Coupling procedure for the Poincaré Gauge Theory of Gravity

Marcin Kaźmierczak

Chair of Theory of Relativity and Gravitation
Institute of Theoretical Physics
Faculty of Physics, University of Warsaw
Poland

Zakopane, June 7, 2009

Outline of the Talk

Standard Yang–Mills theories

Standard Yang–Mills theories

- $S_m[\phi] = \int \mathcal{L}_m \left(\phi, \partial_\mu \phi\right) d^4x = \int \mathfrak{L}_m \left(\phi, d\phi\right)$

Standard Yang–Mills theories

- $S_m[\phi] = \int \mathcal{L}_m(\phi, \partial_\mu \phi)\, d^4x = \int \mathfrak{L}_m(\phi, d\phi)$
- $\phi : M \to \mathcal{V}$,
  M – Minkowski space, $\mathcal{V}$ – a linear space

Standard Yang–Mills theories

- $S_m[\phi] = \int \mathcal{L}_m(\phi, \partial_\mu \phi)\, d^4x = \int \mathfrak{L}_m(\phi, d\phi)$
- $\phi : M \to \mathcal{V}$,
  M – Minkowski space, $\mathcal{V}$ – a linear space
- G – a Lie group, $\pi : Lie(G) \to End(\mathcal{V})$,

Standard Yang–Mills theories

- $S_m[\phi] = \int \mathcal{L}_m\left(\phi, \partial_\mu \phi\right) d^4x = \int \mathfrak{L}_m\left(\phi, d\phi\right)$
- $\phi : M \to \mathcal{V}$,
  M – Minkowski space, $\mathcal{V}$ – a linear space
- G – a Lie group, $\pi : Lie(G) \to End(\mathcal{V})$,
- $\rho\left(\exp(\mathfrak{g})\right) = \exp\left(\pi(\mathfrak{g})\right)$

Standard Yang–Mills theories

- $S_m[\phi] = \int \mathcal{L}_m (\phi, \partial_\mu \phi)\, d^4x = \int \mathfrak{L}_m (\phi, d\phi)$
- $\phi : M \to \mathcal{V}$,
  M – Minkowski space, $\mathcal{V}$ – a linear space
- G – a Lie group, $\pi : Lie(G) \to End(\mathcal{V})$,
- $\rho\,(\exp(\mathfrak{g})) = \exp\,(\pi(\mathfrak{g}))$
- $\mathfrak{L}_m\,(\rho(g)\phi, d\,(\rho(g)\phi)) = \mathfrak{L}_m\,(\phi, d\phi)\,, \quad \forall g \in G$

An interaction associated to G

Allow the group element to depend on space–time point and demand the Lagrangian four–form to be invariant under local action of G. In order to achieve this, replace the differentials by covariant differentials:

- $d\phi \to D\phi = d\phi + \mathbb{A}\phi$,

An interaction associated to G

Allow the group element to depend on space–time point and demand the Lagrangian four–form to be invariant under local action of G. In order to achieve this, replace the differentials by covariant differentials:

- $d\phi \to D\phi = d\phi + \mathbb{A}\phi$,
- $\mathbb{A}$ – a $\pi\,(Lie(G))$–valued one–form field on M,

An interaction associated to G

Allow the group element to depend on space–time point and demand the Lagrangian four–form to be invariant under local action of G. In order to achieve this, replace the differentials by covariant differentials:

- $d\phi \to D\phi = d\phi + \mathbb{A}\phi$,
- $\mathbb{A}$ – a $\pi\,(Lie(G))$–valued one–form field on M,
- $\mathbb{A} \to \mathbb{A}' = \rho(g)\mathbb{A}\rho^{-1}(g) - d\rho(g)\rho^{-1}(g)$.

An interaction associated to G

Allow the group element to depend on space–time point and demand the Lagrangian four–form to be invariant under local action of G. In order to achieve this, replace the differentials by covariant differentials:

- $d\phi \rightarrow D\phi = d\phi + \mathbb{A}\phi$,
- $\mathbb{A}$ – a $\pi\left(Lie(G)\right)$–valued one–form field on M,
- $\mathbb{A} \rightarrow \mathbb{A}' = \rho(g)\mathbb{A}\rho^{-1}(g) - d\rho(g)\rho^{-1}(g)$.
- $D'\phi' = \rho(g)D\phi \ \Rightarrow \ \tilde{\mathfrak{L}}_m\left(\phi, d\phi, \mathbb{A}\right) := \mathfrak{L}_m\left(\phi, D\phi\right)$ invariant under local transformations.

An interaction associated to G

Allow the group element to depend on space–time point and demand the Lagrangian four–form to be invariant under local action of G. In order to achieve this, replace the differentials by covariant differentials:

- $d\phi \to D\phi = d\phi + \mathbb{A}\phi$,
- $\mathbb{A}$ – a $\pi\,(Lie(G))$–valued one–form field on M,
- $\mathbb{A} \to \mathbb{A}' = \rho(g)\mathbb{A}\rho^{-1}(g) - d\rho(g)\rho^{-1}(g)$.
- $D'\phi' = \rho(g)D\phi \;\Rightarrow\; \tilde{\mathfrak{L}}_m\,(\phi, d\phi, \mathbb{A}) := \mathfrak{L}_m\,(\phi, D\phi)$ invariant under local transformations.

  MCP– the Minimal Coupling Procedure

Physical meaning of $\mathbb{A}$

- components of $\mathbb{A}$ – Yang–Mills fields,

Physical meaning of $\mathbb{A}$

- components of $\mathbb{A}$ – Yang–Mills fields,
- $\mathbb{F} := d\mathbb{A} + \mathbb{A} \wedge \mathbb{A}$ – the field–strength two–form,

Physical meaning of $\mathbb{A}$

- components of $\mathbb{A}$ – Yang–Mills fields,
- $\mathbb{F} := d\mathbb{A} + \mathbb{A} \wedge \mathbb{A}$ – the field–strength two–form,
- $\mathbb{F}' = \rho(g)\mathbb{F}\rho^{-1}(g)$,

Physical meaning of $\mathbb{A}$

- components of $\mathbb{A}$ – Yang–Mills fields,
- $\mathbb{F} := d\mathbb{A} + \mathbb{A} \wedge \mathbb{A}$ – the field–strength two–form,
- $\mathbb{F}' = \rho(g)\mathbb{F}\rho^{-1}(g)$,
- $d\mathbb{F} + \mathbb{A} \wedge \mathbb{F} - \mathbb{F} \wedge \mathbb{A} = 0$.

The Lagrangian four–form

- $\mathfrak{L} = \tilde{\mathfrak{L}}_m + \mathfrak{L}_G,$

The Lagrangian four–form

- $\mathfrak{L} = \tilde{\mathfrak{L}}_m + \mathfrak{L}_G$,
- $\mathfrak{L}_G$ – gauge–field part, built of $\mathbb{A}$, invariant under gauge transformations.

The Lagrangian four–form

- $\mathfrak{L} = \tilde{\mathfrak{L}}_m + \mathfrak{L}_G$,
- $\mathfrak{L}_G$ – gauge–field part, built of $\mathbb{A}$, invariant under gauge transformations.
- $\mathfrak{L}_G \sim tr\left(\mathbb{F} \wedge \star\mathbb{F}\right)$

The Lagrangian four–form

- $\mathfrak{L} = \tilde{\mathfrak{L}}_m + \mathfrak{L}_G$,
- $\mathfrak{L}_G$ – gauge–field part, built of $\mathbb{A}$, invariant under gauge transformations.
- $\mathfrak{L}_G \sim tr\left(\mathbb{F} \wedge \star\mathbb{F}\right)$
- (will not work for gravity!)

