A Particle Physicists' View of Gravity

XLIX Cracow School of Theoretical Physics

Nonperturbative Gravity and Quantum Chromodynamics

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Zakopane, Poland

Outline

- Gravity as a gauge theory in the Higgs phase
- Asymptotic Safety: towards an UV completion of Einstein's theory

Nonlinear FT

- embedding (NLSM)
- quotient (Yang Mills)
- both (gravity)

Nonlinear Sigma Model

$\varphi : M \to N$

$$\frac{1}{2g^2}\int d^d x\, \partial_\mu \varphi^\alpha \partial^\mu \varphi^\beta h_{\alpha\beta}(\varphi)$$

$g \sim \text{mass}^{\frac{2-d}{2}}$

$$\frac{1}{2}\int d^d x\, \partial_\mu \bar{\varphi}^\alpha \partial^\mu \bar{\varphi}^\beta h_{\alpha\beta}(g\bar{\varphi})$$

expand $\bar{\varphi} = \bar{\varphi}_0 + \eta$

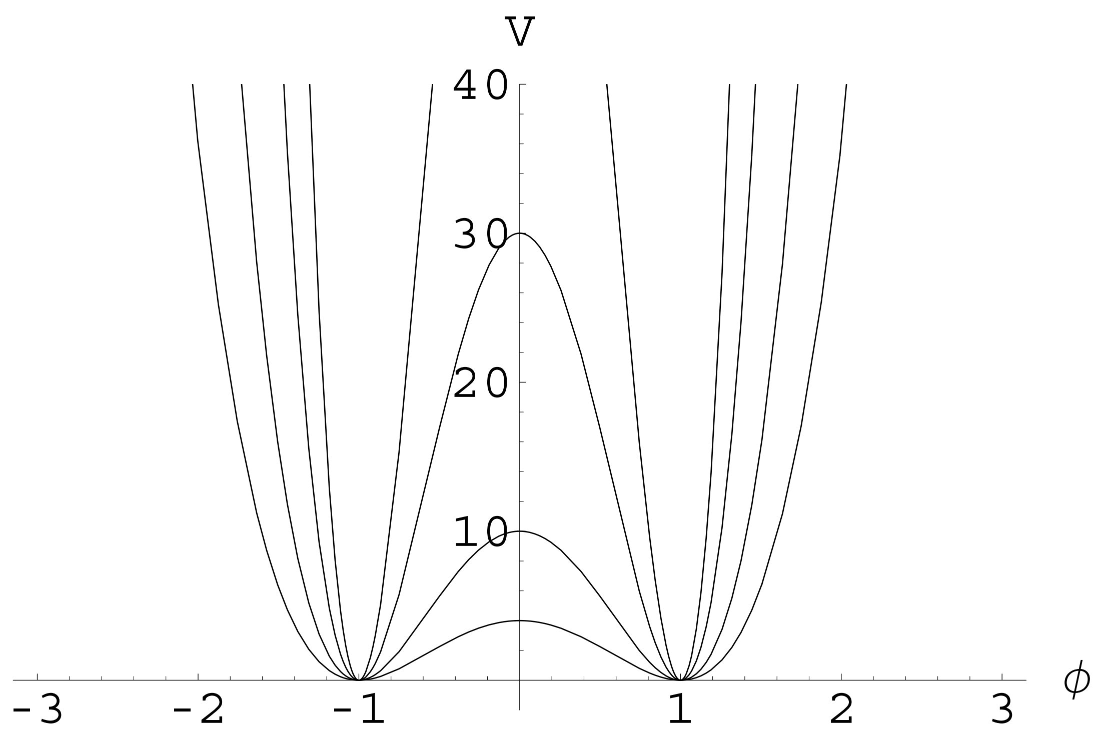
$$h_{\alpha\beta}(g\bar{\varphi}) = h_{\alpha\beta}(g\bar{\varphi}_0) + g\,\partial_\gamma h_{\alpha\beta}(g\bar{\varphi}_0)\eta^\gamma + \frac{1}{2}g^2\partial_\gamma\partial_\delta h_{\alpha\beta}(g\bar{\varphi}_0)\eta^\gamma\eta^\delta + \cdots$$

the coefficient of $\partial\eta\,\partial\eta\,\eta^m$, is $\text{length}^{\frac{m}{2}(d-2)}$.

Strong Coupling limit

$$S = \int d^d x \left[\frac{1}{2} \partial_\mu \phi^a \partial^\mu \phi^a + \frac{\lambda}{4} (\phi^2 - v^2)^2 \right]$$

$\lim_{\lambda \to \infty} V$ with v=const, get NLSM with $g = 1/v$



Chiral Sigma Models

$N = (G_L \times G_R)/\Delta G$

$\varphi(x) \mapsto U(x)$, $\mathrm{Tr} T_a T_b = \delta_{ab}$

$$\frac{1}{2g^2} \int d^d x \, \mathrm{Tr}\,(U^{-1}\partial_\mu U U^{-1} \partial^\mu U)$$

$G = SU(2)$ theory of pions

$f_\pi = 1/g$ pion decay constant

Chiral perturbation theory

χPT

$$S(A,U) = \int dx \, \left[g_2 \mathrm{tr}(U^{-1}\partial U)^2 + g_{41}(\mathrm{tr}(U^{-1}\partial U)^2)^2 + g_{42}\mathrm{tr}(U^{-1}\partial U)^4\right]$$

