Holography for non-relativistic CFTs

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Outline of the lectures

Lecture 1: Non-relativistic scale invariant theories
- Introduction to Galilean scaling symmetries
- Schrödinger algebra and its realizations
- Lifshitz theories
- Real world systems with Schrödinger symmetries

Lecture 2: Galilean holography
- The holographic dual spacetime
- String theory realization of Schrödinger invariant theories

Lecture 3: Applications of the Galilean hologram
- Thermodynamics & Hydrodynamics
- Correlation functions
Motivation

Holographic models for strongly coupled systems

• The AdS/CFT correspondence allows us to probe the physics of strongly coupled gauge theories.
  * Insight into transport properties of QGP, relevant for physics seen in heavy-ion collisions.

• There are other strongly coupled systems discussed in condensed matter literature which exhibit a wide range of extremely interesting physics.

• Use holographic methods to find the classical “Master field” for these theories.
Introduction

Motivation

New insights into Quantum Gravity

- AdS/CFT has a dual role: it allows us to probe quantum aspects of gravity in terms of a non-perturbatively well defined QFT.
- Generalizations of the AdS/CFT correspondence, to new terrains has the potential to unveil important lessons for quantum gravity.

Understanding fluid dynamics

- The mathematical structure of Navier-Stokes equations (non-relativistic) poses interesting challenges.
- Can we reformulate the Fluid-Gravity correspondence in a context relevant for non-relativistic fluids?

Bhattacharyya, Hubeny, Minwalla, MR
New insights into Quantum Gravity

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Bhattacharyya, Hubeny, Minwalla, MR
Introduction

Motivation

Experimental relevance

- There is currently an intensive experimental effort to understand the physics of cold atoms.
- These systems seem to admit an hydrodynamic description in terms of a nearly-ideal fluid.
  - The energy per particle is about 50% of the free value, similar in spirit to the Stephan-Boltzmann saturation of QGP just above the deconfinement transition.
  - Experimental results of elliptic type flow (shear driven relaxation) give $\eta/s \sim 1/\pi$!
- Can we find systems that have holographic duals which share at least some of the symmetries exhibited in these cold atom systems?
• *Proposal for holographic duals*
  * Son: 0804.3972
  * Balasubramanian K, McGreevy: 0804.4053

• *Holographic embedding in string theory, etc.*
  * Herzog, MR, Ross: 0807.1099
  * Maldacena, Martelli, Tachikawa: 0807.1100
  * Adams, Balasubramanian K, McGreevy: 0807.1111

• *Fluid dynamics*
  * MR, Ross, Son, Thompson: 0711.2049

• *Related work*
  * Goldberger: 0705.2867
  * Barbon, Fuertes: 0705.3244

• *Earlier relevant work*
  * Nishida, Son: 0706.3746
  * Hubeny, MR, Ross: hep-th/0504034
The Schrödinger algebra

- The Schrödinger algebra is the symmetry algebra of the free Schrödinger operator in $d + 1$ dimensions.
- It is generated by operators that commute with
  \[ S = i \partial_t + \frac{1}{2m} \partial_i^2 \]
- It is analog of the conformal algebra for relativistic systems – we will see how to relate the two shortly.
- It is believed that the system of cold atoms at unitarity is an example of an interacting QFT which realizes this symmetry.
The Schrödinger group

- One can write down the Schrödinger group as the following set of transformations:

  \[ x \rightarrow x' = \frac{\mathcal{R} x + \mathbf{v} t + \mathbf{a}}{\gamma t + \delta} \]

  \[ t \rightarrow t' = \frac{\alpha t + \beta}{\gamma t + \delta} \]

with \( \alpha \delta - \beta \gamma = 1 \).

- The group includes, spatial translations indicated by \( \mathbf{a} \), rotations captured by \( \mathcal{R} \), Galilean boosts with velocity \( \mathbf{v} \), a scale transformation and a special conformal generator.

- We will derive the algebra momentarily by employing a useful trick.
Apart from the Schrödinger algebra there is another conformal algebra which includes the Galilean algebra as a sub-algebra – this is the Galilean Conformal Algebra (GCA).

The two algebras are quite distinct; they have different numbers of generators and also treat the dilatation generator differently.

One can view the GCA as a contraction of the conformal algebra obtained by sending $c \to \infty$.

The Schrödinger algebra on the other hand requires us to rescale the particle mass as well.

We will mostly focus on the Schrödinger algebra in these lectures.
Light-cone reductions

- Recall that one can get the Galilean algebra in $d$ dimensions by reducing the Poincaré algebra $SO(d + 1, 1)$ on light-cone

  \[ u = t + y, \quad v = t - y \]

- Propagation in light-cone time $u$ respects Galilean invariance.