The Poincaré group as a gauge group

The Poincaré group as a gauge group

- The Poincaré group $\mathcal{P}$ consists of all isometries of M,

The Poincaré group as a gauge group

- The Poincaré group $\mathcal{P}$ consists of all isometries of M,
- $(\Lambda, a)x = \Lambda x + a, \quad x, a \in M, \quad \Lambda \in O(1,3),$

The Poincaré group as a gauge group

- The Poincaré group $\mathcal{P}$ consists of all isometries of M,
- $(\Lambda, a)x = \Lambda x + a, \quad x, a \in M, \quad \Lambda \in O(1,3)$,
- $(\Lambda_1, a_1)(\Lambda_2, a_2) = (\Lambda_1\Lambda_2, \Lambda_1 a_2 + a_1)$.

The Poincaré group as a gauge group

- The Poincaré group $\mathcal{P}$ consists of all isometries of M,
- $(\Lambda, a)x = \Lambda x + a, \quad x, a \in M, \quad \Lambda \in O(1,3)$,
- $(\Lambda_1, a_1)(\Lambda_2, a_2) = (\Lambda_1\Lambda_2, \Lambda_1 a_2 + a_1)$.
- Consider a representation

$$\rho(\Lambda, a) := \rho(a)\rho(\Lambda),$$

$$\rho(a) := \exp\left(a_a P^a\right), \quad \rho\left(\Lambda(\varepsilon)\right) := \exp\left(\frac{1}{2}\varepsilon_{ab}J^{ab}\right)$$

where $P^a, J^{ab} \in \pi\left(Lie(\mathcal{P})\right)$.

- $\rho(\Lambda_1, a_1)\rho(\Lambda_2, a_2) = \rho(\Lambda_1\Lambda_2, \Lambda_1 a_2 + a_1)$

- $\rho(\Lambda_1, a_1)\rho(\Lambda_2, a_2) = \rho(\Lambda_1\Lambda_2, \Lambda_1 a_2 + a_1) \quad \Rightarrow$

- $\rho(\Lambda_1, a_1)\rho(\Lambda_2, a_2) = \rho(\Lambda_1\Lambda_2, \Lambda_1 a_2 + a_1) \quad \Rightarrow$

$$
\begin{aligned}
&\rho(\Lambda, a) P^a \rho^{-1}(\Lambda, a) = \Lambda_c{}^a P^c, \\
&\rho(\Lambda, a) J^{ab} \rho^{-1}(\Lambda, a) = \Lambda_c{}^a \Lambda_d{}^b \left(J^{cd} + a^c P^d - a^d P^c \right), \\
&[P^a, P^b] = 0, \\
&[P^a, J^{cd}] = \eta^{ac} P^d - \eta^{ad} P^c, \\
&[J^{ab}, J^{cd}] = \eta^{ad} J^{bc} + \eta^{bc} J^{ad} - \eta^{bd} J^{ac} - \eta^{ac} J^{bd}.
\end{aligned}
\tag{1}
$$

- $\mathbb{A} = \frac{1}{2}\omega_{ab}J^{ab} + \Gamma_a P^a$ $\pi\left(Lie(\mathcal{P})\right)$-valued one–form on M,

- $\mathbb{A} = \frac{1}{2}\omega_{ab}J^{ab} + \Gamma_a P^a \quad \pi\left(Lie(\mathcal{P})\right)$-valued one–form on M,
- $\omega_{ab} = -\omega_{ba}$

- $\mathbb{A} = \frac{1}{2}\omega_{ab}J^{ab} + \Gamma_a P^a \quad \pi\,(Lie(\mathcal{P}))$-valued one–form on M,
- $\omega_{ab} = -\omega_{ba}$
- $\omega = (\omega^a{}_b)\,, \quad \Gamma = (\Gamma^a)\,,$

- $\mathbb{A} = \frac{1}{2}\omega_{ab}J^{ab} + \Gamma_a P^a \quad \pi\left(Lie(\mathcal{P})\right)$-valued one–form on M,
- $\omega_{ab} = -\omega_{ba}$
- $\omega = \left(\omega^a{}_b\right), \quad \Gamma = \left(\Gamma^a\right),$
- $\mathbb{A}' = \rho(g)\mathbb{A}\rho^{-1}(g) - d\rho(g)\rho^{-1}(g)$ and (1) imply

- $\mathbb{A} = \frac{1}{2}\omega_{ab}J^{ab} + \Gamma_a P^a$ $\pi\,(Lie(\mathcal{P}))$-valued one–form on M,
- $\omega_{ab} = -\omega_{ba}$
- $\omega = (\omega^a{}_b)\,, \quad \Gamma = (\Gamma^a)\,,$
- $\mathbb{A}' = \rho(g)\mathbb{A}\rho^{-1}(g) - d\rho(g)\rho^{-1}(g)$ and (1) imply

$$\omega' = \Lambda\omega\Lambda^{-1} - d\Lambda\Lambda^{-1}, \quad \Gamma' = \Lambda\Gamma - \omega' a - da \quad . \tag{2}$$

General Relativity as a theory of gravitation

- The space–time is a manifold $\mathcal{M}$ with a Lorentzian metric g.

General Relativity as a theory of gravitation

- The space–time is a manifold $\mathcal{M}$ with a Lorentzian metric g.
- tetrad: $\tilde{e}_a = \tilde{e}_a^{\mu} \partial_{\mu}, \quad g\left(\tilde{e}_a, \tilde{e}_b\right) = \eta_{ab}$

General Relativity as a theory of gravitation

- The space–time is a manifold $\mathcal{M}$ with a Lorentzian metric g.
- tetrad: $\tilde{e}_a = \tilde{e}_a^{\mu}\partial_{\mu}, \quad g\left(\tilde{e}_a, \tilde{e}_b\right) = \eta_{ab}$
- cotetrad: $e^a = e^a_{\mu} dx^{\mu}, \quad e^a\left(\tilde{e}_b\right) = \delta^a_b$

General Relativity as a theory of gravitation

- The space–time is a manifold $\mathcal{M}$ with a Lorentzian metric g.
- tetrad: $\tilde{e}_a = \tilde{e}_a^{\mu}\partial_{\mu}, \quad g\left(\tilde{e}_a, \tilde{e}_b\right) = \eta_{ab}$
- cotetrad: $e^a = e_{\mu}^{a} dx^{\mu}, \quad e^a\left(\tilde{e}_b\right) = \delta_b^a$
- $g = \eta_{ab} e^a \otimes e^b$

General Relativity as a theory of gravitation

- The space–time is a manifold $\mathcal{M}$ with a Lorentzian metric g.
- tetrad: $\tilde{e}_a = \tilde{e}^{\mu}_{a}\partial_\mu, \quad g\left(\tilde{e}_a, \tilde{e}_b\right) = \eta_{ab}$
- cotetrad: $e^a = e^a_\mu dx^\mu, \quad e^a\left(\tilde{e}_b\right) = \delta^a_b$
- $g = \eta_{ab} e^a \otimes e^b$
- $$e'^a = \Lambda^a{}_b e^b \quad \equiv \quad e' = \Lambda e, \quad \Lambda \in O(1,3) \tag{3}$$

General Relativity as a theory of gravitation

- The space–time is a manifold $\mathcal{M}$ with a Lorentzian metric g.
- tetrad: $\tilde{e}_a = \tilde{e}_a^\mu \partial_\mu, \quad g\left(\tilde{e}_a, \tilde{e}_b\right) = \eta_{ab}$
- cotetrad: $e^a = e_\mu^a dx^\mu, \quad e^a\left(\tilde{e}_b\right) = \delta_b^a$
- $g = \eta_{ab} e^a \otimes e^b$
-
$$e'^a = \Lambda^a{}_b e^b \quad \equiv \quad e' = \Lambda e, \quad \Lambda \in O(1,3) \tag{3}$$
- connection one–forms: $\omega^a{}_b = \Gamma^a_{bc} e^c, \quad \nabla_{\tilde{e}_c} \tilde{e}_b = \Gamma^a_{bc} \tilde{e}_a$