Higgs mechanism

$A_{\mu ab} = -A_{\mu ba}$ gauge field for $SO(N)$

$\phi^a \in R^N$ carries fundamental of $SO(N)$

$D_\mu \phi^a = \partial_\mu \phi^a + A_{\mu ab}\phi^b$

potential $V = \frac{\lambda}{4}(\phi^2 - \upsilon^2)^2$

has minimum at $\langle \phi^2 \rangle = \upsilon^2 \neq 0$

choose unitary gauge such that $\langle \phi^a \rangle = (\upsilon, 0, \ldots, 0)$

$D_\mu \phi^a D^\mu \phi^a \mapsto \upsilon^2 A_{\mu a1} A^{\mu a1}$

Higgs field $\delta\rho = \sqrt{\phi^a \phi^a} - \upsilon$ has mass $2\lambda\upsilon^2$

Higgsless Higgs mechanism

A_μ gauge field for $O(N)$

φ^α NLSM with value in $S^{N-1} = SO(N)/SO(N-1) \subset R^N$

$D_\mu \varphi^\alpha = \partial_\mu \varphi^\alpha + A_{\mu ab} K^\alpha_{ab}(\varphi)$

choose unitary gauge such that $\langle \varphi^\alpha \rangle = (0, \ldots, 0)$

$\upsilon^2 h_{\alpha\beta}(\varphi) D_\mu \varphi^\alpha D^\mu \varphi^\beta \mapsto \upsilon^2 A_{\mu a 1} A^{\mu a 1}$

only Goldstone bosons are necessary for Higgs mechanism

Higgs particle only necessary for perturbative renormalizability

Low Energy EFT

at momenta $p \ll M_A, M_H$:

- $\delta\rho$ decouples
- in unitary gauge $D_\mu \varphi^\alpha = 0$ is equivalent to $A_{\mu a 1} = 0$

electroweak χPT with $f_\pi \to v$

[Thomas Appelquist, Claude W. Bernard, Phys.Rev.D22:200,1980.]

[Anthony C. Longhitano, Phys.Rev.D22:1166,1980.]
[A. Dobado, M.J. Herrero, Phys.Lett.B228:495,1989]

Gravity with more Variables

a manifold M, $\dim M = n$, a real vectorbundle E of rank n
local bases $\{\partial_\mu\}$, $\{e_a\}$

- fiber metric in E, γ_{ab} signature $+,+,+,+$
- linear connection in E, $A_\mu{}^a{}_b$
- soldering form $\theta^a{}_\mu$, $\det\theta \neq 0$

Induced structures in TM

- $g_{\mu\nu} = \theta^a{}_\mu \, \theta^b{}_\nu \, \gamma_{ab}$
- $\Gamma_\lambda{}^\mu{}_\nu = \theta^{-1}{}_a{}^\mu A_\lambda{}^a{}_b \theta^b{}_\nu + \theta_a^{-1\,\mu} \partial_\lambda \theta^a{}_\nu$

Torsion and Nonmetricity

- $\Theta_{\mu}{}^{a}{}_{\nu} = \partial_{\mu}\theta^{a}{}_{\nu} - \partial_{\nu}\theta^{a}{}_{\mu} + A_{\mu}{}^{a}{}_{b}\,\theta^{b}{}_{\nu} - A_{\nu}{}^{a}{}_{b}\,\theta^{b}{}_{\mu}$
- $\Delta_{\lambda ab} = -\partial_{\lambda}\gamma_{ab} + A_{\lambda}{}^{c}{}_{a}\,\gamma_{cb} + A_{\lambda}{}^{c}{}_{b}\,\gamma_{ac}$

Gauge invariance

$$\begin{aligned}
\theta^{a}{}_{\mu}(x) &\mapsto \theta'^{a}{}_{\mu}(x') = \Lambda^{-1a}{}_{b}(x)\,\theta^{b}{}_{\nu}(x)\frac{\partial x^{\nu}}{\partial x'^{\mu}} \\
\gamma_{ab}(x) &\mapsto \gamma'_{ab}(x') = \Lambda^{c}{}_{a}(x)\,\Lambda^{d}{}_{b}(x)\,\gamma_{cd}(x) \\
A_{\mu}{}^{a}{}_{b}(x) &\mapsto A'_{\mu}{}^{a}{}_{b}(x') = \frac{\partial x^{\nu}}{\partial x'^{\mu}}\left(\Lambda^{-1a}{}_{c}(x)A_{\nu}{}^{c}{}_{d}(x)\Lambda^{d}{}_{b}(x) + \Lambda^{-1a}{}_{c}(x)\partial_{\nu}\Lambda^{c}{}_{b}(x)\right)
\end{aligned} \tag{1}$$

$\mathcal{G} = \mathcal{A}ut E$

$$0 \to \mathcal{A}ut_{M} E \to \mathcal{A}ut E \to \mathcal{D}iff \to 0$$

is split: $\theta_* : \mathcal{D}iff \to \mathcal{A}ut$

$\theta_*(f) = \theta \circ Tf \circ \theta^{-1}$

Metric gauge

$$\theta^a{}_\mu = \delta^a_\mu$$

$$\mathcal{G} = \mathcal{D}iff M$$

$$g_{\mu\nu} = \gamma_{\mu\nu}, \Gamma_\lambda{}^\mu{}_\nu = A_\lambda{}^\mu{}_\nu$$

$$\Theta_\mu{}^a{}_\nu = \Gamma_\mu{}^a{}_\nu - \Gamma_\nu{}^a{}_\mu$$

VIERBEIN GAUGE

$$\gamma_{ab} = \delta_{ab}$$

$\mathcal{G} = \mathcal{A}ut^{SO(4)}M$

$g_{\mu\nu} = \theta^a{}_\mu \, \theta^b{}_\nu \, \delta_{ab}$

$\Delta_{\lambda ab} = A_{\lambda ab} + A_{\lambda ba}$

not enough freedom to fix both simultaneously

Goldstone Bosons

$\gamma(x) \in GL(4)/O(4)$

$\gamma \in \{\text{metrics}\} \approx \mathcal{A}ut_M E / \mathcal{A}ut_M^{SO(n)} E$