- We can similarly reduce the conformal algebra $SO(d + 2, 2)$ in $d + 2$ dimensions on a light-cone to obtain the **Schrödinger** algebra in $d$-spatial dimensions.
The Schrödinger algebra: Generators

Starting from the conformal algebra we keep all generators which commute with the particle number.

- Hamiltonian: $H$
- Spatial rotations: $M_{ij}$
- Spatial momenta: $P_i$
- Galilean boosts: $K_i$
- Dilatation: $D$
- Special conformal generator: $C$
- Particle number: $N$

where we are restricting attention to $d$-spatial dimensions, i.e., \( \{i,j\} \in \{1, \ldots d\} \).
### The Schrödinger algebra from conformal algebra

<table>
<thead>
<tr>
<th>Generator</th>
<th>Galilean</th>
<th>Conformal</th>
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<tr>
<td>Particle number</td>
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<td>P&lt;sub&gt;v&lt;/sub&gt;</td>
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<td>Hamiltonian</td>
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<td>Dilatation</td>
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<tr>
<td>Special conformal</td>
<td>C</td>
<td>K&lt;sub&gt;v&lt;/sub&gt;</td>
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</table>
The Schrödinger algebra: Commutation relations

\[
\begin{align*}
[M_{ij}, M_{kl}] &= i \left( \delta_{ik} M_{jl} - \delta_{jk} M_{il} + \delta_{il} M_{kj} - \delta_{jl} M_{ki} \right) \\
[M_{ij}, P_k] &= i \left( \delta_{ik} P_j - \delta_{jk} P_i \right) \\
[M_{ij}, K_k] &= i \left( \delta_{ik} K_j - \delta_{jk} K_i \right) \\
[M_{ij}, H] &= [M_{ij}, D] = [M_{ij}, C] = 0 \\
[P_i, P_j] &= [K_i, K_j] = 0, \quad [K_i, P_j] = i \delta_{ij} N \\
[D, P_i] &= i P_i, \quad [D, K_i] = -i K_i \\
[H, P_i] &= 0, \quad [H, K_i] = -i P_i \\
[C, P_i] &= i K_i, \quad [C, K_i] = 0 \\
[D, H] &= 2i H, \quad [D, C] = -2i C \\
[H, C] &= -i D
\end{align*}
\]
• From the commutation relations descending from the conformal algebra one can infer that

\[ [H, D] = -2iH \]

• This implies that the Hamiltonian has scaling dimension 2.

• Intuitively, this follows from the fact that non-relativistic systems are first order in time, leading to scaling

\[ t \rightarrow \lambda^2 t, \quad x \rightarrow \lambda x \]
Galilean Conformal Symmetry

Scaling dimensions and representations

• Representation of Schrödinger algebra in terms of highest weight states as usual.

• In particular, we will talk about 2 quantum numbers
  ★ The scaling dimension:

  \[ [D, \varnothing] = i \Delta_\varnothing \varnothing \]

  ★ The particle number:

  \[ [N, \varnothing] = N_\varnothing \varnothing \]

• We have \( \Delta_H = 2 \) and \( \Delta_P = 1 \).
Scaling dimensions and representations

- We will realize highest weight representations in terms of primary operators which have a given conformal dimension $\Delta_\mathcal{O}$ and particle number $N_\mathcal{O}$.

- As usual the spacetime dependence of the operator can be inferred via translation:

  $$\mathcal{O}(t, x) = e^{iHt - iP_i x_i} \mathcal{O}(0) e^{-iHt + iP_i x_i}$$

- The primary operators are defined so that lowering operators $K$ and $C$ (which have scaling dimensions $-1$ and $-2$ respectively) annihilate it i.e.,

  $$[K_i, \mathcal{O}] = [C, \mathcal{O}] = 0$$
Scaling dimensions and representations

One can give a simple representation of the algebra in terms using the usual derivative representation. For an operator $\mathcal{O}(t, x)$:

\[
\begin{align*}
[H, \mathcal{O}] &= -i \partial_t \mathcal{O} \\
[P_i, \mathcal{O}] &= i \partial_i \mathcal{O} \\
[D, \mathcal{O}] &= i (2t \partial_t + x_i \partial_i + \Delta_{\mathcal{O}}) \mathcal{O} \\
[K_i, \mathcal{O}] &= (-i t \partial_i + N_{\mathcal{O}} x_i) \mathcal{O} \\
[C, \mathcal{O}] &= -i (t^2 p_t + t x_i \partial_i + t \Delta_{\mathcal{O}}) \mathcal{O}
\end{align*}
\]

which in particular implies that the quasi-primary operators satisfy

\[
e^{-i \lambda D} \mathcal{O}(t, x) e^{i \lambda D} = e^{\lambda \Delta_{\mathcal{O}}} \mathcal{O} \left(e^{2 \lambda t}, e^{\lambda x}\right)
\]
State-operator correspondence