General Relativity as a theory of gravitation

- The space–time is a manifold $\mathcal{M}$ with a Lorentzian metric g.
- tetrad: $\tilde{e}_a = \tilde{e}^{\mu}_a \partial_\mu, \quad g\left(\tilde{e}_a, \tilde{e}_b\right) = \eta_{ab}$
- cotetrad: $e^a = e^a_\mu dx^\mu, \quad e^a\left(\tilde{e}_b\right) = \delta^a_b$
- $g = \eta_{ab} e^a \otimes e^b$
- $$e'^a = \Lambda^a{}_b e^b \quad \equiv \quad e' = \Lambda e, \quad \Lambda \in O(1,3) \tag{3}$$
- connection one–forms: $\omega^a{}_b = \Gamma^a_{bc} e^c, \quad \nabla_{\tilde{e}_c} \tilde{e}_b = \Gamma^a_{bc} \tilde{e}_a$
- $\omega' = \Lambda \omega \Lambda^{-1} - d\Lambda \Lambda^{-1}$,

General Relativity as a theory of gravitation

- The space–time is a manifold $\mathcal{M}$ with a Lorentzian metric g.
- tetrad: $\tilde{e}_a = \tilde{e}_a^{\mu}\partial_\mu, \quad g\left(\tilde{e}_a, \tilde{e}_b\right) = \eta_{ab}$
- cotetrad: $e^a = e^a_\mu dx^\mu, \quad e^a\left(\tilde{e}_b\right) = \delta^a_b$
- $g = \eta_{ab} e^a \otimes e^b$
- $$e'^a = \Lambda^a{}_b e^b \quad \equiv \quad e' = \Lambda e, \quad \Lambda \in O(1,3) \tag{3}$$
- connection one–forms: $\omega^a{}_b = \Gamma^a_{bc} e^c, \quad \nabla_{\tilde{e}_c}\tilde{e}_b = \Gamma^a_{bc}\tilde{e}_a$
- $\omega' = \Lambda\omega\Lambda^{-1} - d\Lambda\Lambda^{-1}$,
- $\omega_{ab} = -\omega_{ba} \quad \Leftrightarrow \quad \nabla g = 0.$

In order to construct a cotetrad within the framework of the Poincaré gauge theory, one has to introduce a vector–valued zero–form y^a on $\mathcal{M}$ transforming under the gauge transformations as

- $y' = \Lambda y + a.$

In order to construct a cotetrad within the framework of the Poincaré gauge theory, one has to introduce a vector–valued zero–form y^a on $\mathcal{M}$ transforming under the gauge transformations as

- $y' = \Lambda y + a.$
- $e := \Gamma + Dy, \quad Dy = dy + \omega y \quad \Rightarrow \quad e' = \Lambda e.$

In order to construct a cotetrad within the framework of the Poincaré gauge theory, one has to introduce a vector–valued zero–form y^a on $\mathcal{M}$ transforming under the gauge transformations as

- $y' = \Lambda y + a.$
- $e := \Gamma + Dy, \quad Dy = dy + \omega y \quad \Rightarrow \quad e' = \Lambda e.$
- If $\mathfrak{L} = \mathfrak{L}\left(e(\Gamma, y, \omega), \omega, \phi\right)$, then it is justified to acknowledge e as a fundamental field.

In order to construct a cotetrad within the framework of the Poincaré gauge theory, one has to introduce a vector–valued zero–form y^a on $\mathcal{M}$ transforming under the gauge transformations as

- $y' = \Lambda y + a.$
- $e := \Gamma + Dy, \quad Dy = dy + \omega y \quad \Rightarrow \quad e' = \Lambda e.$
- If $\mathfrak{L} = \mathfrak{L}\left(e(\Gamma, y, \omega), \omega, \phi\right)$, then it is justified to acknowledge e as a fundamental field.
- $\mathfrak{L} = \mathfrak{L}_G + \tilde{\mathfrak{L}}_m, \quad \mathfrak{L}_G = -\frac{1}{4k}\epsilon_{abcd}e^a \wedge e^b \wedge \Omega^{cd},$
  $\Omega^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b.$

In order to construct a cotetrad within the framework of the Poincaré gauge theory, one has to introduce a vector–valued zero–form y^a on $\mathcal{M}$ transforming under the gauge transformations as

- $y' = \Lambda y + a.$
- $e := \Gamma + Dy, \quad Dy = dy + \omega y \quad \Rightarrow \quad e' = \Lambda e.$
- If $\mathfrak{L} = \mathfrak{L}\left(e(\Gamma, y, \omega), \omega, \phi\right)$, then it is justified to acknowledge e as a fundamental field.
- $\mathfrak{L} = \mathfrak{L}_G + \tilde{\mathfrak{L}}_m, \quad \mathfrak{L}_G = -\frac{1}{4k}\epsilon_{abcd} e^a \wedge e^b \wedge \Omega^{cd},$
  $\Omega^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b.$
- GR: $\frac{1}{2} T^a{}_{bc} e^b \wedge e^c := de^a + \omega^a{}_b \wedge e^b = 0$

In order to construct a cotetrad within the framework of the Poincaré gauge theory, one has to introduce a vector–valued zero–form y^a on $\mathcal{M}$ transforming under the gauge transformations as

- $y' = \Lambda y + a.$
- $e := \Gamma + Dy, \quad Dy = dy + \omega y \quad \Rightarrow \quad e' = \Lambda e.$
- If $\mathfrak{L} = \mathfrak{L}\left(e(\Gamma, y, \omega), \omega, \phi\right)$, then it is justified to acknowledge e as a fundamental field.
- $\mathfrak{L} = \mathfrak{L}_G + \tilde{\mathfrak{L}}_m, \quad \mathfrak{L}_G = -\frac{1}{4k}\epsilon_{abcd}e^a \wedge e^b \wedge \Omega^{cd}$, $\Omega^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b.$
- GR: $\frac{1}{2}T^a{}_{bc}e^b \wedge e^c := de^a + \omega^a{}_b \wedge e^b = 0$

The field equations of the Einstein–Cartan theory

$$\frac{\delta \mathfrak{L}_G}{\delta e^a} + \frac{\delta \tilde{\mathfrak{L}}_m}{\delta e^a} = 0 \quad \Leftrightarrow \quad G^a{}_b := R^a{}_b - \frac{1}{2} R \delta^a_b = k\, t_b{}^a$$

$$\frac{\delta \mathfrak{L}_G}{\delta \omega^{ab}} + \frac{\delta \tilde{\mathfrak{L}}_m}{\delta \omega^{ab}} = 0 \quad \Leftrightarrow \quad T^{cab} - T^a \eta^{bc} + T^b \eta^{ac} = k S^{abc}$$

$$\frac{\delta \tilde{\mathfrak{L}}_m}{\delta \phi} = 0$$

where $R^a{}_b := \eta^{ac} R^d{}_{cdb}$, $R := R^a{}_a$, $T^a := T^{ba}{}_b$ and the dynamical definitions of energy–momentum and spin density tensors on Riemann–Cartan space are

$$t_{ab} e^b := - \star \frac{\delta \tilde{\mathfrak{L}}_m}{\delta e^a}, \quad S^{abc} e_c := 2 \star \frac{\delta \tilde{\mathfrak{L}}_m}{\delta \omega_{ab}}.$$

The addition of a divergence to the flat–space Lagrangian density is a symmetry transformation

$$\mathfrak{L}_m = \mathcal{L}_m d^4x,$$

The addition of a divergence to the flat–space Lagrangian density is a symmetry transformation

$$\mathfrak{L}_m = \mathcal{L}_m d^4x,$$

$$\mathcal{L}_m \rightarrow \mathcal{L}'_m = \mathcal{L}_m + \partial_\mu V^\mu, \tag{4}$$

The addition of a divergence to the flat–space Lagrangian density is a symmetry transformation

$$\mathfrak{L}_m = \mathcal{L}_m d^4x,$$

$$\mathcal{L}_m \to \mathcal{L}'_m = \mathcal{L}_m + \partial_\mu V^\mu, \tag{4}$$

$$\partial_\mu V^\mu \, d^4x = \pounds_V \, d^4x = d(V \lrcorner \, d^4x),$$

The addition of a divergence to the flat–space Lagrangian density is a symmetry transformation

$$\mathfrak{L}_m = \mathcal{L}_m d^4x,$$

$$\mathcal{L}_m \to \mathcal{L}'_m = \mathcal{L}_m + \partial_\mu V^\mu, \tag{4}$$

$$\partial_\mu V^\mu \, d^4x = \pounds_V \, d^4x = d(V \lrcorner \, d^4x),$$

- $\pounds$ denotes the Lie derivative, $\lrcorner$ the internal product, $\star$ the Hodge star of Minkowski metric.