$\theta \in \{\text{isomorphisms } TM \to E\} \approx \mathcal{A}ut_M E$

First Hint of a Higgs Mechanism

flat background: $A = 0$, $\theta = \mathbf{1}$, $\gamma = \mathbf{1}$

Palatini action

$$S_P(A, \gamma, \theta) = \frac{1}{16\pi G} \int d^4x \, \sqrt{|\det g|} \, {\theta^{-1}}_a{}^\mu \theta^b{}_\rho \, g^{\rho\nu} F_{\mu\nu}{}^a{}_b$$

contains

$$\frac{1}{16\pi G} \int d^4x \, \delta_a{}^\mu \delta^b{}_\rho \, \delta^{\rho\nu} (A_\mu{}^a{}_c A_\nu{}^c{}_b - A_\nu{}^a{}_c A_\mu{}^c{}_b)$$

Levi–Civita Connection

given γ, θ there is a unique $\bar{A}$ s.t. $\bar{\Theta} = 0$, $\bar{\Delta} = 0$

$$\bar{A} = \frac{1}{2}\left(\theta^{-1}{}_c{}^\lambda\, \partial_\lambda \gamma_{ab} + \theta^{-1}{}_a{}^\lambda\, \partial_\lambda \gamma_{bc} - \theta^{-1}{}_b{}^\lambda\, \partial_\lambda \gamma_{ac}\right) + \frac{1}{2}\left(C_{abc} + C_{bac} - C_{cab}\right)$$

where $C_{abc} = \gamma_{ad}\, \theta^d{}_\lambda \left(\theta^{-1}{}_b{}^\mu\, \partial_\mu \theta^{-1}{}_c{}^\lambda - \theta^{-1}{}_c{}^\mu\, \partial_\mu \theta^{-1}{}_b{}^\lambda\right)$

Any connection A can be split uniquely in $A = \bar{A}(\gamma, \theta) + \Phi$

then $S(A, \gamma, \theta) = S(\bar{A}(\gamma, \theta) + \Phi, \gamma, \theta) = S'(\Phi, \gamma, \theta)$

Example: Einstein Theory

$$S(A,\gamma,\theta) = S_P(A,\gamma,\theta) + S_{\mathrm{m}}(A,\gamma,\theta)$$

where

$$S_{\mathrm{m}} = \int d^4x \sqrt{|\det g|}\, \left[A^{\mu}{}_{a}{}^{\nu\rho}{}_{b}{}^{\sigma} \Theta_{\mu}{}^{a}{}_{\nu} \Theta_{\rho}{}^{b}{}_{\sigma} + B^{\mu ab\nu cd} \Delta_{\mu ab} \Delta_{\nu cd} + C^{\mu}{}_{a}{}^{\nu\rho cd} \Theta_{\mu}{}^{a}{}_{\nu} \Delta_{\rho cd} \right]$$

Rewrite

$$\begin{aligned} \Theta_{\mu}{}^{a}{}_{\nu} &= \Phi_{\mu}{}^{a}{}_{\nu} - \Phi_{\nu}{}^{a}{}_{\mu} \\ \Delta_{\mu ab} &= \Phi_{\mu ab} + \Phi_{\mu ba} \\ F_{\mu\nu}{}^{a}{}_{b} &= \bar{F}_{\mu\nu}{}^{a}{}_{b} + \tilde{\nabla}_{\mu}\Phi_{\nu}{}^{a}{}_{b} - \tilde{\nabla}_{\nu}\Phi_{\mu}{}^{a}{}_{b} + \Phi_{\mu}{}^{a}{}_{c}\,\Phi_{\nu}{}^{c}{}_{b} - \Phi_{\nu}{}^{a}{}_{c}\,\Phi_{\mu}{}^{c}{}_{b} \end{aligned}$$

therefore

$$S(A,\gamma,\theta) = S(\bar{A}+\Phi,\gamma,\theta) = S_H(\gamma,\theta) + S_Q(\Phi,\gamma,\theta)$$

where

$$S_Q(\Phi,\gamma,\theta) = \frac{1}{2}\int d^4x\ \sqrt{|\det g|}\ Q^{\mu}{}_{a}{}^{b\nu}{}_{c}{}^{d}\,\Phi_{\mu}{}^{a}{}_{b}\,\Phi_{\nu}{}^{c}{}_{d}$$

More generally

- if we add F^2 or higher, still at low energy torsion and nonmetricity can be neglected

General Gravitational Higgs Phenomenon

- gravity is a gauge theory of $GL(4)$ with two Goldstone bosons
- there are two unitary gauges
- Higgs phenomenon occurs at Planck scale, giving mass to Φ
- at low energy $A = \bar{A}(\gamma, \theta)$
- $\Theta = 0$ and $\Delta = 0$ are the analogs of $D\varphi^\alpha = 0$
- in a unitary gauge, at low energy A is a function of the surviving Goldstone boson (e.g. in metric gauge $A_\mu{}^\lambda{}_\nu = \{_\mu{}^\lambda{}_\nu\}$) (similar to CP^n etc)

