Holography for non-relativistic CFTs

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Outline of the lectures

Lecture 1: Non-relativistic scale invariant theories

- Introduction to Galilean scaling symmetries
- Schrödinger algebra and its realizations
- Liftshitz theories
- Real world systems with Schrödinger symmetries

Lecture 2: Galilean holography

- The holographic dual spacetime
- String theory realization of Schrödinger invariant theories

Lecture 3: Applications of the Galilean hologram

- Thermodynamics & Hydrodynamics
- Correlation functions

Motivation

Holographic models for strongly coupled systems

- The AdS/CFT correspondence allows us to probe the physics of strongly coupled gauge theories.
  - ⋆ Insight into transport properties of QGP, relevant for physics seen in heavy-ion collisons.
- There are other strongly coupled systems discussed in condensed matter literature which exhibit a wide range of extremely interesting physics.
- Use holographic methods to find the classical "Master field" for these theories.

Motivation

New insights into Quantum Gravity

- AdS/CFT has a dual role: it allows us to probe quantum aspects of gravity in terms of a non-perturbatively well defined QFT.
- Generalizations of the AdS/CFT correspondence, to new terrains has the potential to unveil important lessons for quantum gravity.

Understanding fluid dynamics

- The mathematical structure of Navier-Stokes equations (non-relativistic) poses interesting challenges.
- Can we reformulate the Fluid-Gravity correspondence in a context relevant for non-relativisitic fluids? Bhattacharyya, Hubeny, Minwalla, MR

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Understanding fluid dynamics

- The mathematical structure of Navier-Stokes equations (non-relativistic) poses interesting challenges.
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Motivation

Experimental relevance

- There is currently an intensive experimental effort to understand the physics of cold atoms.
- These systems seem to admit an hydrodynamic description in terms of a nearly-ideal fluid.
  - ⋆ The energy per particle is about 50% of the free value, similar in spirit to the Stephan-Boltzmann saturation of QGP just above the deconfinement transition.
  - ⋆ Experimental results of elliptic type flow (shear driven relaxation) give $\eta/\mathrm{s} \sim 1/\pi$!
- Can we find systems that have holographic duals which share at least some of the symmetries exhibited in these cold atom systems?

References

- *Proposal for holographic duals*
  - ⋆ Son: 0804.3972
  - ⋆ Balasubramanian K, McGreevy: 0804.4053
- *Holographic embedding in string theory, etc..*
  - ⋆ Herzog, MR, Ross: 0807.1099
  - ⋆ Maldacena, Martelli, Tachikawa: 0807.1100
  - ⋆ Adams, Balasubramanian K, McGreevy: 0807.1111
- *Fluid dynamics*
  - ⋆ MR, Ross, Son, Thompson: 0711.2049
- *Related work*
  - ⋆ Goldberger: 0705.2867
  - ⋆ Barbon, Fuertes: 0705.3244
- *Earlier relevant work*
  - ⋆ Nishida, Son: 0706.3746
  - ⋆ Hubeny, MR, Ross: hep-th/0504034

The Schrödinger algebra

- The Schrödinger algebra is the symmetry algebra of the free Schrödinger operator in $d + 1$ dimensions.
- It is generated by operators that commute with

$$S = i\,\partial_t + \frac{1}{2\,m}\,\partial_i^2$$

- It is analog of the conformal algebra for relativistic systems – we will see how to relate the two shortly.
- It is believe that the system of cold atoms at unitarity is an example of an interacting QFT which realizes this symmetry.

The Schrödinger group

- One can write down the Schrödinger group as the following set of transformations:

$$\mathrm{x} \to \mathrm{x}' = \frac{\mathfrak{R}\,\mathrm{x} + \mathfrak{v}\,\mathrm{t} + \mathfrak{a}}{\gamma\,\mathrm{t} + \delta}$$

$$\mathrm{t} \to \mathrm{t}' = \frac{\alpha\,\mathrm{t} + \beta}{\gamma\,\mathrm{t} + \delta}$$

  with $\alpha\,\delta - \beta\,\gamma = 1$.

- The group includes, spatial translations indicated by $\mathfrak{a}$, rotations captured by $\mathfrak{R}$, Galilean boosts with velocity $\mathfrak{v}$, a scale transformation and a special conformal generator.
- We will derive the algebra momentarily by employing a useful trick.

Aside: Galilean Conformal Algebra

- Apart from the Schrödinger algebra there is another conformal algebra which includes the Galilean algebra as a sub-algebra – this is the Galilean Conformal Algebra (GCA). Bagchi, Gopakumar
- The two algebras are quite distinct; they have different numbers of generators and also treat the dilatation generator differently.
- One can view the GCA as a contraction of the conformal algebra obtained by sending $c \to \infty$.
- The Schrödinger algebra on the other hand requires us to rescale the particle mass as well.
- We will mostly focus on the Schrödinger algebra in these lectures.

Galileo & Poincaré

Light-cone reductions

- Recall that one can get the Galilean algebra in d dimensions by reducing the Poincaré algebra $SO(d+1,1)$ on light-cone

$$u = t + y \,, \qquad v = t - y$$

- Propagation in light-cone time u respects Galilean invariance.
- We can similarly reduce the conformal algebra $SO(d+2,2)$ in $d+2$ dimensions on a light-cone to obtain the Schrödinger algebra in d-spatial dimensions.

The Schrödinger algebra: Generators

Starting from the conformal algebra we keep all generators which commute with the particle number.