The addition of a divergence to the flat–space Lagrangian density is a symmetry transformation

$$\mathfrak{L}_m = \mathcal{L}_m d^4x,$$

$$\mathcal{L}_m \to \mathcal{L}'_m = \mathcal{L}_m + \partial_\mu V^\mu, \tag{4}$$

$$\partial_\mu V^\mu \, d^4x = \pounds_V \, d^4x = d(V \lrcorner \, d^4x),$$

- $\pounds$ denotes the Lie derivative, $\lrcorner$ the internal product, $\star$ the Hodge star of Minkowski metric.
- The field equations, as well as integrated Noether energy and momenta, remain unchanged.

Introducing gravity

- How to construct $\tilde{\mathfrak{L}}_m(\phi, d\phi, \mathbb{A})$ from $\mathfrak{L}_m(\phi, d\phi)$?

Introducing gravity

- How to construct $\tilde{\mathfrak{L}}_m(\phi, d\phi, \mathbb{A})$ from $\mathfrak{L}_m(\phi, d\phi)$?
- A consistency requirement – the transformation (4) is a symmetry of the resulting theory with gravity.

Introducing gravity

- How to construct $\tilde{\mathfrak{L}}_m(\phi, d\phi, \mathbb{A})$ from $\mathfrak{L}_m(\phi, d\phi)$?
- A consistency requirement – the transformation (4) is a symmetry of the resulting theory with gravity.
- MCP does not satisfy this requirement!

Introducing gravity

- How to construct $\tilde{\mathfrak{L}}_m\left(\phi, d\phi, \mathbb{A}\right)$ from $\mathfrak{L}_m\left(\phi, d\phi\right)$?
- A consistency requirement – the transformation (4) is a symmetry of the resulting theory with gravity.
- MCP does not satisfy this requirement!
- $\partial_\mu V^\mu d^4x \quad \xrightarrow{MCP} \quad d\left(V \lrcorner \epsilon\right) - T_a V^a \epsilon, \qquad T_a := T^b{}_{ab}.$

Introducing gravity

- How to construct $\tilde{\mathfrak{L}}_m(\phi, d\phi, \mathbb{A})$ from $\mathfrak{L}_m(\phi, d\phi)$?
- A consistency requirement – the transformation (4) is a symmetry of the resulting theory with gravity.
- MCP does not satisfy this requirement!
- $\partial_\mu V^\mu d^4x \quad \xrightarrow{MCP} \quad d\left(V \lrcorner \epsilon\right) - T_a V^a \epsilon, \qquad T_a := T^b{}_{ab}.$

Introducing gravity

- How to construct $\tilde{\mathfrak{L}}_m(\phi, d\phi, \mathbb{A})$ from $\mathfrak{L}_m(\phi, d\phi)$?
- A consistency requirement – the transformation (4) is a symmetry of the resulting theory with gravity.
- MCP does not satisfy this requirement!
- $\partial_\mu V^\mu d^4x \quad \xrightarrow{MCP} \quad d\left(V \lrcorner \epsilon\right) - T_a V^a \epsilon, \qquad T_a := T^b{}_{ab}.$

Introducing gravity

- How to construct $\tilde{\mathfrak{L}}_m(\phi, d\phi, \mathbb{A})$ from $\mathfrak{L}_m(\phi, d\phi)$?
- A consistency requirement – the transformation (4) is a symmetry of the resulting theory with gravity.
- MCP does not satisfy this requirement!
- $\partial_\mu V^\mu d^4x \quad \xrightarrow{MCP} \quad d\left(V \lrcorner \epsilon\right) - T_a V^a \epsilon, \qquad T_a := T^b{}_{ab}.$

Introducing gravity

- How to construct $\tilde{\mathfrak{L}}_m\left(\phi, d\phi, \mathbb{A}\right)$ from $\mathfrak{L}_m\left(\phi, d\phi\right)$?
- A consistency requirement – the transformation (4) is a symmetry of the resulting theory with gravity.
- MCP does not satisfy this requirement!
- $\partial_\mu V^\mu d^4x \quad \xrightarrow{MCP} \quad d\left(V \lrcorner \epsilon\right) - T_a V^a \epsilon, \qquad T_a := T^b{}_{ab}.$
- Any physical consequences?

An example – the Dirac field

- $\mathfrak{L}_{F0} = -i\,(\star dx_\mu) \wedge \overline{\psi}\gamma^\mu d\psi - m\overline{\psi}\psi\, d^4x$

An example – the Dirac field

- $\mathfrak{L}_{F0} = -i\,(\star dx_\mu) \wedge \overline{\psi}\gamma^\mu d\psi - m\overline{\psi}\psi\, d^4x$
- $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}, \quad \overline{\psi} := \psi^\dagger\gamma^0.$

An example – the Dirac field

- $\mathfrak{L}_{F0} = -i\,(\star dx_\mu) \wedge \overline{\psi}\gamma^\mu d\psi - m\overline{\psi}\psi\, d^4x$
- $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}, \quad \overline{\psi} := \psi^\dagger\gamma^0.$
- Invariant under the global action of $\mathcal{P}$

An example – the Dirac field

- $\mathfrak{L}_{F0} = -i\,(\star dx_\mu) \wedge \overline{\psi}\gamma^\mu d\psi - m\overline{\psi}\psi\, d^4x$
- $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}, \quad \overline{\psi} := \psi^\dagger\gamma^0.$
- Invariant under the global action of $\mathcal{P}$

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu, \quad \psi \to \psi' = S(\Lambda)\psi,$$

$$S(\Lambda(\varepsilon)) := \exp\left(-\frac{i}{4}\varepsilon_{\mu\nu}\Sigma^{\mu\nu}\right), \; \Sigma^{\mu\nu} := \frac{i}{2}[\gamma^\mu, \gamma^\nu].$$

An example – the Dirac field

- $\mathfrak{L}_{F0} = -i\,(\star dx_\mu) \wedge \overline{\psi}\gamma^\mu d\psi - m\overline{\psi}\psi\, d^4x$
- $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}, \quad \overline{\psi} := \psi^\dagger\gamma^0.$
- Invariant under the global action of $\mathcal{P}$

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu, \quad \psi \to \psi' = S(\Lambda)\psi,$$
$$S(\Lambda(\varepsilon)) := \exp\left(-\frac{i}{4}\varepsilon_{\mu\nu}\Sigma^{\mu\nu}\right), \; \Sigma^{\mu\nu} := \frac{i}{2}[\gamma^\mu, \gamma^\nu].$$

- $\mathfrak{L}_{F0} \quad \xrightarrow{MCP} \quad \tilde{\mathfrak{L}}_{F0} = -i\,(\star e_a) \wedge \overline{\psi}\gamma^a D\psi - m\overline{\psi}\psi\,\epsilon$

An example – the Dirac field

- $\mathfrak{L}_{F0} = -i\,(\star dx_\mu) \wedge \overline{\psi}\gamma^\mu d\psi - m\overline{\psi}\psi\, d^4x$
- $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}, \quad \overline{\psi} := \psi^\dagger\gamma^0.$
- Invariant under the global action of $\mathcal{P}$

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu, \quad \psi \to \psi' = S(\Lambda)\psi,$$

$$S(\Lambda(\varepsilon)) := \exp\left(-\frac{i}{4}\varepsilon_{\mu\nu}\Sigma^{\mu\nu}\right), \; \Sigma^{\mu\nu} := \frac{i}{2}[\gamma^\mu, \gamma^\nu].$$