Useful to:

- understand gravity in particle physics terms
- understand issue of Chern–Simons terms in topologically massive gravity
- understand Bardeen Zumino anomaly counterterm
- understand transformation of spinors under diffeomorphisms
- string duality realized linearly (Siegel 1993)

General Framework for Unification

let $\dim M = 4$, enlarge fibers of E to have dimension N

gauge theory of $GL(N)$

extended metric gauge

$$\theta = \begin{bmatrix} \mathbf{1}_4 \\ 0 \end{bmatrix} \qquad , \qquad \gamma = \begin{bmatrix} g & 0 \\ 0 & \mathbf{1}_{N-4} \end{bmatrix}$$

$$A_\lambda = \begin{bmatrix} A_\lambda^{(4)} & H_\lambda \\ K_\lambda & A_\lambda^{(N-4)} \end{bmatrix}$$

give mass to $A^{(4)}$, H and K

$SO(14)$

$$\mathbf{64} = \mathbf{2} \times \mathbf{16} + \bar{\mathbf{2}} \times \overline{\mathbf{16}}$$

[R.P. Phys. Lett. **B 144**, 37 (1984)]

[R.P. Nucl. Phys. **B 353**, 271, (1991)]

[F. Nesti and R.P. J. Phys. A: Math. Theor. 41 075405 (15pp) (2008) arXiv:0706.3307 [hep-th]]

Low Energy EFTs

χPT

$$S(A,U) = \int dx \, \left[g_2(U^{-1}\partial U)^2 + g_4(U^{-1}\partial U)^4 + g_6(U^{-1}\partial U)^6 + \ldots\right]$$

similarly for gravity

$$S(g_{\mu\nu}) = \int dx\sqrt{g} \, \left[g_0 + g_2 R + g_4 R^2 + g_6 R^3 + \ldots\right]$$

with $R = g^{\mu\nu} F_{\lambda\mu}{}^{\lambda}{}_{\nu}$, Riemann$= \partial\bar{A} + \bar{A}\bar{A} \sim (g^{-1}\partial g)^2$

Low Energy EFTs

χPT breaks down at $p \approx 4\pi\sqrt{g_2}$

- pion χPT $4\pi\sqrt{g_2} \approx 4\pi f_\pi \approx 1GeV$
- EW χPT $4\pi\sqrt{g_2} \approx 4\pi v \approx 3TeV$
- gravity $4\pi\sqrt{g_2} \approx 10^{19}GeV$

Look for UV completion

- relax nonlinear constraints on θ and γ
- anholonomic, difficult anyway
- maybe UV completion can be found in the nonlinear theory
- asymptotic safety

Perturbation theory

$$\Gamma_k = \sum_i g_i \mathcal{O}_i$$

$$= \int d^d x \sqrt{g} \left[2Z\Lambda - ZR + \frac{1}{2\lambda} C^2 + \frac{1}{\xi} R^2 + \frac{1}{\rho} E + \frac{1}{\tau} \nabla^2 R + \dots \right]$$

$$Z = \frac{1}{16\pi G} \; ; \qquad \frac{1}{\xi} = -\frac{\omega}{3\lambda} \; ; \qquad \frac{1}{\rho} = \frac{\theta}{\lambda}$$

$$\frac{1}{\epsilon} \int d^4 x \sqrt{g} \left[\frac{7}{10} R_{\mu\nu} R^{\mu\nu} + \frac{1}{60} R^2 + \frac{53}{45} E - \frac{19}{15} \nabla^2 R \right] \qquad \text{1 loop}$$

$$\frac{1}{\epsilon} \int d^4 x \sqrt{g} \, R^{\mu\nu}{}_{\rho\sigma} R^{\rho\sigma}{}_{\alpha\beta} R^{\alpha\beta}{}_{\mu\nu} \qquad \text{2 loops}$$

The Problems of QG

- interaction strength grows like $\tilde{G} = Gk^2$
- lack of predictivity

Possible RG solution

- $\tilde{G} = G(k)k^2 \to \tilde{G}_*$ (Fixed Point)
- more generally, for all essential couplings we must have $\tilde{g}_i \to \tilde{g}_{i*}$ where $\tilde{g}_i = k^{-d_i} g_i$
- QFT description OK if we are on trajectory that hits the FP for $k \to \infty$
- The RG trajectories that flow into the FP for $t = \log \frac{k}{k_0} \to \infty$ form the UV critical surface $\mathcal{S}$.
- theory is predictive if $\mathcal{S}$ is finite dimensional

Asymptotic Safety

- If (1) a FP exist, and (2) UV critical surface is finite-dimensional, then (a) reaction rates remain finite (in units of k) for $t \to \infty$ and (b) theory is predictive
- In practice can determine $T_* \mathcal{S}$
- Example: QCD. Gaußian Fixed Point at $\tilde{g}_{i*} = 0$.
- $\tilde{\beta}_i = \partial_t \tilde{g}_i = -d_i \tilde{g}_i + k^{-d_i} \beta_i$
- Linearize flow around FP: $M_{ij}\big|_* = \frac{\partial \tilde{\beta}_i}{\partial \tilde{g}_j}\big|_* = -d_i \delta_{ij}$
- UV critical surface=span{renormalizable couplings}.

