+1)(j+3k+4) a_{j+1,k}(E) + 3(k+2)^2 a_{j-2,k+2}(E) = 0.$$

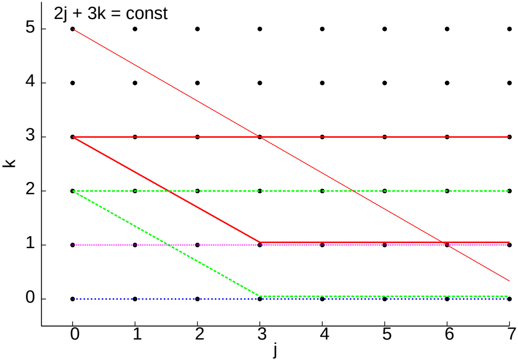

Figure: Schematic structure of solutions of the $SU(3)$ model.

Exact solutions - $SU(3)$

The general solutions in the infinite cut-off limit read,

$$
\begin{aligned}
|E\rangle_{2k} &= \frac{1}{\mathcal{N}}\sum_{j=0}^{\infty} L_j^{3(2k+1)}(E)\Big(|j,2k\rangle + \sum_{q=1}^{k}\alpha_q|j+3q,2k-2q\rangle\Big)\\
\alpha_q &= (-\frac{4}{3})^{q+1}\frac{1}{(q+1)!}\frac{(2k-q-1)!}{(2k)!}\Big(\frac{k!}{(k-q-1)!}\Big)^2.
\end{aligned}
$$

For example,

$$
|E\rangle_2 = \frac{1}{\mathcal{N}}\sum_{j=0}^{\infty} L_j^9(E)\big(|j,2\rangle - \frac{2}{3}|j+3,0\rangle\big).
$$

In position representation we have,

$$
\langle R|E\rangle_0 = \sum_j \langle R|j\rangle\langle j|E\rangle \sim {}_0F_1\big(4,-\frac{ER}{4}\big).
$$

Exact solutions vs numerical results

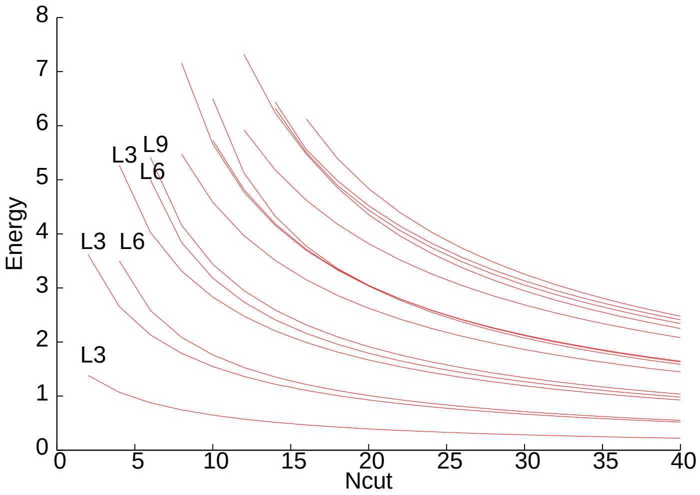

Figure: Spectrum of the $SU(3)$ model in the $F = 0$ sector.

Conclusions

Summary:

- numerical algorithm permits calculations of the spectra for any N and in any fermionic sector
- analytic solutions with possible generalizations to more complicated models

Possible further directions:

- generalize to higher spatial dimensions - the free spectrum and eigenstates of the $D = 4$ and $D = 10$ SYMQM
- large N limit possible
- perturbative expansion of the interacting model around the free solutions

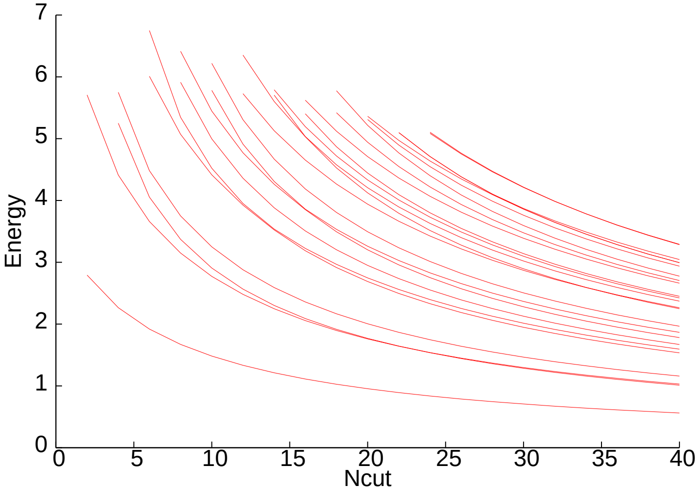

Figure: Spectrum of the $SU(4)$ model in the $F = 0$ sector.

Backup slides

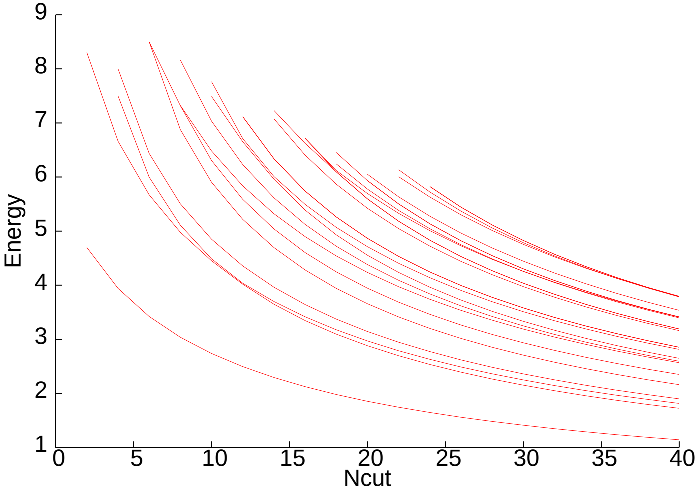

Figure: Spectrum of the $SU(5)$ model in the $F = 0$ sector.

Backup slides

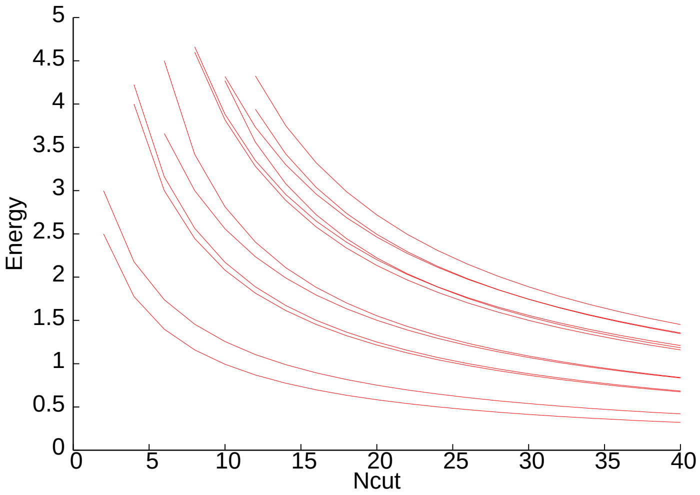

Figure: Spectrum of the $SU(3)$ model in the $F = 1$ sector.