- $\mathfrak{L}_{F0} \quad \xrightarrow{MCP} \quad \tilde{\mathfrak{L}}_{F0} = -i\,(\star e_a) \wedge \overline{\psi}\gamma^a D\psi - m\overline{\psi}\psi\,\epsilon$
- $D\psi = d\psi - \frac{i}{4}\omega_{ab}\Sigma^{ab}\psi, \qquad \epsilon = e^0 \wedge e^1 \wedge e^2 \wedge e^3.$

The Dirac field

- $$\frac{\delta \tilde{\mathfrak{L}}_{F0}}{\delta \psi} = 0 \quad \not\Leftrightarrow \quad \frac{\delta \tilde{\mathfrak{L}}_{F0}}{\delta \overline{\psi}} = 0$$

- $\dfrac{\delta \tilde{\mathfrak{L}}_{F0}}{\delta \psi} = 0 \quad \not\Leftrightarrow \quad \dfrac{\delta \tilde{\mathfrak{L}}_{F0}}{\delta \overline{\psi}} = 0$
- Commonly accepted solution:

- $\frac{\delta \tilde{\mathfrak{L}}_{F0}}{\delta \psi} = 0 \quad \not\Leftrightarrow \quad \frac{\delta \tilde{\mathfrak{L}}_{F0}}{\delta \overline{\psi}} = 0$
- Commonly accepted solution:
- $\mathfrak{L}_{FR} = -\frac{i}{2} (\star dx_\mu) \wedge \left(\overline{\psi}\gamma^\mu d\psi - \overline{d\psi}\gamma^\mu \psi\right) - m\overline{\psi}\psi d^4x$

- $\frac{\delta \tilde{\mathfrak{L}}_{F0}}{\delta \psi} = 0 \quad \not\Leftrightarrow \quad \frac{\delta \tilde{\mathfrak{L}}_{F0}}{\delta \overline{\psi}} = 0$
- Commonly accepted solution:
- $\mathfrak{L}_{FR} = -\frac{i}{2} (\star dx_{\mu}) \wedge \left(\overline{\psi} \gamma^{\mu} d\psi - \overline{d\psi} \gamma^{\mu} \psi \right) - m \overline{\psi} \psi d^4 x$
- justified, since $\mathcal{L}_{FR} = \mathcal{L}_{F0} + \partial_{\mu} V^{\mu}, \quad V^{\mu} = -\frac{i}{2} \overline{\psi} \gamma^{\mu} \psi$

- $\frac{\delta \tilde{\mathfrak{L}}_{F0}}{\delta \psi} = 0 \quad \not\Leftrightarrow \quad \frac{\delta \tilde{\mathfrak{L}}_{F0}}{\delta \overline{\psi}} = 0$
- Commonly accepted solution:
- $\mathfrak{L}_{FR} = -\frac{i}{2} (\star dx_\mu) \wedge \left(\overline{\psi} \gamma^\mu d\psi - \overline{d\psi} \gamma^\mu \psi \right) - m \overline{\psi} \psi d^4 x$
- justified, since $\mathcal{L}_{FR} = \mathcal{L}_{F0} + \partial_\mu V^\mu, \quad V^\mu = -\frac{i}{2} \overline{\psi} \gamma^\mu \psi$
- $MCP \;\Rightarrow \tilde{\mathfrak{L}}_{FR} = -\frac{i}{2} (\star e_a) \wedge \left(\overline{\psi} \gamma^a D\psi - \overline{D\psi} \gamma^a \psi \right) - m \overline{\psi} \psi \, \epsilon.$

- $\frac{\delta \tilde{\mathfrak{L}}_{F0}}{\delta \psi} = 0 \quad \not\Leftrightarrow \quad \frac{\delta \tilde{\mathfrak{L}}_{F0}}{\delta \overline{\psi}} = 0$
- Commonly accepted solution:
- $\mathfrak{L}_{FR} = -\frac{i}{2} (\star dx_\mu) \wedge \left(\overline{\psi} \gamma^\mu d\psi - \overline{d\psi} \gamma^\mu \psi \right) - m \overline{\psi} \psi d^4 x$
- justified, since $\mathcal{L}_{FR} = \mathcal{L}_{F0} + \partial_\mu V^\mu, \quad V^\mu = -\frac{i}{2} \overline{\psi} \gamma^\mu \psi$
- $MCP \;\Rightarrow \tilde{\mathfrak{L}}_{FR} = -\frac{i}{2} (\star e_a) \wedge \left(\overline{\psi} \gamma^a D\psi - \overline{D\psi} \gamma^a \psi \right) - m \overline{\psi} \psi \, \epsilon.$
- $\tilde{\mathfrak{L}}_{FR}$ is real and the theory is well defined. But

- $\dfrac{\delta\tilde{\mathfrak{L}}_{F0}}{\delta\psi} = 0 \quad \not\Leftrightarrow \quad \dfrac{\delta\tilde{\mathfrak{L}}_{F0}}{\delta\overline{\psi}} = 0$
- Commonly accepted solution:
- $\mathfrak{L}_{FR} = -\dfrac{i}{2}(\star dx_\mu) \wedge \left(\overline{\psi}\gamma^\mu d\psi - \overline{d\psi}\gamma^\mu\psi\right) - m\overline{\psi}\psi d^4x$
- justified, since $\mathcal{L}_{FR} = \mathcal{L}_{F0} + \partial_\mu V^\mu, \quad V^\mu = -\frac{i}{2}\overline{\psi}\gamma^\mu\psi$
- $MCP \;\Rightarrow \tilde{\mathfrak{L}}_{FR} = -\dfrac{i}{2}(\star e_a) \wedge \left(\overline{\psi}\gamma^a D\psi - \overline{D\psi}\gamma^a\psi\right) - m\overline{\psi}\psi\,\epsilon.$
- $\tilde{\mathfrak{L}}_{FR}$ is real and the theory is well defined. But
- $\mathcal{L}_{FR} \longrightarrow \mathcal{L}_{FR} + \partial_\mu V^\mu,$

- $\frac{\delta \tilde{\mathfrak{L}}_{F0}}{\delta \psi} = 0 \quad \not\Leftrightarrow \quad \frac{\delta \tilde{\mathfrak{L}}_{F0}}{\delta \overline{\psi}} = 0$
- Commonly accepted solution:
- $\mathfrak{L}_{FR} = -\frac{i}{2}(\star dx_\mu) \wedge \left(\overline{\psi}\gamma^\mu d\psi - \overline{d\psi}\gamma^\mu \psi\right) - m\overline{\psi}\psi d^4x$
- justified, since $\mathcal{L}_{FR} = \mathcal{L}_{F0} + \partial_\mu V^\mu, \quad V^\mu = -\frac{i}{2}\overline{\psi}\gamma^\mu\psi$
- $MCP \Rightarrow \tilde{\mathfrak{L}}_{FR} = -\frac{i}{2}(\star e_a) \wedge \left(\overline{\psi}\gamma^a D\psi - \overline{D\psi}\gamma^a \psi\right) - m\overline{\psi}\psi\, \epsilon.$
- $\tilde{\mathfrak{L}}_{FR}$ is real and the theory is well defined. But
- $\mathcal{L}_{FR} \longrightarrow \mathcal{L}_{FR} + \partial_\mu V^\mu,$
- $V^\mu = aJ^\mu_{(V)} + bJ^\mu_{(A)}, \quad a, b \in \mathbb{R},$

- $\dfrac{\delta\tilde{\mathfrak{L}}_{F0}}{\delta\psi} = 0 \quad \not\Leftrightarrow \quad \dfrac{\delta\tilde{\mathfrak{L}}_{F0}}{\delta\overline{\psi}} = 0$
- Commonly accepted solution:
- $\mathfrak{L}_{FR} = -\dfrac{i}{2}\,(\star dx_\mu) \wedge \left(\overline{\psi}\gamma^\mu d\psi - \overline{d\psi}\gamma^\mu\psi\right) - m\overline{\psi}\psi d^4x$
- justified, since $\mathcal{L}_{FR} = \mathcal{L}_{F0} + \partial_\mu V^\mu, \quad V^\mu = -\frac{i}{2}\overline{\psi}\gamma^\mu\psi$
- $MCP \ \Rightarrow \tilde{\mathfrak{L}}_{FR} = -\dfrac{i}{2}\,(\star e_a) \wedge \left(\overline{\psi}\gamma^a D\psi - \overline{D\psi}\gamma^a\psi\right) - m\overline{\psi}\psi\,\epsilon.$
- $\tilde{\mathfrak{L}}_{FR}$ is real and the theory is well defined. But
- $\mathcal{L}_{FR} \longrightarrow \mathcal{L}_{FR} + \partial_\mu V^\mu,$
- $V^\mu = aJ^\mu_{(V)} + bJ^\mu_{(A)}, \quad a, b \in \mathbb{R},$