- Hamiltonian: H
- Spatial rotations: $\mathrm{M_{ij}}$
- Spatial momenta: $\mathrm{P_i}$
- Galilean boosts: $\mathrm{K_i}$
- Dilatation: D
- Special conformal generator: C
- Particle number: N

where we are restricting attention to d-spatial dimensions, i.e., $\{\mathrm{i},\mathrm{j}\} \in \{1,\cdots \mathrm{d}\}$.

The Schrödinger algebra from conformal algebra

Generator	Galilean	Conformal
Particle number	N	P_v
Hamiltonian	H	P_u
Momenta	P_i	P_i
Angular momenta	M_{ij}	M_{ij}
Galilean boost	K_i	M_{iv}
Dilatation	D	$D + M_{uv}$
Special conformal	C	K_v

The Schrödinger algebra: Commutation relations

$$
\begin{aligned}
[M_{ij}, M_{kl}] &= i\,(\delta_{ik} M_{jl} - \delta_{jk} M_{il} + \delta_{il} M_{kj} - \delta_{jl} M_{ki}) \\
[M_{ij}, P_k] &= i\,(\delta_{ik} P_j - \delta_{jk} P_i) \\
[M_{ij}, K_k] &= i\,(\delta_{ik} K_j - \delta_{jk} K_i) \\
[M_{ij}, H] &= [M_{ij}, D] = [M_{ij}, C] = 0 \\
[P_i, P_j] &= [K_i, K_j] = 0\ , \qquad [K_i, P_j] = i\,\delta_{ij}\, N
\end{aligned}
$$

$$
\begin{aligned}
[D, P_i] &= i\,P_i\ , & [D, K_i] &= -i\,K_i \\
[H, P_i] &= 0\ , & [H, K_i] &= -i\,P_i \\
[C, P_i] &= i\,K_i\ , & [C, K_i] &= 0 \\
[D, H] &= 2i\,H\ , & [D, C] &= -2i\,C\ , \\
[H, C] &= -i\,D
\end{aligned}
$$

Scaling dimensions and representations

- From the commutation relations descending from the conformal algebra one can infer that

$$[\mathrm{H}, \mathrm{D}] = -2\,\mathrm{i}\,\mathrm{H}$$

- This implies that the Hamiltonian has scaling dimension 2.
- Intuitively, this follows from the fact that non-relativistic systems are first order in time, leading to scaling

$$\mathrm{t} \to \lambda^2\,\mathrm{t}\ , \qquad \mathrm{x} \to \lambda \mathrm{x}$$

Scaling dimensions and representations

- Representation of Schrödinger algebra in terms of highest weight states as usual. Nishida, Son
- In particular, we will talk about 2 quantum numbers
  - ⋆ The scaling dimension:

$$[\mathrm{D}, \mathcal{O}] = \mathrm{i}\, \Delta_{\mathcal{O}}\, \mathcal{O}$$

  - ⋆ The particle number:

$$[\mathrm{N}, \mathcal{O}] = \mathrm{N}_{\mathcal{O}}\, \mathcal{O}$$

- We have $\Delta_{\mathrm{H}} = 2$ and $\Delta_{\mathrm{P}} = 1$.

Scaling dimensions and representations

- We will realize highest weight representations in terms of primary operators which have a given conformal dimension $\Delta_{\mathcal{O}}$ and particle number $N_{\mathcal{O}}$.
- As usual the spacetime dependence of the operator can be inferred via translation:

$$\mathcal{O}(t, x) = e^{i H t - i P_i x_i} \, \mathcal{O}(0) \, e^{-i H t + i P_i x_i}$$

- The primary operators are defined so that lowering operators K and C (which have scaling dimensions -1 and -2 respectively annihilate it i.e.,

$$[K_i, \mathcal{O}] = [C, \mathcal{O}] = 0$$

Scaling dimensions and representations

One can give a simple representation of the algebra in terms using the usual derivative representation. For an operator $\mathcal{O}(t, x)$:

$$\begin{aligned}
[H, \mathcal{O}] &= -i\, \partial_t \mathcal{O} \\
[P_i, \mathcal{O}] &= i\, \partial_i \mathcal{O} \\
[D, \mathcal{O}] &= i\, (2\, t\, \partial_t + x_i\, \partial_i + \Delta_{\mathcal{O}})\, \mathcal{O} \\
[K_i, \mathcal{O}] &= \left(-i\, t\, \partial_i + N_{\mathcal{O}}\, x_i\right) \mathcal{O} \\
[C, \mathcal{O}] &= -i \left(t^2\, p_t + t\, x_i\, \partial_i + t\, \Delta_{\mathcal{O}}\right) \mathcal{O}
\end{aligned}$$

which in particular implies that the quasi-primary operators satisfy

$$e^{-i\, \lambda\, D}\, \mathcal{O}(t, x)\, e^{i\, \lambda\, D} = e^{\lambda\, \Delta_{\mathcal{O}}}\, \mathcal{O}\left(e^{2\, \lambda}\, t, e^{\lambda}\, x\right)$$