- Primary operators are in one-one correspondence with the eigenstates of a quantum system in a harmonic trap.
- The state
  \[ |\psi_\mathcal{O}\rangle = e^{-\mathcal{H}_\mathcal{O}^\dagger} |0\rangle \]
  is an eigenstate of the Hamiltonian \( H_{\text{osc}} = H + C \) with eigenvalue \( \Delta_\mathcal{O} \).
- The Schrödinger algebra has a \( SL(2, R) \) sub-algebra generated by \{D, H, C\}.

\[
H_{\text{osc}} = \frac{1}{2} (H + C) \\
\mathcal{A}^\dagger = \frac{1}{2} (H - C + iD) \\
a = \frac{1}{2} (H - C - iD)
\]
Lifshitz points

- We can also consider more general scaling, but not conformal symmetries.
- These are described by a real number $z = 1 + \nu$.
- We assign weight $-\nu$ to $K_i$ and $1 + \nu$ to $H$.
- The commutation relations are deformed to

$$[D, H] = i(1 + \nu)H, \quad [D, N] = -i(\nu - 1)N$$

$$[D, K_i] = -i\nu K_i$$

- For $\nu \neq 1$ we don’t have a conserved particle number and the special conformal generator $C$ does not exist in the algebra.
- These describe generalized scaling

$$t \rightarrow \lambda^{1+\nu} t \quad x \rightarrow \lambda x$$
Fermions at unitarity

- Cold atom systems are an increasingly interesting arena to explore a wide range of physical phenomena.
- Fermionic Li\textsuperscript{6} or K\textsuperscript{40} in optical traps provide systems of fermionic gases where inter-atomic interactions can be externally tuned to produce different phases.
- The quantity of interest is the s-wave scattering length $a$; tuning $a$ one can pass from a BEC condensate to a BCS superfluid.
  - Small negative $a$ leads to weak attractive interaction – BCS limit.
  - As $a \to \infty$ we achieve the unitarity limit as the s-wave cross section is saturated.
  - For positive scattering length is the BEC phase where the fermions form deeply bound molecules.
Fermions at unitarity

- Tuning \( a \) is achieved by external magnetic field with the fermionic atoms in an optical trap.
- Exactly at threshold one obtains a massless bound state, and the theory is supposed to be described as a non-relativistic conformal field theory with Schrödinger symmetry.
- Experimental studies of this fluid suggest that it is another example of a nearly-ideal fluid with \( \eta/s \sim 1/\pi \).
- Fixed points are known to exist in \( \epsilon \) expansion around 2 dimensions and 4 dimensions.

Schäfer, Teaney
Nishida, Son.
Statement of the AdS/CFT correspondence

AdS/CFT

Quantum gravity on asymptotically d-dimensional Anti-deSitter spacetime is described by a d − 1 dimensional gauge theory sans gravity.

A particularly appealing and testable form of the conjecture:

Type IIB string theory on AdS$_5 \times S^5$ spacetime.

$R_{\text{AdS}}, g_s$ $\leftrightarrow$ $g_{\text{YM}}^2, N$
**Statement of the AdS/CFT correspondence**

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**AdS/CFT**

Quantum gravity on asymptotically d-dimensional Anti-deSitter spacetime is described by a \( d - 1 \) dimensional gauge theory sans gravity.

A particularly appealing and testable form of the conjecture:

\[
\text{Type IIB string theory on } \text{AdS}_5 \times S^5 \quad \equiv \quad \text{4-dimensional superconformal Yang-Mills gauge theory.}
\]

\[
\text{R}_{\text{AdS}}, g_s \quad \leftrightarrow \quad g_{YM}^2, N
\]
AdS/CFT continued

$$g_{YM}^2 N \gg 1$$

$$N \gg 1$$
\[ g_{YM}^2 N \ll 1 \]
\[ N \gg 1 \]
Motivating the correspondence

- Start with $N$ D3-branes in flat space. The world-volume is $\mathbb{R}^{3,1} \subset \mathbb{R}^{9,1}$.

- This has two equivalent descriptions in string theory:
  - As open strings ending on D3 interacting with closed strings in the bulk
  - As purely closed strings in a back-reacted spacetime.