  $J^\mu_{(V)} = \overline{\psi}\gamma^\mu\psi, \quad J^\mu_{(A)} = \overline{\psi}\gamma^\mu\gamma^5\psi$

Effective Lagrangian for Einstein–Cartan theory with fermions

Exploiting the algebraic invertible relation between spin and torsion
$T^{cab} - T^{a}\eta^{bc} + T^{b}\eta^{ac} = kS^{abc}$, $\quad S^{abc}e_c := 2\star\frac{\delta\tilde{\mathfrak{L}}_m}{\delta\omega_{ab}}$, one obtaines

$$\mathfrak{L}_{eff} = \mathring{\mathfrak{L}}_G + \mathring{\tilde{\mathfrak{L}}}_{FR} + \left(C_{AA}\, J^{(A)}_a J^a_{(A)} + C_{AV}\, J^{(A)}_a J^a_{(V)} + C_{VV}\, J^{(V)}_a J^a_{(V)}\right)\epsilon\ ,$$

Effective Lagrangian for Einstein–Cartan theory with fermions

Exploiting the algebraic invertible relation between spin and torsion

$T^{cab} - T^a\eta^{bc} + T^b\eta^{ac} = kS^{abc}, \qquad S^{abc}e_c := 2\star\frac{\delta\tilde{\mathfrak{L}}_m}{\delta\omega_{ab}},$ one obtaines

$$\mathfrak{L}_{eff} = \mathring{\mathfrak{L}}_G + \mathring{\tilde{\mathfrak{L}}}_{FR} + \left(C_{AA}\, J_a^{(A)} J^a_{(A)} + C_{AV}\, J_a^{(A)} J^a_{(V)} + C_{VV}\, J_a^{(V)} J^a_{(V)}\right)\epsilon\ ,$$

$$C_{AA} = \frac{3k}{16}\left(1 - 4b^2\right)\ , \qquad C_{AV} = -\frac{3k}{2}ab\ , \qquad C_{VV} = -\frac{3k}{4}a^2\ ,$$

Effective Lagrangian for Einstein–Cartan theory with fermions

Exploiting the algebraic invertible relation between spin and torsion

$T^{cab}-T^{a}\eta^{bc}+T^{b}\eta^{ac}=kS^{abc}, \qquad S^{abc}e_{c}:=2\star\frac{\delta\tilde{\mathfrak{L}}_{m}}{\delta\omega_{ab}},$ one obtaines

$$\mathfrak{L}_{eff}=\mathring{\mathfrak{L}}_{G}+\mathring{\tilde{\mathfrak{L}}}_{FR}+\left(C_{AA}\,J^{(A)}_{a}J^{a}_{(A)}+C_{AV}\,J^{(A)}_{a}J^{a}_{(V)}+C_{VV}\,J^{(V)}_{a}J^{a}_{(V)}\right)\epsilon\ ,$$

$$C_{AA}=\frac{3k}{16}\left(1-4b^{2}\right)\ , \qquad C_{AV}=-\frac{3k}{2}ab\ , \qquad C_{VV}=-\frac{3k}{4}a^{2}\ ,$$

$$\left(i\gamma^{a}\mathring{\nabla}_{a}-m\right)\psi+\left[-2C_{AA}J^{(A)}_{a}\gamma^{5}+C_{AV}\left(J^{(A)}_{a}-J^{(V)}_{a}\gamma^{5}\right)+2C_{VV}J^{(V)}_{a}\right]\gamma^{a}\psi=0\ .$$

Effective Lagrangian for Einstein–Cartan theory with fermions

Exploiting the algebraic invertible relation between spin and torsion

$T^{cab} - T^a \eta^{bc} + T^b \eta^{ac} = k S^{abc}, \qquad S^{abc} e_c := 2 \star \frac{\delta \tilde{\mathfrak{L}}_m}{\delta \omega_{ab}},$ one obtaines

$$\mathfrak{L}_{eff} = \mathring{\mathfrak{L}}_G + \mathring{\tilde{\mathfrak{L}}}_{FR} + \left(C_{AA}\, J_a^{(A)} J^a_{(A)} + C_{AV}\, J_a^{(A)} J^a_{(V)} + C_{VV}\, J_a^{(V)} J^a_{(V)} \right) \epsilon \ ,$$

$$C_{AA} = \frac{3k}{16}\left(1 - 4b^2\right) \ , \qquad C_{AV} = -\frac{3k}{2} ab \ , \qquad C_{VV} = -\frac{3k}{4} a^2 \ ,$$

$$\left(i\gamma^a \mathring{\nabla}_a - m \right)\psi + \left[-2C_{AA} J_a^{(A)} \gamma^5 + C_{AV}\left(J_a^{(A)} - J_a^{(V)} \gamma^5 \right) + 2C_{VV} J_a^{(V)} \right] \gamma^a \psi = 0 \ .$$

Loop Quantum Gravity: $\mathfrak{L}_G = -\frac{1}{4k} \epsilon_{abcd} e^a \wedge e^b \wedge \Omega^{cd} + \frac{1}{2k\beta} e^a \wedge e^b \wedge \Omega_{ab} \ ,$

$$C_{AA} = \frac{3k\beta}{16(1+\beta^2)}\left[4b + \beta(1 - 4b^2)\right], \quad C_{AV} = \frac{3k\beta}{4(1+\beta^2)} a(1 - 2\beta b), \quad C_{VV} = -\frac{3k\beta^2}{4(1+\beta^2)} a^2.$$

Outline of the Talk

1. Introduction — The Poincaré Gauge Theory
2. The ambiguity of MCP in the presence of torsion
3. Removing the ambiguity by modifying coupling procedure
   - The general idea
   - The Dirac field
   - Conclusions

The generalisation of the YM construction

- $S_m[\phi] = \int \mathcal{L}_m\left(\phi, \partial_\mu \phi\right) d^4x = \int \mathfrak{L}_m\left(\phi, d\phi\right)$

The generalisation of the YM construction

- $S_m[\phi] = \int \mathcal{L}_m \left(\phi, \partial_\mu \phi\right) d^4x = \int \mathfrak{L}_m \left(\phi, d\phi\right)$
- $d\phi \to \mathcal{D}\phi = d\phi + \mathcal{A}\phi$,
  where $\mathcal{A}$ is a $Lin(\mathcal{V})$–valued one–form field on M,

The generalisation of the YM construction

- $S_m[\phi] = \int \mathcal{L}_m\left(\phi, \partial_\mu \phi\right) d^4x = \int \mathfrak{L}_m\left(\phi, d\phi\right)$
- $d\phi \to \mathcal{D}\phi = d\phi + \mathcal{A}\phi$,
  where $\mathcal{A}$ is a $Lin(\mathcal{V})$–valued one–form field on M,
- $\mathcal{A} \to \mathcal{A}' = \rho(g)\mathcal{A}\rho^{-1}(g) - d\rho(g)\rho^{-1}(g)$.

The generalisation of the YM construction

- $S_m[\phi] = \int \mathcal{L}_m \left(\phi, \partial_\mu \phi\right) d^4x = \int \mathfrak{L}_m \left(\phi, d\phi\right)$
- $d\phi \to \mathcal{D}\phi = d\phi + \mathcal{A}\phi$,
  where $\mathcal{A}$ is a $Lin(\mathcal{V})$–valued one–form field on M,
- $\mathcal{A} \to \mathcal{A}' = \rho(g)\mathcal{A}\rho^{-1}(g) - d\rho(g)\rho^{-1}(g)$.
- $\mathcal{A} = \mathbb{A} + \mathbb{B}(\mathbb{A}, e)$,
  where $\mathbb{A}$ is $Ran(\pi)$–valued and $\mathbb{B}(\mathbb{A}, e)$ is $Ran(\pi)^{\perp}$–valued.