State-operator correspondence

- Primary operators are in one-one correspondence with the eigenstates of a quantum system in a harmonic trap.
- The state
$$|\,\psi_{\mathcal{O}}\,\rangle = \mathrm{e}^{-\mathrm{H}}\,\mathcal{O}^{\dagger}\,|\,0\,\rangle$$
is an eigenstate of the Hamiltonian $\mathrm{H}_{\mathrm{osc}} = \mathrm{H} + \mathrm{C}$ with eigenvalue $\Delta_{\mathcal{O}}$.
- The Schrödinger algebra has a $\mathrm{SL}(2,\mathrm{R})$ sub-algebra generated by $\{\mathrm{D},\mathrm{H},\mathrm{C}\}$.
$$\mathrm{H}_{\mathrm{osc}} = \frac{1}{2}\,(\mathrm{H}+\mathrm{C})$$
$$\mathrm{a}^{\dagger} = \frac{1}{2}\,(\mathrm{H}-\mathrm{C}+\mathrm{i}\,\mathrm{D})$$
$$\mathrm{a} = \frac{1}{2}\,(\mathrm{H}-\mathrm{C}-\mathrm{i}\,\mathrm{D})$$

Liftshitz points

- We can also consider more general scaling, but not conformal symmetries.
- These are described by a real number $z = 1 + \nu$.
- We assign weight $-\nu$ to K_i and $1 + \nu$ to H.
- The commutation relations are deformed to

$$[D, H] = i\,(1 + \nu)\,H\ , \qquad [D, N] = -i\,(\nu - 1)\,N$$

$$[D, K_i] = -i\,\nu\,K_i$$

- For $\nu \neq 1$ we don't have a conserved particle number and the special conformal generator C does not exist in the algebra.
- These describe generalized scaling

$$t \to \lambda^{1+\nu}\,t \qquad x \to \lambda\,x$$

Fermions at unitarity

- Cold atom systems are an increasingly interesting arena to explore a wide range of physical phenomena.
- Fermionic Li^6 or K^{40} in optical traps provide systems of fermionic gases where inter-atomic interactions can be externally tuned to produce different phases.
- The quantity of interest is the s-wave scattering length a; tuning a one can pass from a BEC condensate to a BCS superfluid.
  - ⋆ Small negative a leads to weak attractive interaction – BCS limit.
  - ⋆ As $a \to \infty$ we achieve the unitarity limit as the s-wave cross section is saturated.
  - ⋆ For positive scattering length is the BEC phase where the fermions form deeply bound molecules.

Fermions at unitarity

- Tuning a is achieved by external magnetic field with the fermionic atoms in an optical trap.
- Exactly at threshold one obtains a massless bound state, and the theory is supposed to be described as a non-relativistic conformal field theory with Schrödinger symmetry.
- Experimental studies of this fluid suggest that it is another example of a nearly-ideal fluid with $\eta/\mathrm{s} \sim 1/\pi$. Schäfer, Teaney
- Fixed points are known to exist in ϵ expansion around 2 dimensions and 4 dimensions. Nishida, Son.

Statement of the AdS/CFT correspondence

AdS/CFT

Quantum gravity on asymptotically d-dimensional Anti-deSitter spacetime is described by a $d-1$ dimensional gauge theory sans gravity.

A particularly appealing and testable form of the conjecture:

Type IIB string theory on $AdS_5 \times S^5$ spacetime.	$\equiv$	4-dimensional superconformal Yang-Mills gauge theory.
R_{AdS}, g_s	$\leftrightarrow$	g_{YM}^2, N

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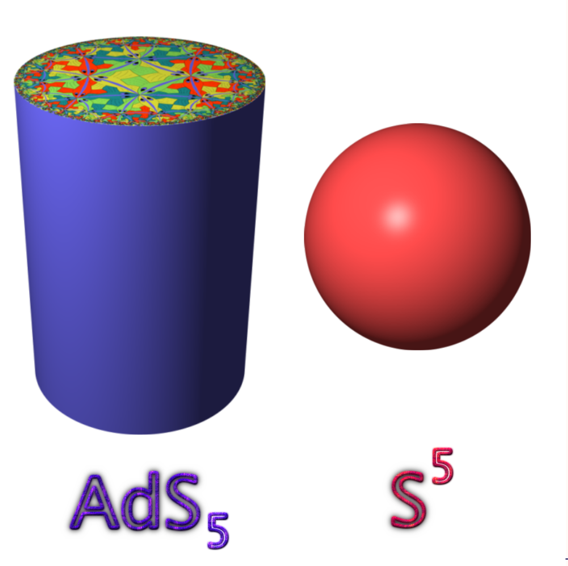
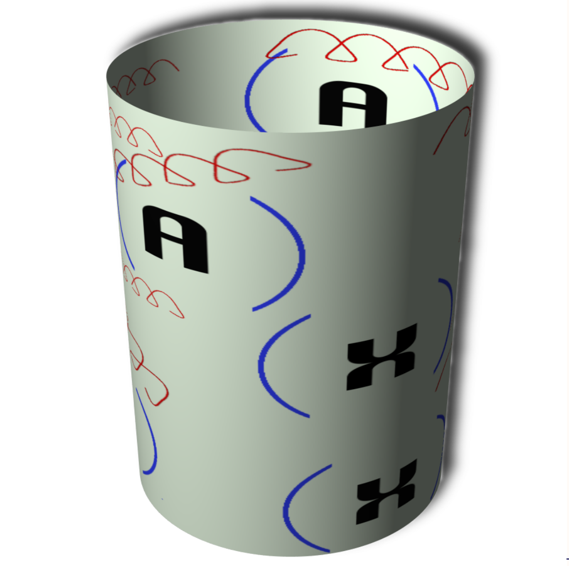
AdS/CFT continued



$$g_{YM}^2 N \gg 1$$

$$N \gg 1$$

AdS/CFT continued



$$g_{YM}^2 N \ll 1$$

$$N \gg 1$$

Motivating the correspondence

- Start with N D3-branes in flat space. The world-volume is $R^{3,1} \subset R^{9,1}$.
- This has two equivalent descriptions in string theory:
  - ⋆ As open strings ending on D3 interacting with closed strings in the bulk
  - ⋆ As purely closed strings in a back-reacted spacetime.
- A suitable decoupling limit $\ell_s \to 0$ zooms in onto the dynamics of just the open strings whilst in the geometric picture we focus on a region of the full spacetime.
- Effectively, closing the holes on the world-sheet leads to a pure closed string description.