- A suitable decoupling limit $\ell_s \to 0$ zooms in onto the dynamics of just the open strings whilst in the geometric picture we focus on a region of the full spacetime.

- Effectively, closing the holes on the world-sheet leads to a pure closed string description.
Salient features of the AdS/CFT correspondence

- Symmetry matching: the $SO(4, 2) \times SO(6) \subset PSU(2, 2|4)$ global symmetry of field theory are realized as isometries of the spacetime.

- Local gauge invariant single trace operators of the field theory such as $\mathcal{O} = \text{Tr}(X \cdots X)$ are mapped to single particle states in the super-gravity description.

- There exists a precise prescription to compute the generating function of correlation functions for these gauge invariant operators:

$$\langle e^{\mathcal{B} \varphi_0(x) \mathcal{O}(x)} \rangle_{\text{CFT}} = \mathcal{Z}_{\text{string}} [\varphi(r, x)|_{\mathcal{B}} = \varphi_0(x)]$$
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AdS/CFT basics

- Consider the geometry of $\text{AdS}_{d+3}$
  \[
  ds^2 = -r^2 \, dt^2 + r^2 \, dx^2 + \frac{dr^2}{r^2}
  \]
  which is the metric covering the Poincaré patch of AdS.
- $\text{AdS}_{d+3}$ has the $SO(d + 2, 2)$ isometry algebra of which we can look at the scaling symmetry
  \[
  t \rightarrow \lambda \, t \, , \quad x \rightarrow \lambda \, x \, , \quad r \rightarrow \frac{1}{\lambda} \, r
  \]
AdS/CFT basics

- This is the familiar scale transformations for the relativistic CFT on $\mathbb{R}^{1,d+1}$ which is the boundary of AdS$_{d+3}$ in Poincaré coordinates.
- The radial direction is holographically said to correspond to a energy scale in the field theory, cf., the holographic renormalization group.
- We can map out the other symmetries as well similarly in terms of AdS isometries.
Holography for non-relativistic CFTs: DLCQ

- Consider the scaling symmetry

\[ t \rightarrow \lambda^{\nu+1} t , \quad x \rightarrow \lambda x \]

- This can be achieved by starting from AdS_{d+3} in light-cone coordinates

\[ ds^2 = r^2 \left( -2 \, du \, dv + dx^2 \right) + \frac{dr^2}{r^2} \]

and define an unconventional scaling

\[ u \rightarrow \lambda^{\nu+1} u , \quad v \rightarrow \lambda^{1-\nu} v , \quad x \rightarrow \lambda x , \quad r \rightarrow \frac{1}{\lambda} r \]

and interpreting u as time.
This Galilean symmetry is familiar from DLCQ.

In fact, this is essentially the observation that DLCQ of any relativistic theory gives a Galilean invariant model in a sector with fixed light-cone momentum.

However, we should be careful about the zero mode.

Finally, the underlying theory is relativistic – the Galilean symmetry is an artifact of our choice of light-cone quantization.
Holography for non-relativistic CFTs

- To motivate a dual that has manifest Galilean scaling consider

\[ ds^2 = r^2 \left( -2 \, du \, dv - \beta^2 \, r^2 \nu \, du^2 + dx^2 \right) + \frac{dr^2}{r^2} \]

which naturally has the required scaling

\[ u \rightarrow \lambda^{\nu+1} \, u , \quad v \rightarrow \lambda^{1-\nu} \, v , \quad x \rightarrow \lambda \, x , \quad r \rightarrow \frac{1}{\lambda} \, r \]

★ \( \nu = 0 \) is pure AdS_{d+3}.
★ \( \nu = 1 \) corresponds to the Schrödinger algebra.
★ \( \nu = 2 \) is relevant for lightlike non-commutative SYM.
★ We will call such spacetimes Schr_{d+3}.
Holography for non-relativistic CFTs

- The metric with \( \beta \neq 0 \) is sourced by null energy momentum \( T_{uu} \).
- This can be shown to be a solution of Einstein-Hilbert action with negative cosmological constant, with a massive vector field providing the appropriate stress tensor.
- In fact, this spacetime has naturally a Galilean causal structure.
- Technically, it belongs to a class of spacetimes that is known as non-distinguishing.

Before we discuss this issue lets take a classical gravity detour.
**Causality conditions I: Top-Down**

1. **Global hyperbolicity:** A spacetime is said to be globally hyperbolic if it admits a Cauchy surface.