Should we expect AS? I

in NLSM g cannot be eliminated by field redefinition

in gravity

$$S(g_{\mu\nu}) = \frac{1}{16\pi G} \int dx \sqrt{g}\, g^{\mu\nu} R_{\lambda\mu}{}^{\lambda}{}_{\nu}$$

$g_{\mu\nu} \to (16\pi G) g_{\mu\nu}$ eliminate G completely

but

in so doing change also cutoff. Rescalings not allowed.

G is essential coupling

[R.P. D. Perini, Class. Quantum Grav. 21, 5035-5041, (2004)]

SHOULD WE EXPECT AS? II

Consider NLSM or gravity in $d = 4$

$$g_2(k) = \frac{1}{2g^2} \; ; \qquad (g^2 = 8\pi G)$$

$$\partial_t g_2 = B_1 k^2$$

$$\partial_t g^2 = -2B_1 g^4 k^2$$

$$\partial_t \tilde{g}^2 = 2\tilde{g}^2 - 2B_1 \tilde{g}^4$$

if $B_1 > 0$ fixed point at $\tilde{g}^2 = 1/B_1$

Should we expect AS? III

in local QFT only finitely many relevant operators

expect quantum corrections to be finite

One Loop Corrections in Einstein's Theory II

$$G(r) = G\left[1 - \frac{167}{30\pi}\frac{G\hbar}{r^2}\right]$$

[N.E.J. Bjerrum-Bohr, J.F. Donoghue and B.R. Holstein, Phys. Rev. D 68, 084005 (2003)]

$$\beta_{\tilde{G}} = 2\tilde{G} - \frac{167}{15\pi}\tilde{G}^2$$

ERGE I

$$e^{-W_k[J]} = \int (D\phi) \mathrm{e}xp(-(S + \Delta S_k + \int J\phi))$$

$$\Delta S_k(\phi) = \frac{1}{2} \int d^4 q \phi(-q) R_k(q^2) \phi(q)$$

modified inverse propagator $P_k(q^2) = q^2 + R_k(q^2)$

$$\tilde{\Gamma}_k[\phi] = W_k[J] - \int J\phi$$

$$\Gamma_k[\phi] = \tilde{\Gamma}_k[\phi] - \Delta S_k$$

ERGE II: One loop

$$\Gamma^{(1)} = S + \frac{1}{2} \ln \det \frac{\delta^2 S}{\delta\phi\delta\phi}$$

$$\Gamma_k^{(1)} = S + \frac{1}{2} \ln \det \left(\frac{\delta^2 S}{\delta\phi\delta\phi} + R_k \right)$$

$$\partial_t \Gamma_k^{(1)} = \frac{1}{2} \mathrm{Tr} \left(\frac{\delta^2 S}{\delta\phi\delta\phi} + R_k \right)^{-1} \partial_t R_k$$

ERGE III

$$\partial_t W_k = \partial_t \langle \Delta S_k \rangle = \frac{1}{2} \mathrm{Tr} \langle \phi\phi \rangle \partial_t \mathcal{R}_k$$

$$\begin{aligned}\partial_t \Gamma_k =& \partial_t W_k - \partial_t \Delta S_k = \\ &= \frac{1}{2} \mathrm{Tr} (\langle \phi\phi \rangle - \langle \phi \rangle \langle \phi \rangle) \partial_t \mathcal{R}_k \\ &= -\frac{1}{2} \mathrm{Tr} \frac{\delta^2 W_k}{\delta J \delta J} \partial_t \mathcal{R}_k\end{aligned}$$

$$\frac{\delta^2 W_k}{\delta J \delta J} = -\left(\frac{\delta^2 \tilde{\Gamma}_k}{\delta\phi\delta\phi} \right)^{-1}$$

$$\boxed{\frac{d}{dt}\Gamma_k = \frac{1}{2} \mathrm{Tr} \left(\frac{\delta^2 \Gamma_k}{\delta\phi\delta\phi} + \mathcal{R}_k \right)^{-1} \frac{d\mathcal{R}_k}{dt}}$$

[C. Wetterich, Phys. Lett. B **301** (1993) 90.]

ERGE IV: Beta functions

$$\Gamma_k(\phi) = \sum_i g_i(k)\mathcal{O}_i(\phi)$$

$$\partial_t \Gamma_k = \sum_i \partial_t g_i \mathcal{O}_i = \sum_i \beta_{g_i} \mathcal{O}_i$$

compare with

$$\partial_t \Gamma_k = \frac{1}{2}\mathrm{Tr}\left(\frac{\delta^2 \Gamma_k}{\delta\phi\delta\phi} + R_k\right)^{-1} \partial_t R_k$$

read off beta functions. Generally scheme-dependent.