Salient features of the AdS/CFT correspondence

- Symmetry matching: the $SO(4,2) \times SO(6) \subset PSU(2,2|4)$ global symmetry of field theory are realized as isometries of the spacetime.
- Local gauge invariant single trace operators of the field theory such as $\mathcal{O} = \text{Tr}\,(X \cdots X)$ are mapped to single particle states in the super-gravity description.
- There exists a precise prescription to compute the generating function of correlation functions for these gauge invariant operators:

$$\langle\, e^{\int_{\mathcal{B}}\ \varphi_0(x)\, \mathcal{O}(x)}\, \rangle_{CFT} = \mathcal{Z}_{\text{string}} \left[\varphi(r,x)\big|_{\mathcal{B}} = \varphi_0(x) \right]$$

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AdS/CFT basics

- Consider the geometry of AdS_{d+3}

$$ds^2 = -r^2\,dt^2 + r^2\,dx^2 + \frac{dr^2}{r^2}$$

which is the metric covering the Poincaré patch of AdS.

- AdS_{d+3} has the $SO(d+2,2)$ isometry algebra of which we can look at the scaling symmetry

$$t \to \lambda\,t\ , \qquad x \to \lambda\,x\ , \qquad r \to \frac{1}{\lambda}\,r$$

AdS/CFT basics

- This is the familiar scale transformations for the relativistic CFT on $R^{1,d+1}$ which is the boundary of AdS_{d+3} in Poincaré coordinates.
- The radial direction is holographically said to correspond to a energy scale in the field theory, cf., the holographic renormalization group.
- We can map out the other symmetries as well similarly in terms of AdS isometries.

Holography for non-relativistic CFTs: DLCQ

- Consider the scaling symmetery

$$t \to \lambda^{\nu+1}\, t \ , \qquad x \to \lambda\, x$$

- This can be achieved by starting from AdS_{d+3} in light-cone coordinates

$$ds^2 = r^2 \left(-2\, du\, dv + dx^2\right) + \frac{dr^2}{r^2}$$

and define an unconventional scaling

$$u \to \lambda^{\nu+1}\, u \ , \qquad v \to \lambda^{1-\nu}\, v \ , \qquad x \to \lambda\, x \ , \qquad r \to \frac{1}{\lambda}\, r$$

and interpreting u as time.

Holography for non-relativistic CFTs: DLCQ

- This Galilean symmetry is familiar from DLCQ.
- In fact, this is essentially the observation that DLCQ of any relativistic theory gives a Galilean invariant model in a sector with fixed light-cone momentum.
- However, we should be careful about the zero mode.
- Finally, the underlying theory is relativistic – the Galilean symmetry is an artifact of our choice of light-cone quantization.

Holography for non-relativistic CFTs

- To motivate a dual that has manifest Galilean scaling consider

Son; Balasubramanian K, McGreevy

$$ds^2 = r^2\left(-2\,du\,dv - \beta^2\,r^{2\,\nu}\,du^2 + dx^2\right) + \frac{dr^2}{r^2}$$

which naturally has the required scaling

$$u \to \lambda^{\nu+1}\,u\,, \qquad v \to \lambda^{1-\nu}\,v\,, \qquad x \to \lambda\,x\,, \qquad r \to \frac{1}{\lambda}\,r$$

- ⋆ $\nu = 0$ is pure AdS_{d+3}.
- ⋆ $\nu = 1$ corresponds to the Schrödinger algebra.
- ⋆ $\nu = 2$ is relevant for lightlike non-commutative SYM.
- ⋆ We will call such spacetimes $Schr_{d+3}$.

Holography for non-relativistic CFTs

- The metric with $\beta \neq 0$ is sourced by null energy momentum T_{uu}.
- This can be shown to be a solution of Einstein-Hilbert action with negative cosmological constant, with a massive vector field providing the appropriate stress tensor. Son
- In fact, this spacetime has naturally a Galilean causal structure.
- Technically, it belongs to a class of spacetimes that is known as non-distinguishing. Hubeny, MR, Ross

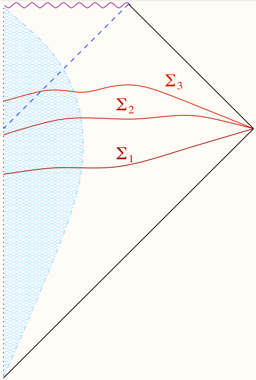
Before we discuss this issue lets take a classical gravity detour.

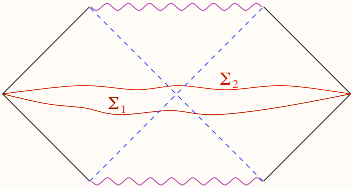
Causality conditions I: Top-Down

1. *Global hyperbolicity:* A spacetime is said to be globally hyperbolic if it admits a Cauchy surface.

Examples

Minkowski space, Schwarzschild black hole.