**Examples**

- Minkowski space, Schwarzschild black hole.
Causality conditions I: Top-Down

1. **Global hyperbolicity**: A spacetime is said to be globally hyperbolic if it admits a Cauchy surface.

**Not-examples**

AdS, plane wave geometries.
Causality conditions I: Top-Down

1. **Global hyperbolicity:** A spacetime is said to be globally hyperbolic if it admits a Cauchy surface.

2. **Stable causality:** A stably causal spacetime is one that admits a time-function, i.e.,

   \[ \exists \text{ smooth } t : \mathcal{M} \to \mathbb{R}, \text{ with } \| \nabla a t \|^2 < 0 \text{ everywhere} \]

**Examples**

Minkowski space, AdS, plane wave spacetimes.
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1. **Global hyperbolicity:** A spacetime is said to be globally hyperbolic if it admits a Cauchy surface.

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\[ \exists \text{ smooth } t : M \rightarrow \mathbb{R}, \text{ with } \| \nabla_a t \|^2 < 0 \text{ everywhere} \]

3. **Strong causality:** For point \( p \in M \), causal curves passing close to \( p \) do not come arbitrarily close to being CCCs.
Causality conditions II: Bottom-Up

1. **Causal:** A causal spacetime is one which is devoid of closed causal curves.

**Examples**

Minkowski space, AdS, plane wave spacetimes.
1. **Causal**: A causal spacetime is one which is devoid of closed causal curves.

**Not Examples**

Gödel, Minkowski space with periodic time identification.
Causality conditions II: Bottom-Up

1. **Causal:** A causal spacetime is one which is devoid of closed causal curves.
2. **Distinguishing:** A spacetime is said to be distinguishing if we can distinguish points on the manifold $\mathcal{M}$ based on their causal sets. For $p, q \in \mathcal{M}$,
   \[ J^\pm(p) = J^\pm(q) \Rightarrow p = q \]

**Examples**

Minkowski space, AdS, plane wave spacetimes.
1. **Causal:** A causal spacetime is one which is devoid of closed causal curves.

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**Not Examples**

A large class of pp-wave spacetimes are non-distinguishing.
Hierarchy of causality conditions

The hierarchy

The causality conditions are inclusive:

Causal $\iff$ Distinguishing $\iff$ Strong causality
$\iff$ Stable causality $\iff$ Global hyperbolicity
Non-distinguishing \emph{pp-wave} spacetimes

\textbf{pp-wave}

pp-wave spacetimes are those that admit a covariantly constant, null Killing field, say \((\frac{\partial}{\partial v})^a\)

\[
ds^2 = -2 \, du \, dv - f(u, x^i) \, du^2 + dx^i dx^i
\]

\textbf{Non-distinguishing pp-waves}

- If \(f(u, x^i)\) grows super-quadratically in \(x^i\) or is singular at some \(x^i = x^i_0\) then the pp-wave is non-distinguishing.
- Require that \(f(u, x^i)\) diverges to \(+\infty\).

Flores, Sanchez, HRR

Example

The Schrödinger spacetime is conformal to a pp-wave and hence is non-distinguishing.
**Non-distinguishing pp-wave spacetimes**

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Flores, Sanchez, HRR
Non-distinguishing pp-wave spacetimes

Non-distinguishing pp-waves

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- Require that \( f(u, x^i) \) diverges to \(+\infty\).

Example

The \( \text{Schr}_{d+3} \) spacetime is conformal to a pp-wave and hence is non-distinguishing.
Why is the spacetime non-distinguishing?

- The causal future of \( p = (u_0, v_0, r_0, \vec{x}_0) \) is the set of points with \( u > u_0 \).
- So every point on a plane of constant \( u \) shares the same causal future.
Why is the spacetime non-distinguishing?

- The geometry despite having local Lorentzian tangent space, achieves a global Galilean light-cone by its non-distinguishing character.
The spacetime dual to Galilean CFTs can be generated from known solutions by a solution generating technique.

This technique Null Melvin Twist or TsT transformation maps an asymptotically AdS geometry and converts it into a deformed spacetime with $\beta \neq 0$.

$$\text{TsT} = \text{T-duality} + \text{shift} + \text{T-duality}$$

Starting from $\text{AdS}_{d+3} \times X$ with $X$ having one $U(1)$ isometry we generate $\text{Schr}_{d+3} \times_w X$.  