The generalisation of the YM construction

- $S_m[\phi] = \int \mathcal{L}_m(\phi, \partial_\mu \phi)\, d^4x = \int \mathfrak{L}_m(\phi, d\phi)$
- $d\phi \to \mathcal{D}\phi = d\phi + \mathcal{A}\phi$,
  where $\mathcal{A}$ is a $Lin(\mathcal{V})$–valued one–form field on M,
- $\mathcal{A} \to \mathcal{A}' = \rho(g)\mathcal{A}\rho^{-1}(g) - d\rho(g)\rho^{-1}(g)$.
- $\mathcal{A} = \mathbb{A} + \mathbb{B}(\mathbb{A}, e)$,
  where $\mathbb{A}$ is $Ran(\pi)$–valued and $\mathbb{B}(\mathbb{A}, e)$ is $Ran(\pi)^\perp$–valued.
- $\mathbb{A}' = \rho(g)\mathbb{A}\rho^{-1}(g) - d\rho(g)\rho^{-1}(g), \qquad e' = \Lambda e,$

The generalisation of the YM construction

- $S_m[\phi] = \int \mathcal{L}_m(\phi, \partial_\mu \phi)\, d^4x = \int \mathfrak{L}_m(\phi, d\phi)$
- $d\phi \to \mathcal{D}\phi = d\phi + \mathcal{A}\phi$,
  where $\mathcal{A}$ is a $Lin(\mathcal{V})$–valued one–form field on M,
- $\mathcal{A} \to \mathcal{A}' = \rho(g)\mathcal{A}\rho^{-1}(g) - d\rho(g)\rho^{-1}(g)$.

- $\mathcal{A} = \mathbb{A} + \mathbb{B}(\mathbb{A}, e)$,
  where $\mathbb{A}$ is $Ran(\pi)$–valued and $\mathbb{B}(\mathbb{A}, e)$ is $Ran(\pi)^{\perp}$–valued.

- $\mathbb{A}' = \rho(g)\mathbb{A}\rho^{-1}(g) - d\rho(g)\rho^{-1}(g), \qquad e' = \Lambda e,$
- $\mathbb{B}(\mathbb{A}', e') = \rho(g)\mathbb{B}(\mathbb{A}, e)\rho^{-1}(g)$.

What is $Ran(\pi)^{\perp}$?

What is $Ran(\pi)^{\perp}$?

- $Lin(\mathcal{V}) = Ran(\pi) \oplus Ran(\pi)^{\perp} \quad \Rightarrow \quad \mathbb{A}$ and $\mathbb{B}(\mathbb{A}, e)$ are uniquely determined by $\mathcal{A}$.

What is $Ran(\pi)^{\perp}$?

- $Lin(\mathcal{V}) = Ran(\pi) \oplus Ran(\pi)^{\perp} \quad \Rightarrow \quad \mathbb{A}$ and $\mathbb{B}(\mathbb{A}, e)$ are uniquely determined by $\mathcal{A}$.
- $Ran(\pi)^{\perp}$ is not uniquely determined by this requirement.

What is $Ran(\pi)^{\perp}$?

- $Lin(\mathcal{V}) = Ran(\pi) \oplus Ran(\pi)^{\perp} \quad \Rightarrow \quad \mathbb{A}$ and $\mathbb{B}(\mathbb{A}, e)$ are uniquely determined by $\mathcal{A}$.
- $Ran(\pi)^{\perp}$ is not uniquely determined by this requirement.
- if $\mathcal{V}$ admits a ρ–invariant scalar product $\langle , \rangle_{\rho}$, such that $\forall v, w \in \mathcal{V}, \quad g \in G, \quad \langle \rho(g)v, \rho(g)w \rangle_{\rho} = \langle v, w \rangle_{\rho}\,,$

What is $Ran(\pi)^{\perp}$?

- $Lin(\mathcal{V}) = Ran(\pi) \oplus Ran(\pi)^{\perp} \quad \Rightarrow \quad \mathbb{A}$ and $\mathbb{B}(\mathbb{A}, e)$ are uniquely determined by $\mathcal{A}$.
- $Ran(\pi)^{\perp}$ is not uniquely determined by this requirement.
- if $\mathcal{V}$ admits a ρ–invariant scalar product $\langle , \rangle_{\rho}$, such that $\forall v, w \in \mathcal{V}, \quad g \in G, \quad \langle \rho(g)v, \rho(g)w \rangle_{\rho} = \langle v, w \rangle_{\rho}$,
  (e.g. $\overline{\psi}\gamma^0\phi$ for Dirac representation, $\eta_{\mu\nu}V^{\mu}W^{\nu}$ for vector representation etc.)

What is $Ran(\pi)^{\perp}$?

- $Lin(\mathcal{V}) = Ran(\pi) \oplus Ran(\pi)^{\perp} \quad \Rightarrow \quad \mathbb{A}$ and $\mathbb{B}(\mathbb{A}, e)$ are uniquely determined by $\mathcal{A}$.
- $Ran(\pi)^{\perp}$ is not uniquely determined by this requirement.
- if $\mathcal{V}$ admits a ρ–invariant scalar product $\langle , \rangle_\rho$, such that
  $\forall v, w \in \mathcal{V}, \quad g \in G, \quad \langle \rho(g)v, \rho(g)w \rangle_\rho = \langle v, w \rangle_\rho \,,$
  (e.g. $\overline{\psi}\gamma^0\phi$ for Dirac representation, $\eta_{\mu\nu}V^\mu W^\nu$ for vector representation etc.)
  then the induced product $\langle\!\langle \, , \, \rangle\!\rangle_\rho$ on $Lin(\mathcal{V})$ satisfying
  $\langle\!\langle \, \rho(g)X\rho^{-1}(g), \rho(g)Y\rho^{-1}(g) \, \rangle\!\rangle_\rho = \langle\!\langle \, X, Y \, \rangle\!\rangle_\rho$ can be used.

What is $Ran(\pi)^{\perp}$?

- $Lin(\mathcal{V}) = Ran(\pi) \oplus Ran(\pi)^{\perp} \quad \Rightarrow \quad \mathbb{A}$ and $\mathbb{B}(\mathbb{A}, e)$ are uniquely determined by $\mathcal{A}$.
- $Ran(\pi)^{\perp}$ is not uniquely determined by this requirement.
- if $\mathcal{V}$ admits a ρ–invariant scalar product $\langle , \rangle_\rho$, such that
  $\forall v, w \in \mathcal{V}, \quad g \in G, \quad \langle \rho(g)v, \rho(g)w \rangle_\rho = \langle v, w \rangle_\rho \, ,$
  (e.g. $\overline{\psi}\gamma^0\phi$ for Dirac representation, $\eta_{\mu\nu}V^\mu W^\nu$ for vector representation etc.)
  then the induced product $\langle\!\langle \, , \, \rangle\!\rangle_\rho$ on $Lin(\mathcal{V})$ satisfying
  $\langle\!\langle \rho(g)X\rho^{-1}(g), \rho(g)Y\rho^{-1}(g) \rangle\!\rangle_\rho = \langle\!\langle X, Y \rangle\!\rangle_\rho$ can be used.
- if the subspace $Ran(\pi) \subset Lin(\mathcal{V})$ is nondegenerate with respect to $\langle\!\langle \, , \, \rangle\!\rangle_\rho$, then $Lin(\mathcal{V}) = Ran(\pi) \oplus Ran(\pi)^{\perp}$.