ERGE and Asymptotic Safety

no need for UV regulator

choose functional space, compute β_i, choose initial point, evolve

no a priori specification of bare action

bottom up approach

ERGE and Gravity

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$$

$$S_{GF}(h, \bar{g}) = \int d^4x \sqrt{\bar{g}}\, \chi_\mu \bar{g}^{\mu\nu} \chi_\nu$$

$$\chi_\nu = \bar{\nabla}^\mu h_{\mu\nu} + \beta \bar{\nabla}_\nu h$$

$$\Delta S_k(h, \bar{g}) = \frac{1}{2} \int d^4x \sqrt{\bar{g}}\; h_{\mu\nu} \bar{g}^{\mu\rho} \bar{g}^{\nu\sigma} R_k(-\bar{\nabla}^2) h_{\rho\sigma}$$

$$\Gamma_k(\bar{g}_{\mu\nu}, \langle h_{\mu\nu} \rangle)$$

$$\Gamma_k(g_{\mu\nu}) = \Gamma_k(g_{\mu\nu}, 0)$$

[M. Reuter, Phys. Rev. D **57** 971(1998)]

[D. Dou and R. P., Class. and Quantum Grav. **15** 3449 (1998)]

Cutoff types

$$\frac{\delta^2\Gamma_k}{\delta\phi\delta\phi} = \boldsymbol{\Delta} = -\nabla^2 + \mathbf{E}_1(R) + \mathbf{E}_2(g_i, R)$$

$$P_k(z) = z + R_k(z)$$

type I

$$\boldsymbol{\Delta} + R_k(-\nabla^2) = P_k(-\nabla^2) + \mathbf{E}_1(R) + \mathbf{E}_2(g_i, R)$$

type II

$$\boldsymbol{\Delta} + R_k(-\nabla^2 + \mathbf{E}_1) = P_k(-\nabla^2 + \mathbf{E}_1) + \mathbf{E}_2$$

type III

$$\boldsymbol{\Delta} + R_k(\boldsymbol{\Delta}) = P_k(\boldsymbol{\Delta})$$

Heat Kernel Technique I

$$\mathrm{Tr}W(\Delta) = Q_{\frac{d}{2}}(W)B_0(\Delta) + Q_{\frac{d}{2}-1}(W)B_2(\Delta) + \ldots + Q_0(W)B_{2d}(\Delta) + \ldots$$

$$Q_n[W] = \frac{1}{\Gamma(n)}\int_0^\infty dz\, z^{n-1}W(z)\ ; \qquad Q_{-n}[W] = (-1)^n W^{(n)}(0)\ . \qquad n > 0$$

[A.H. Chamseddine and A. Connes, Comm. Math. Phys. **186**, 731 (1997)]

[M. Reuter, Phys. Rev. D **57**, 971 (1998)]

Heat Kernel Technique II

$$\begin{aligned}\partial_t \Gamma_k &= \frac{1}{2}\mathrm{Tr}\left(\frac{\partial_t R_k(\Delta)}{P_k(\Delta)}\right)\\ &= \frac{1}{2}\Big[Q_{\frac{d}{2}}\left(\frac{\partial_t R_k}{P_k}\right)B_0(\Delta) + Q_{\frac{d}{2}-1}\left(\frac{\partial_t R_k}{P_k}\right)B_2(\Delta) + \ldots\Big]\end{aligned}$$

with optimized cutoff $R_k(z) = (k^2 - z)\theta(k^2 - z)$

[D.Litim, Phys.Rev.D64:105007,2001]

$$Q_n\left(\frac{\partial_t P}{P^\ell}\right) = \frac{2}{n}k^{2(n-\ell+1)}\ ; \qquad Q_0\left(\frac{\partial_t P}{P^\ell}\right) = 2k^{2(-\ell+1)}$$