Realization in string theory

• Starting from AdS$_5 \times S^5$ and writing $S^5$ as $S^1$ fibration over CP$^2$ (with fibre $\psi$) we obtain via NMT

$$ds^2 = r^2 \left(-2 \, du \, dv - r^2 \, du^2 + dx^2\right) + \frac{dr^2}{r^2} + (d\psi + A)^2 + d\Sigma_4^2,$$

$$F_{(5)} = 2 \left(1 + \star\right) d\psi \wedge J \wedge J,$$

$$B_{(2)} = r^2 \, du \wedge (d\psi + A),$$

• This geometry can be reduced to a solution of a 5 dimensional effective theory which is a consistent truncation of IIB supergravity involving a massive vector and 3 scalars.  

Maldacena, Martelli, Tachikawa
The Dual Field Theory

- The NMT also allows us to infer the dual field theory since we can follow the solution generating technique on the open string side.
- The field theory (for $\nu = 1$) is $N = 4$ SYM deformed by a (heterotic) star product

$$f \star g = e^{i \beta (V^f R^g - V^g R^f)} f g$$

where $V$ is the v-momentum of the field and $R$ refers to a global $U(1)_R$ charge.
Effective Lagrangian

• For purposes of discussing thermodynamics issues we can however truncate to a one scalar model with action

\[
16\pi G_5 S = \int d^5 x \sqrt{-g} \left( R - \frac{4}{3} (\partial_\mu \phi)(\partial^\mu \phi) - V(\phi) \right) + \int d^5 x \sqrt{-g} \left( \frac{1}{4} e^{-8\phi/3} F_{\mu\nu} F^{\mu\nu} - 4 A_\mu A^\mu \right)
\]

\[
V(\phi) = 4 e^{2\phi/3} (e^{2\phi} - 4)
\]

• This action needs to be supplemented with appropriate boundary terms.
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Duals for Lifshitz points

- For $\nu \neq 1$ one can write down holographic duals for theories which have anisotropic scaling.

$$ds^2 = -r^{2\nu+2} dt^2 + r^2 dx^2 + \frac{dr^2}{r^2}$$

- These spacetimes haven’t yet been embedded into string theory; however it is possible to write down low energy effective actions which have these spacetimes as solutions.
One can also realize variants of the Schrodinger spacetimes which
different spatio-temporal scaling:

\[ ds^2 = r^2 (-2 \, du \, dv - \beta^2 r^2 \nu \, du^2 + dx^2) + \frac{dr^2}{r^2} \]

Various values of \( \nu \) are realized in supergravity theories.

For some of these embeddings one can indeed find the dual field
theory; typically these are non-local deformations of known field
theories.
Upshot of stringy embedding

- Given any superconformal theory with U(1) R-symmetry, the twist procedure described above can be used to deform the theory.
- In the holographic context we want to consider the theory at strong coupling $\lambda \gg 1$ and restrict to the planar limit $N \gg 1$.
- This is an interesting class of non-local quantum field theories which provide examples of Schrödinger invariant theories. Rather different from fermions at unitarity.
Effective Lagrangian for Schrödinger spacetimes

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\[
16\pi G_5 S = \int d^5x \sqrt{-g} \left( R - \frac{4}{3} (\partial_\mu \phi)(\partial^\mu \phi) - V(\phi) \right) \\
+ \int d^5x \sqrt{-g} \left( \frac{1}{4} e^{-8\phi/3} F_{\mu\nu} F^{\mu\nu} - 4 A_\mu A^\mu \right)
\]

\[
V(\phi) = 4 e^{2\phi/3} (e^{2\phi} - 4)
\]

- This action needs to be supplemented with appropriate boundary terms.
Black Hole solution

\[ ds^2_E = r^2 k(r)^{-\frac{2}{3}} \left( \left[ \frac{1 - f(r)}{4\beta^2} - r^2 f(r) \right] du^2 + \frac{\beta^2 r^4}{r^4} dv^2 - \left[ 1 + f(r) \right] du dv \right) + k(r)^{\frac{1}{3}} \left( r^2 dx^2 + \frac{dr^2}{r^2 f(r)} \right) , \]

\[ A = \frac{r^2}{k(r)} \left( \frac{1 + f(r)}{2} du - \frac{\beta^2 r^4}{r^4} dv \right) , \]

\[ e^\phi = \frac{1}{\sqrt{k(r)}} , \]

\[ f(r) = 1 - \frac{r^4_+}{r^4} , \quad k(r) = 1 + \frac{\beta^2 r^4_+}{r^2} \]
Thermodynamics

- The NMT/TsT does not change the entropy

\[ S = \frac{r_+^3 \beta}{4 G_5} \Delta v V \]

- Note that the canonically normalized Killing generator of the horizon is

\[ \xi^a = \left( \frac{\partial}{\partial u} \right)^a + \frac{1}{2 \beta^2} \left( \frac{\partial}{\partial v} \right)^a \]

- This gives the temperature:

\[ T = \frac{r_+}{\pi \beta} \]

- Moreover, the system is in a grand canonical ensemble with (particle number) chemical potential

\[ \mu = \frac{1}{2 \beta^2} \]
Thermodynamics contd.