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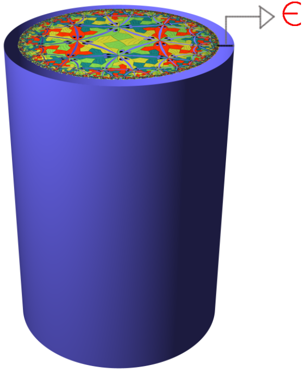
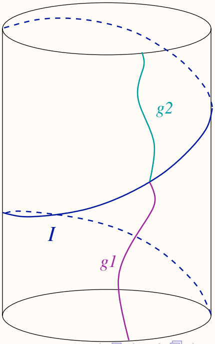
Not-examples

AdS, plane wave geometries.





Causality conditions I: Top-Down

1. *Global hyperbolicity:* A spacetime is said to be globally hyperbolic if it admits a Cauchy surface.
2. *Stable causality:* A stably causal spacetime is one that admits a time-function, i.e.,

$$\exists \text{ smooth } t : \mathcal{M} \to \mathrm{R}, \text{with } \|\nabla_a t\|^2 < 0 \text{ everywhere}$$

Examples

Minkowski space, AdS, plane wave spacetimes.

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3. *Strong causality:* For point $p \in \mathcal{M}$, causal curves passing close to p do not come arbitrarily close to being CCCs.

Causality conditions II: Bottom-Up

1. *Causal:* A causal spacetime is one which is devoid of closed causal curves.

Examples

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Causality conditions II: Bottom-Up

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Not Examples

Gödel,Minkowski space with periodic time identification.

Causality conditions II: Bottom-Up

1. *Causal:* A causal spacetime is one which is devoid of closed causal curves.
2. *Distinguishing:* A spacetime is said to be distinguishing if we can distinguish points on the manifold $\mathcal{M}$ based on their causal sets. For $p, q \in \mathcal{M}$,

$$\mathcal{J}^{\pm}(p) = \mathcal{J}^{\pm}(q) \Rightarrow p = q$$

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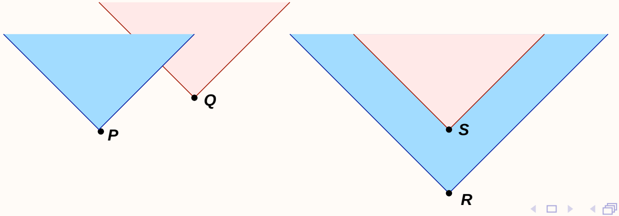

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Not Examples

A large class of pp-wave spacetimes are non-distinguishing.

Hierarchy of causality conditions

The hierarchy

The causality conditions are inclusive:

$$\begin{aligned} \text{Causal} \quad &\Leftarrow \quad \text{Distinguishing} \Leftarrow \text{Strong causality} \\ &\Leftarrow \quad \text{Stable causality} \Leftarrow \text{Global hyperbolicity} \end{aligned}$$

Non-distinguishing pp-wave spacetimes

pp-wave

pp-wave spacetimes are those that admit a covariantly constant, null Killing field, say $\left(\frac{\partial}{\partial v}\right)^a$

$$ds^2 = -2\,du\,dv - f(u, x^i)\,du^2 + dx^i dx^i$$

Non-distinguishing pp-waves

- If $f(u, x^i)$ grows super-quadratically in x^i or is singular at some $x^i = x^i_0$ then the pp-wave is non-distinguishing.
- Require that $f(u, x^i)$ diverges to $+\infty$. Flores, Sanchez, HRR

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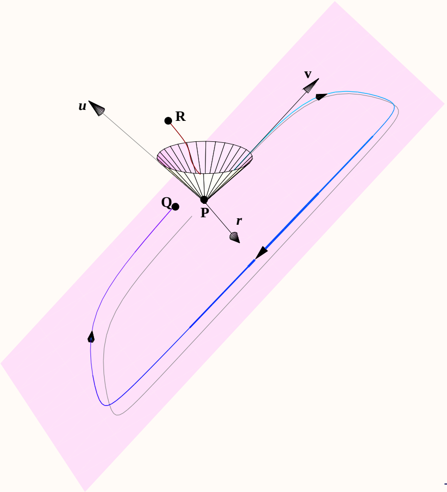
Example

The $Schr_{d+3}$ spacetime is conformal to a pp-wave and hence is non-distinguishing.

Why is the spacetime non-distinguishing?



- The causal future of $p = (u_0, v_0, r_0, \vec{x}_0)$ is the set of points with $u > u_0$.
- So every point on a plane of constant u shares the same causal future.

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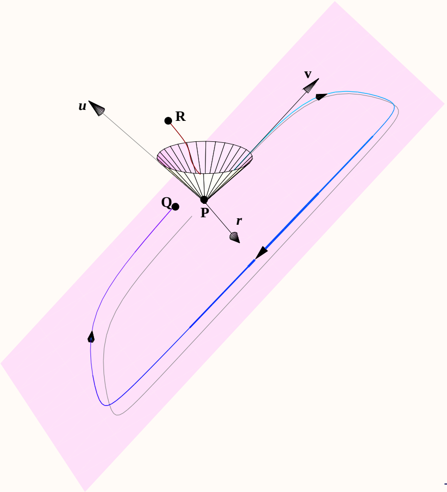

- The geometry despite having local Lorentzian tangent space, achieves a global Galilean light-cone by its non-distinguishing character.

Realization in string theory

- The spacetime dual to Galilean CFTs can be generated from known solutions by a solution generating techinque.
- This technique Null Melvin Twist or TsT transformation maps an asymptotically AdS geometry and converts it into a deformed spacetime with $\beta \neq 0$.

$$\text{TsT} = \text{ T-duality} + \text{shift} + \text{T-duality}$$

- Starting from $\mathrm{AdS}_{\mathrm{d+3}} \times \mathrm{X}$ with X having one U(1) isometry we generate $\mathrm{Schr}_{\mathrm{d+3}} \times_{\mathrm{w}} \mathrm{X}$.