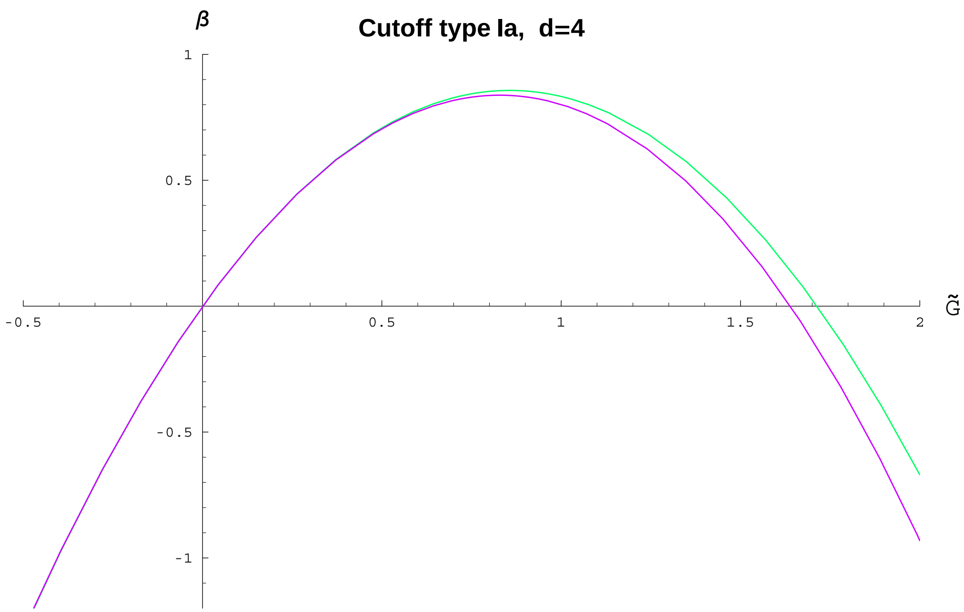
Einstein–Hilbert truncation I

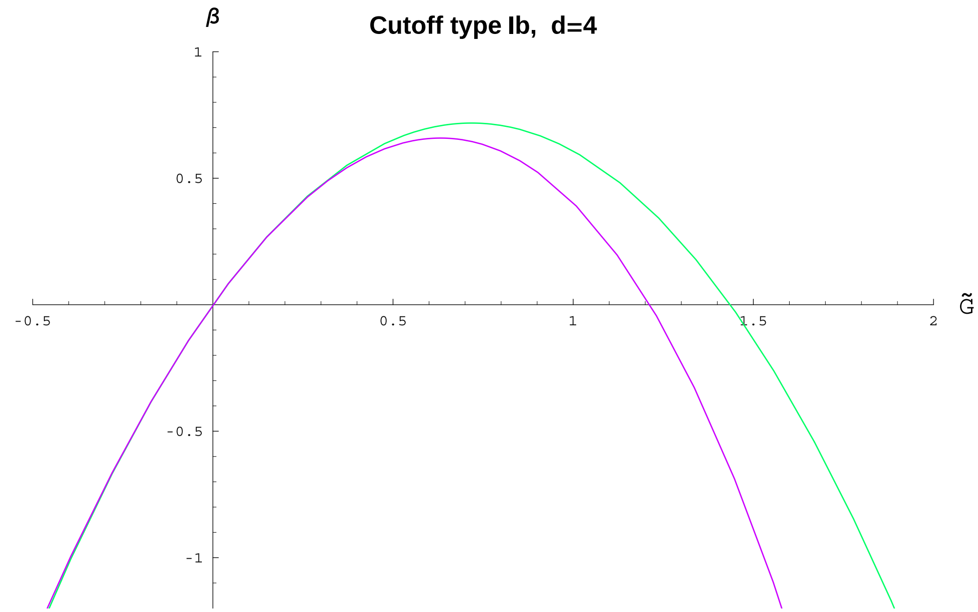
$$\Gamma_k(g) = \frac{1}{16\pi G}\int dx\sqrt{g}(2\Lambda - R)$$

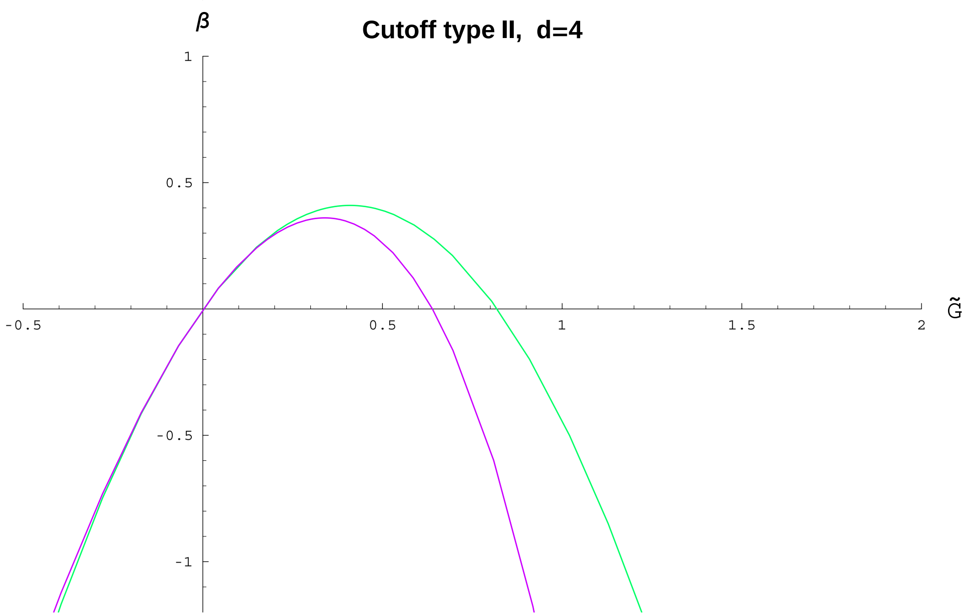
$$\beta_{\tilde\Lambda} = \frac{-2(1-2\tilde\Lambda)^2\tilde\Lambda + \frac{36-41\tilde\Lambda+42\tilde\Lambda^2-600\tilde\Lambda^3}{72\pi}\tilde G + \frac{467-572\tilde\Lambda}{288\pi^2}\tilde G^2}{(1-2\tilde\Lambda)^2 - \frac{29-9\tilde\Lambda}{72\pi}\tilde G}$$

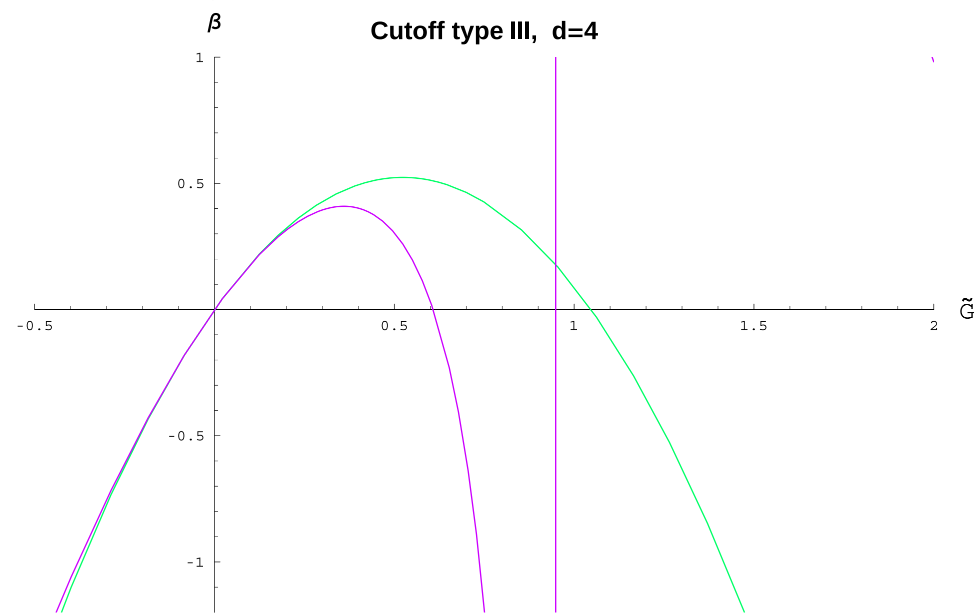
$$\beta_{\tilde G} = \frac{2(1-2\tilde\Lambda)^2\tilde G - \frac{373-654\tilde\Lambda+600\tilde\Lambda^2}{72\pi}\tilde G^2}{(1-2\tilde\Lambda)^2 - \frac{29-9\tilde\Lambda}{72\pi}\tilde G}$$