- To determine the Gibbs potential of this grand canonical ensemble, we can do an "Euclidean action" computation.
- Analytically continuation of $t$ gives a complex geometry, which leads to a real Euclidean action.

$$I = -\frac{\beta r_+^3}{16 G_5} \Delta v V$$

- This action is the identical to the on-shell action (regulated) for the Schwarzschild-AdS black hole.
  - The NMT/TsT does not change the leading large $N$ thermodynamic properties (follows from star product).
- Careful analysis of boundary counter-terms required to obtain the result.
Equation of state

- From the Gibbs potential easy to read off

\[ \langle E \rangle = \frac{\pi^3 T^4}{64 G_5 \mu^2} \Delta v \ V \]

\[ \langle N \rangle = P_v \frac{\Delta v}{2\pi} = \frac{\pi^2 T^4}{64 G_5 \mu^3} (\Delta v)^2 \ V \]

- This leads to an equation of state

\[ E = P \ V \]

which is the non-relativistic conformal equation of state in 2 spatial dimensions.

- Generalizes to all dimensions easily.

Herzog, MR, Ross; Kovtun, Nickel.
Linearized fluctuations

- Study the two point function of the spatial stress tensor $\Pi_{ij}(u,x)$ to learn about $\eta$.
- Gravitational computation involves fluctuation analysis about the black hole solution.
- While generically $\delta g, \delta A$ and $\delta \phi$ give a coupled system: the shear mode $\delta g_{x_1 x_2}$ decouples.
- In fact $\delta g_{x_1 x_2}$ satisfies massless, minimally coupled wave equation (for zero spatial momentum).
Shear viscosity of the conformal plasma

• Remembering that the stress tensor has zero particle number $P_v = 0$, the wave equation in fact reduces to that in the Schwarzschild-AdS background, modulo

$$\omega_{\text{AdS}} = \beta \omega_{\text{Schr}}$$

• One can easily compute $\langle \Pi_{x_1 x_2} \Pi_{x_1 x_2} \rangle$ at zero spatial momentum and read off $\eta$ using a Kubo formula.

• One finds

$$\frac{\eta}{s} = \frac{1}{4\pi}$$

• Finally, note that non-relativisititc conformal invariance requires that the bulk viscosity vanish; $\zeta = 0$. 
Non-relativistic hydrodynamics

**Aim:** Derive the hydrodynamic equations for the non-relativistic plasma from gravity using the fluid-gravity correspondence.

**The Hard Way**

- Take the asymptotically Schrödinger black hole and generalize it to a \( d + 2 \) parameter solution (\( d \) Galilean velocities \( v_i \)).
- Promote \( r_+ \), \( \beta \) and \( v_i \) to fields depending on \( \{u, x\} \).
- Solve bulk gravity equations order by order in derivatives of \( \{u, x\} \) for asymptotically Schrödinger \( d + 3 \) solutions.
- Gravity constraint equations → Navier-Stokes equations.
- Asymptotic fall-off conditions → ‘boundary’ stress tensor.
**Non-relativistic hydrodynamics**

*Aim:* Derive the hydrodynamic equations for the non-relativistic plasma from gravity using the fluid-gravity correspondence.

**The Short-Cut**

- Leading planar physics of the non-relativistic theory is the same as the parent relativistic theory.
- Obtain the stress tensor complex for the non-relativistic theory by reducing the corresponding relativistic stress tensor on the light-cone (along $v$).
- The bulk metric is obtained by TsT transformation of the asymptotically AdS fluid black hole solutions (with $\partial_v$ being the null Killing vector).
**Equations for ideal relativistic hydrodynamics:** These are just conservation of energy-momentum tensor and are $d + 2$ equations for $d + 2$ variables (fluids on $\mathbb{R}^{d+1,1}$)

\[
\nabla_\mu T^{\mu\nu} = 0.
\]

\[
T^{\mu\nu} = (\epsilon_{\text{rel}} + P_{\text{rel}}) u^\mu u^\nu + P_{\text{rel}} \eta^{\mu\nu},
\]
Equations for ideal non-relativistic hydrodynamics: These are again conservation equations:

Continuity equation: \[ \partial_t \rho + \partial_i (\rho \mathbf{v}^i) = 0, \]