The Dirac field

The Dirac field

- $\mathcal{V} = \mathbb{C}^4$, $Ran(\pi)$ is spanned by $\Sigma^{ab} = \frac{i}{2}[\gamma^a, \gamma^b]$,

The Dirac field

- $\mathcal{V} = \mathbb{C}^4$, $Ran(\pi)$ is spanned by $\Sigma^{ab} = \frac{i}{2}[\gamma^a, \gamma^b]$,
- $\langle\!\langle X, Y \rangle\!\rangle_\rho = trace\left(\gamma^0 X^\dagger \gamma^0 Y\right)$, $Ran(\pi)^\perp$ is spanned by $\mathbf{1}, \gamma^5, \gamma^a, \gamma^5\gamma^a$.

The Dirac field

- $\mathcal{V} = \mathbb{C}^4$, $Ran(\pi)$ is spanned by $\Sigma^{ab} = \frac{i}{2}[\gamma^a, \gamma^b]$,
- $\langle\!\langle X, Y \rangle\!\rangle_\rho = trace\left(\gamma^0 X^\dagger \gamma^0 Y\right)$, $Ran(\pi)^\perp$ is spanned by $\mathbf{1}, \gamma^5, \gamma^a, \gamma^5\gamma^a$.
- $$\mathcal{D}\psi = d\psi + \mathcal{A}\psi, \quad \mathcal{A} = \mathbb{A} + \mathbb{B},$$
  $$\mathbb{A} = -\frac{i}{4}\omega_{ab}\Sigma^{ab}, \quad \mathbb{B} = \chi\mathbf{1} + \kappa\gamma^5 + \tau_a\gamma^a + \rho_a\gamma^5\gamma^a,$$
  where χ, κ, τ_a, ρ_a are complex–valued one–forms on space–time.

We will require that the Leibniz rule hold

We will require that the Leibniz rule hold

- $$(\mathcal{D}\overline{\psi})\gamma^a\psi + \overline{\psi}\gamma^a\mathcal{D}\psi = dJ^a_{(V)} + \tilde{\omega}^a{}_b J^b_{(V)},$$
$$(\mathcal{D}\overline{\psi})\gamma^a\gamma^5\psi + \overline{\psi}\gamma^a\gamma^5\mathcal{D}\psi = dJ^a_{(A)} + \tilde{\omega}^a{}_b J^b_{(A)},$$

where $\mathcal{D}\overline{\psi} := (\mathcal{D}\psi)^\dagger\gamma^0$ and $\tilde{\omega}^a{}_b$ is a modified connection on space–time. The solution:

We will require that the Leibniz rule hold

- $$(\mathcal{D}\overline{\psi})\gamma^a\psi + \overline{\psi}\gamma^a\mathcal{D}\psi = dJ^a_{(V)} + \tilde{\omega}^a{}_b J^b_{(V)},$$
$$(\mathcal{D}\overline{\psi})\gamma^a\gamma^5\psi + \overline{\psi}\gamma^a\gamma^5\mathcal{D}\psi = dJ^a_{(A)} + \tilde{\omega}^a{}_b J^b_{(A)},$$

where $\mathcal{D}\overline{\psi} := (\mathcal{D}\psi)^\dagger\gamma^0$ and $\tilde{\omega}^a{}_b$ is a modified connection on space–time. The solution:

- $$\tilde{\omega}^a{}_b = \omega^a{}_b + \lambda\delta^a_b, \quad \mathbb{B} = \frac{1}{2}\lambda\mathbf{1} + i\mu_1\mathbf{1} + i\mu_2\gamma^5,$$

where λ, μ_1, μ_2 are real–valued one–forms.

We will require that the Leibniz rule hold

- $$(\mathcal{D}\overline{\psi})\gamma^a\psi + \overline{\psi}\gamma^a\mathcal{D}\psi = dJ^a_{(V)} + \tilde{\omega}^a{}_b J^b_{(V)},$$
$$(\mathcal{D}\overline{\psi})\gamma^a\gamma^5\psi + \overline{\psi}\gamma^a\gamma^5\mathcal{D}\psi = dJ^a_{(A)} + \tilde{\omega}^a{}_b J^b_{(A)},$$

where $\mathcal{D}\overline{\psi} := (\mathcal{D}\psi)^\dagger\gamma^0$ and $\tilde{\omega}^a{}_b$ is a modified connection on space–time. The solution:

- $$\tilde{\omega}^a{}_b = \omega^a{}_b + \lambda\delta^a_b, \quad \mathbb{B} = \frac{1}{2}\lambda\mathbf{1} + i\mu_1\mathbf{1} + i\mu_2\gamma^5,$$

where λ, μ_1, μ_2 are real–valued one–forms.

- μ_1 and μ_2 do not influence $\tilde{\omega}$. They could be hidden in the gauge fields corresponding to the localization of the global symmetry $\psi \to e^{i\alpha}\psi$ and the approximate symmetry $\psi \to e^{i\alpha\gamma^5}\psi$. We set $\mu_1 = \mu_2 = 0$.

- The procedure would be free of the ambiguity iff $\lambda = \mathbb{T}$, where $\mathbb{T} = T_a e^a$ is the torsion–trace–one–form.

- The procedure would be free of the ambiguity iff $\lambda = \mathbb{T}$, where $\mathbb{T} = T_a e^a$ is the torsion–trace–one–form.
- $\tilde{\omega}^a{}_b = \omega^a{}_b + \mathbb{T}\delta^a_b$ – a modified connection on $\mathcal{M}(\omega, e)$,

- The procedure would be free of the ambiguity iff $\lambda = \mathbb{T}$, where $\mathbb{T} = T_a e^a$ is the torsion–trace–one–form.
- $\tilde{\omega}^a{}_b = \omega^a{}_b + \mathbb{T}\delta^a_b$ – a modified connection on $\mathcal{M}\left(\omega, e\right)$,
- $\mathcal{D}\psi = d\psi - \frac{i}{4}\omega_{ab}\Sigma^{ab} + \frac{1}{2}\mathbb{T}$.

- The procedure would be free of the ambiguity iff $\lambda = \mathbb{T}$, where $\mathbb{T} = T_a e^a$ is the torsion–trace–one–form.
- $\tilde{\omega}^a{}_b = \omega^a{}_b + \mathbb{T}\delta^a_b$ – a modified connection on $\mathcal{M}\,(\omega, e)$,
- $\mathcal{D}\psi = d\psi - \frac{i}{4}\omega_{ab}\Sigma^{ab} + \frac{1}{2}\mathbb{T}$.
-
$$\mathfrak{L}_{eff} = \overset{\circ}{\mathfrak{L}}_G + \overset{\circ}{\tilde{\mathfrak{L}}}_{FR} + \frac{3k}{16}\, J^{(A)}_a J^a_{(A)}\, \epsilon\ .$$

Conclusions

Conclusions

- The modified coupling procedure provides a consistent method for coupling gravity to other field theories within the framework of the Poincaré gauge theory of gravity.

Conclusions

- The modified coupling procedure provides a consistent method for coupling gravity to other field theories within the framework of the Poincaré gauge theory of gravity.
- As opposed to MCP, the results obtained do not depend on the choice of flat space Lagrangian from the class of equivalence corresponding to the possibility of adding divergence.

Conclusions

- The modified coupling procedure provides a consistent method for coupling gravity to other field theories within the framework of the Poincaré gauge theory of gravity.
- As opposed to MCP, the results obtained do not depend on the choice of flat space Lagrangian from the class of equivalence corresponding to the possibility of adding divergence.
- In particular, the predictions of EC theory with fermions are made unique – they agree with those derived in earlier accounts for a particular choice of fermionic Lagrangian. The same concerns the predictions of EC theory modified by the presence of the Holst term.

M.K. “Nonuniqueness of gravity induced fermion interaction in the Einstein-Cartan theory”,
Phys. Rev. D **78**, 124025 (2008) arXiv:0811.1932v3.

M.K. “Einstein–Cartan gravity with Holst term and fermions”,
Phys. Rev. D **79**, 064029 (2009) arXiv:0812.1298v4.

M.K. “Modified coupling procedure for the Poincaré gauge theory of gravity”
(to be published in Phys. Rev. D).