Realization in string theory

- Starting from $AdS_5 \times S^5$ and writing S^5 as S^1 fibration over CP^2 (with fibre ψ) we obtain via NMT

$$\begin{aligned} ds^2 &= r^2 \left(-2\,du\,dv - r^2\,du^2 + dx^2\right) + \frac{dr^2}{r^2} + (d\psi + A)^2 + d\Sigma_4^2, \\ F_{(5)} &= 2\,(1+\star)\,d\psi \wedge J \wedge J, \\ B_{(2)} &= r^2\,du \wedge (d\psi + A), \end{aligned}$$

- This geometry can be reduced to a solution of a 5 dimensional effective theory which is a consistent truncation of IIB supergravity involving a massive vector and 3 scalars. Maldacena, Martelli, Tachikawa

The Dual Field Theory

- The NMT also allows us to infer the dual field theory since we can follow the solution generating technique on the open string side.
- The field theory (for $\nu = 1$) is $\mathcal{N} = 4$ SYM deformed by a (heterotic) star product

$$\mathrm{f} \star \mathrm{g} = \mathrm{e}^{\mathrm{i}\,\beta\left(\mathcal{V}^{\mathrm{f}}\,\mathrm{R}^{\mathrm{g}} - \mathcal{V}^{\mathrm{g}}\,\mathrm{R}^{\mathrm{f}}\right)}\,\mathrm{f}\,\mathrm{g}$$

where $\mathcal{V}$ is the v-momentum of the field and R refers to a global $\mathrm{U}(1)_{\mathrm{R}}$ charge.

Effective Lagrangian

- For purposes of discussing thermodynamics issues we can however truncate to a one scalar model with action

$$
\begin{aligned}
16\pi\, G_5\, \mathcal{S} \;&=\; \int d^5x\sqrt{-g}\left(R - \frac{4}{3}(\partial_\mu\phi)(\partial^\mu\phi) - V(\phi)\right) \\
&+\; \int d^5x\sqrt{-g}\left(\frac{1}{4}e^{-8\phi/3}F_{\mu\nu}F^{\mu\nu} - 4\,A_\mu A^\mu\right)
\end{aligned}
$$

$$
V(\phi) = 4\,e^{2\phi/3}(e^{2\phi} - 4)
$$

- This action needs to be supplemented with appropriate boundary terms.

Outline of the lectures

Lecture 1: Non-relativistic scale invariant theories

- Introduction to Galilean scaling symmetries
- Schrödinger algebra and its realizations
- Liftshitz theories
- Real world systems with Schrödinger symmetries

Lecture 2: Galilean holography

- The holographic dual spacetime
- String theory realization of Schrödinger invariant theories

Lecture 3: Applications of the Galilean hologram

- Thermodynamics & Hydrodynamics
- Correlation functions

Duals for Liftshitz points

- For $\nu \neq 1$ one can write down holographic duals for theories which have anisotropic scaling.

$$\mathrm{ds}^2 = -\mathrm{r}^{2\nu+2}\,\mathrm{dt}^2 + \mathrm{r}^2\,\mathrm{dx}^2 + \frac{\mathrm{dr}^2}{\mathrm{r}^2}$$

Kachru, Liu, Mulligan

- These spacetimes haven't yet been embedded into string theory; however it is possible to write down low energy effective actions which have these spacetimes as solutions.

Duals for Liftshitz points

- One can also realize variants of the Schrodinger spacetimes which different spatio-temporal scaling:

$$ds^2 = r^2 \left(-2\, du\, dv - \beta^2\, r^{2\,\nu}\, du^2 + dx^2\right) + \frac{dr^2}{r^2}$$

- Various values of ν are realized in supergravity theories.
- For some of these embeddings one can indeed find the dual field theory; typically these are non-local deformations of known field theories.

Upshot of stringy embedding

- Given any superconformal theory with U(1) R-symmetry, the twist procedure described above can be used to deform the theory.
- In the holographic context we want to consider the theory at strong coupling $\lambda \gg 1$ and restrict to the planar limit $\mathrm{N} \gg 1$.
- This is an interesting class of non-local quantum field theories which provide examples of Schrödinger invariant theories. Rather different from fermions at unitarity.

Effective Lagrangian for Schrödinger spacetimes

- For purposes of discussing thermodynamics issues we can however truncate to a one scalar model with action

$$16\pi\, G_5\, \mathcal{S} = \int d^5x\sqrt{-g}\left(R - \frac{4}{3}(\partial_\mu\phi)(\partial^\mu\phi) - V(\phi)\right) + \int d^5x\sqrt{-g}\left(\frac{1}{4}e^{-8\phi/3}F_{\mu\nu}F^{\mu\nu} - 4\, A_\mu A^\mu\right)$$

$$V(\phi) = 4\, e^{2\phi/3}(e^{2\phi} - 4)$$

- This action needs to be supplemented with appropriate boundary terms.