Momentum conservation: \[ \partial_t (\rho \mathbf{v}^i) + \partial_j \Pi_{ij} = 0, \]

Energy conservation: \[ \partial_t \left( \varepsilon + \frac{1}{2} \rho \mathbf{v}^2 \right) + \partial_i j_{\varepsilon}^i = 0, \]

where we have defined

spatial stress tensor: \[ \Pi_{ij} = \rho \mathbf{v}^i \mathbf{v}^j + \delta^{ij} \mathbf{P} \]

energy flux: \[ j_{\varepsilon}^i = \frac{1}{2} (\varepsilon + \mathbf{P}) \mathbf{v}^2 \mathbf{v}^i \]
Consider the relativistic stress tensor in light-cone coordinates \( x^\pm = \{u, v\} \).

\[
\partial_+ T^{++} + \partial_i T^{+i} = 0, \quad \partial_+ T^{+i} + \partial_j T^{ij} = 0, \quad \partial_+ T^{+-} + \partial_i T^{-i} = 0,
\]

which allows us to identify

\[
T^{++} = \rho, \quad T^{+i} = \rho v^i, \quad T^{ij} = \Pi^{ij},
\]

\[
T^{+-} = \varepsilon + \frac{1}{2} \rho v^2, \quad T^{-i} = j^i_{\varepsilon}.
\]
**Light-cone reduction of ideal relativistic hydrodynamics**

The map between relativistic and non-relativistic variables:

\[
\begin{align*}
    u^+ &= \sqrt{\frac{1}{2} \frac{\rho}{\varepsilon + P}}, \\
    u^i &= u^+ v^i, \\
    P_{\text{rel}} &= P, \\
    \epsilon_{\text{rel}} &= 2 \varepsilon + P.
\end{align*}
\]

The component of the relativistic velocity \( u^- \) can be determined using the normalization condition \( u_\mu u^\mu = -1 \) to be

\[
    u^- = \frac{1}{2} \left( \frac{1}{u^+} + u^+ v^2 \right).
\]
**Light-cone reduction of viscous relativistic hydrodynamics**

- The map can be extended to incorporate dissipative effects.
- The conformal relativistic stress tensor at first order reads:

\[ T^{\mu\nu} = (\epsilon_{\text{rel}} + P_{\text{rel}}) u^\mu u^\nu + \eta^{\mu\nu} P_{\text{rel}} - 2 \eta_{\text{rel}} \tau^{\mu\nu} \]

with \( \tau^{\mu\nu} \) being the shear tensor.
- Light-cone reduction is as before, with derivative corrections to the map between velocities.
- Can use the map to derive the non-relativistic transport coefficients at first order.
Light-cone reduction of viscous relativistic hydrodynamics

Non-relativistic transport coefficients:

- We find for the shear viscosity

\[ \eta_{\text{rel}} = \frac{\eta}{u^+} . \]

- The heat conductivity is given by

\[ \kappa = 2 \eta \frac{\varepsilon + P}{\rho T} . \]

- The dimensionless ratio Prandtl number defined as the ratio of kinematic viscosity \( \nu \) to thermal diffusivity \( \chi \) is 1.

\[ \text{Pr} = \frac{\nu}{\chi} , \quad \nu = \frac{\eta}{\rho} , \quad \chi = \frac{\kappa}{\rho c_p} , \]
**Schrödinger correlators**

- Can use the Galilean hologram to discuss correlation functions of quasi-primary operators.

- Schrödinger algebra constrains two point functions:

$$\langle \Theta(t, x) \Theta^\dagger(0, 0) \rangle \propto t^{-\Delta_\Theta} e^{-i N_\Theta \frac{x^2}{2t}}$$

- Can derive correlation functions using a minor modification of AdS/CFT:

$$\langle e^{\int_B \varphi_0(x) \Theta(x)} \rangle_{CFT} = \mathcal{Z}_{string} [\varphi(r, x) \bigg|_B = \varphi_0(x)]$$

where we impose boundary conditions at $r = R_c \gg 1$.

Balasubramainan K, McGreevy; Fuertes, Moroz; Volovich, Wen
Salient points

- Holographic dual for system with Galilean conformal invariance, using D-brane construction.
- D-branes probing a Null Melvin geometry naturally give rise to such non-relativistic CFTs.
- Discussed thermodynamics and some hydrodynamic properties of such plasmas.
- As usual, brane engineering leads to systems where $\eta/s$ takes on the universal value $1/4\pi$.
- Can discuss conformal non-relativistic hydrodynamics for the system: derived transport coefficients at first order and constructed dual gravity solutions.