Black Hole solution

$$\begin{aligned} ds_E^2 &= r^2\, k(r)^{-\frac{2}{3}} \left(\left[\frac{1-f(r)}{4\beta^2} - r^2\, f(r) \right] du^2 + \frac{\beta^2 r_+^4}{r^4}\, dv^2 - [1+f(r)]\, du\, dv \right) \\ &+ k(r)^{\frac{1}{3}} \left(r^2 dx^2 + \frac{dr^2}{r^2\, f(r)} \right), \end{aligned}$$

$$\begin{aligned} A &= \frac{r^2}{k(r)} \left(\frac{1+f(r)}{2}\, du - \frac{\beta^2 r_+^4}{r^4}\, dv \right), \\ e^{\phi} &= \frac{1}{\sqrt{k(r)}}\,, \end{aligned}$$

$$f(r) = 1 - \frac{r_+^4}{r^4}\,, \qquad k(r) = 1 + \frac{\beta^2\, r_+^4}{r^2}$$

Thermodynamics

- The NMT/TsT does not change the entropy

$$S = \frac{r_+^3\,\beta}{4\,G_5}\Delta v\,V$$

- Note that the canonically normalized Killing generator of the horizon is

$$\xi^a = \left(\frac{\partial}{\partial u}\right)^a + \frac{1}{2\,\beta^2}\,\left(\frac{\partial}{\partial v}\right)^a$$

- This gives the temperature:

$$T = \frac{r_+}{\pi\,\beta}$$

- Moreover, the system is in a grand canonical ensemble with (particle number) chemical potential

$$\mu = \frac{1}{2\,\beta^2}$$

Thermodynamics contd.

- To determine the Gibbs potential of this grand canonical ensemble, we can do an "Euclidean action" computation.
- Analytically continuation of t gives a complex geometry, which leads to a real Euclidean action.

$$I = -\frac{\beta\, r_+^3}{16\, G_5} \Delta v\, V$$

- This action is the identical to the on-shell action (regulated) for the Schwarzschild-AdS black hole.
  - ⋆ The NMT/TsT does not change the leading large N thermodynamic properties (follows from star product).
- Careful analysis of boundary counter-terms required to obtain the result.

Equation of state

- From the Gibbs potential easy to read off

$$\langle \mathrm{E} \rangle = \frac{\pi^3\, \mathrm{T}^4}{64\, \mathrm{G}_5\, \mu^2}\, \Delta \mathrm{v}\, \mathrm{V}$$

$$\langle \mathrm{N} \rangle = \mathrm{P}_{\mathrm{v}} \frac{\Delta \mathrm{v}}{2\,\pi} = \frac{\pi^2\, \mathrm{T}^4}{64\, \mathrm{G}_5\, \mu^3}\, (\Delta \mathrm{v})^2\, \mathrm{V}$$

- This leads to an equation of state

$$\mathrm{E} = \mathrm{P}\, \mathrm{V}$$

which is the non-relativisitc conformal equation of state in 2 spatial dimensions.

- Generalizes to all dimensions easily. Herzog, MR, Ross; Kovtun, Nickel.

Linearized fluctuations

- Study the two point function of the spatial stress tensor $\Pi_{ij}(u, x)$ to learn about η.
- Gravitational computation involves fluctuation analysis about the black hole solution.
- While generically δg, δA and $\delta \phi$ give a coupled system: the shear mode $\delta g_{x_1 x_2}$ decouples.
- In fact $\delta g_{x_1 x_2}$ satisfies massless, minimally coupled wave equation (for zero spatial momentum).

Shear viscosity of the conformal plasma

- Remembering that the stress tensor has zero particle number $\mathrm{P_v} = 0$, the wave equation in fact reduces to that in the Schwarzschild-AdS background, modulo

$$\omega_{\rm AdS} = \beta\, \omega_{\rm Schr}$$

- One can easily compute $\langle\, \Pi_{x_1 x_2}\, \Pi_{x_1 x_2}\, \rangle$ at zero spatial momentum and read off η using a Kubo formula.
- One finds

$$\frac{\eta}{s} = \frac{1}{4\,\pi}$$

- Finally, note that non-relativisitic conformal invariance requires that the bulk viscosity vanish; $\zeta = 0$.

Non-relativistic hydrodynamics

Aim: Derive the hydrodynamic equations for the non-relativistic plasma from gravity using the fluid-gravity correspondence.

The Hard Way

- Take the asymptotically Schr_{d+3} black hole and generalize it to a $d + 2$ parameter solution (d Galilean velocities $\mathfrak{v}_i$.)
- Promote r_+, β and $\mathfrak{v}_i$ to fields depending on $\{u, x\}$.
- Solve bulk gravity equations order by order in derivatives of $\{u, x\}$ for asymptotically Schr_{d+3} solutions.
- Gravity constraint equations → Navier-Stokes equations.
- Asymptotic fall-off conditions → 'boundary' stress tensor.

Non-relativistic hydrodynamics

Aim: Derive the hydrodynamic equations for the non-relativistic plasma from gravity using the fluid-gravity correspondence.

The Short-Cut

- Leading planar physics of the non-relativistic theory is the same as the parent relativistic theory.
- Obtain the stress tensor complex for the non-relativistic theory by reducing the corresponding relativistic stress tensor on the light-cone (along v).
- The bulk metric is obtained by TsT transformation of the asymptotically AdS fluid black hole solutions (with $\partial_{\rm v}$ being the null Killing vector).

Relativistic & non-relativistic hydrodynamics

Equations for ideal relativistic hydrodynamics: These are just conservation of energy-momentum tensor and are $d + 2$ equations for $d + 2$ variables (fluids on $R^{d+1,1}$)

$$\nabla_\mu T^{\mu\nu} = 0.$$

$$T^{\mu\nu} = (\epsilon_{rel} + P_{rel})\, u^\mu\, u^\nu + P_{rel}\, \eta^{\mu\nu} \ ,$$

Relativistic & non-relativistic hydrodynamics

Equations for ideal non-relativistic hydrodynamics: These are again conservation equations:

Continuity equation: $\partial_t \rho + \partial_i \left(\rho\, \mathfrak{v}^i\right) = 0,$

Momentum conservation: $\partial_t(\rho\, \mathfrak{v}^i) + \partial_j \Pi^{ij} = 0,$

Energy conservation: $\partial_t \left(\varepsilon + \frac{1}{2}\,\rho\, \mathfrak{v}^2\right) + \partial_i\, j^i_\varepsilon = 0,$

where we have defined

spatial stress tensor: $\Pi^{ij} = \rho\, \mathfrak{v}^i\, \mathfrak{v}^j + \delta^{ij} P$

energy flux: $j^i_\varepsilon = \frac{1}{2}\,(\varepsilon + P)\, \mathfrak{v}^2\, \mathfrak{v}^i$

Light-cone reduction of ideal relativistic hydrodynamics

Consider the relativistic stress tensor in light-cone coordinates $x^{\pm} = \{u, v\}$.

$$\partial_+ T^{++} + \partial_i T^{+i} = 0 \ , \qquad \partial_+ T^{+i} + \partial_j T^{ij} = 0 \ , \qquad \partial_+ T^{+-} + \partial_i T^{-i} = 0,$$

which allows us to identify

$$T^{++} = \rho, \quad T^{+i} = \rho\, \mathfrak{v}^i, \quad T^{ij} = \Pi^{ij},$$
$$T^{+-} = \varepsilon + \frac{1}{2}\, \rho\, \mathfrak{v}^2, \qquad T^{-i} = j^i_{\varepsilon}.$$

Light-cone reduction of ideal relativistic hydrodynamics

The map between relativistic and non-relativistic variables:

$$\begin{aligned} \mathfrak{u}^+ &= \sqrt{\frac{1}{2}\,\frac{\rho}{\varepsilon + \mathrm{P}}}\,, \qquad && \mathfrak{u}^{\mathrm{i}} = \mathfrak{u}^+\,\mathfrak{v}^{\mathrm{i}}, \\ \mathrm{P}_{\mathbf{rel}} &= \mathrm{P}\,, && \epsilon_{\mathbf{rel}} = 2\,\varepsilon + \mathrm{P}. \end{aligned}$$

The component of the relativistic velocity $\mathfrak{u}^-$ can be determined using the normalization condition $\mathfrak{u}_\mu\,\mathfrak{u}^\mu = -1$ to be

$$\mathfrak{u}^- = \frac{1}{2}\left(\frac{1}{\mathfrak{u}^+} + \mathfrak{u}^+\,\mathfrak{v}^2\right).$$

Light-cone reduction of viscous relativistic hydrodynamics

- The map can be extended to incorporate dissipative effects.
- The conformal relativistic stress tensor at first order reads:

$$T^{\mu\nu} = (\epsilon_{\rm rel} + P_{\rm rel})\, u^\mu\, u^\nu + \eta^{\mu\nu} P_{\rm rel} - 2\, \eta_{\rm rel}\, \tau^{\mu\nu}$$

  with $\tau^{\mu\nu}$ being the shear tensor.
- Light-cone reduction is as before, with derivative corrections to the map between velocities.
- Can use the map to derive the non-relativistic transport coefficients at first order.

Light-cone reduction of viscous relativistic hydrodynamics

Non-relativistic transport coefficients:

- We find for the shear viscosity

$$\eta_{\text{rel}} = \frac{\eta}{\mathfrak{u}^+} \; .$$

- The heat conductivity is given by

$$\kappa = 2\,\eta\,\frac{\varepsilon + \mathrm{P}}{\rho\,\mathrm{T}} \; .$$

- The dimensionless ratio Prandtl number defined as the ratio of kinematic viscosity ν to thermal diffusivity χ is 1.

$$\mathrm{Pr} = \frac{\nu}{\chi} \; , \qquad \nu = \frac{\eta}{\rho} \; , \qquad \chi = \frac{\kappa}{\rho\,c_{\mathrm{p}}}$$

,

Schrödinger correlators

- Can use the Galilean hologram to discuss correlation functions of quasi-primary operators.
- Schrödinger algebra constrains two point functions:

$$\langle\, \mathcal{O}(t, x)\, \mathcal{O}^\dagger(0,0)\,\rangle \propto t^{-\Delta_{\mathcal{O}}}\, e^{-i\, N_{\mathcal{O}}\, \frac{x^2}{2\,t}}$$

- Can derive correlation functions using a minor modification of AdS/CFT:

$$\langle\, e^{\int_{\mathcal{B}}\, \varphi_0(x)\, \mathcal{O}(x)}\,\rangle_{\text{CFT}} = \mathcal{Z}_{\text{string}}\left[\varphi(r, x)\big|_{\mathcal{B}} = \varphi_0(x)\right]$$

where we impose boundary conditions at $r = R_c \gg 1$.

Balasubramainan K, McGreevy; Fuertes, Moroz; Volovich, Wen

Salient points

- Holographic dual for system with Galilean conformal invariance, using D-brane construction.
- D-branes probing a Null Melvin geometry naturally give rise to such non-relativistic CFTs.
- Discussed thermodynamics and some hydrodynamic properties of such plasmas.
- As usual, brane engineering leads to systems where η/s takes on the universal value $1/4\pi$.
- Can discuss conformal non-relativistic hydrodynamics for the system: derived transport coefficients at first order and constructed dual gravity solutions.