Einstein–Hilbert truncation II

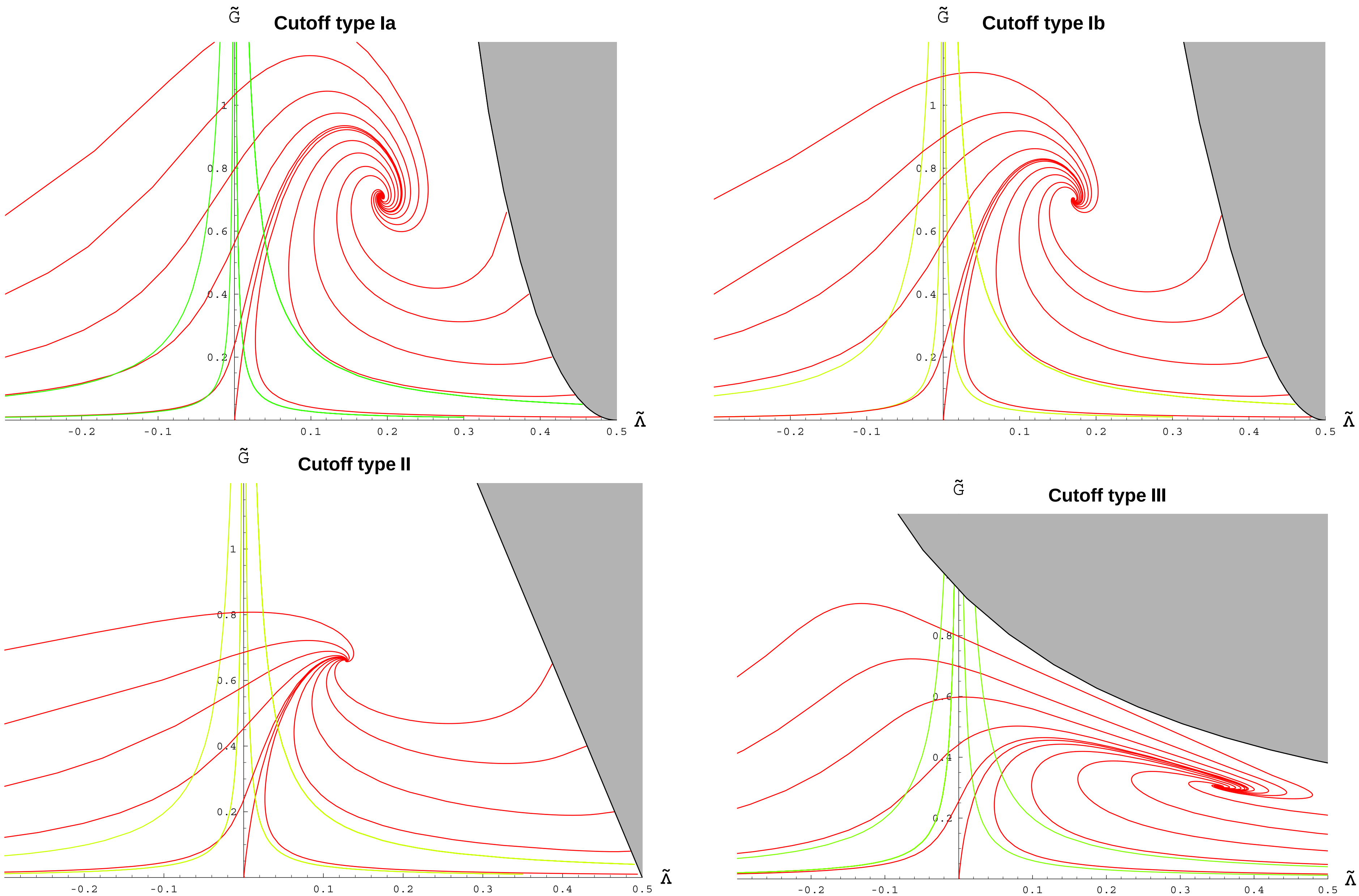








EINSTEIN–HILBERT TRUNCATION III



Divergences in Einstein theory

$$k\frac{d\Gamma_k}{dk}\bigg|_{\sim R^2} = \frac{1}{16\pi^2}\int d^4x\,\sqrt{g}\left[\frac{7}{10}R_{\mu\nu}R^{\mu\nu} + \frac{1}{60}R^2 + \frac{53}{45}E - \frac{19}{15}\nabla^2 R\right]$$

$$g(k) \approx \log k$$

$$k\frac{d\Gamma_k}{dk}\bigg|_{\sim R^3} = \frac{c}{k^2}\frac{1}{16\pi^2}\int d^4x\,\sqrt{g}\,R^{\mu\nu}{}_{\rho\sigma}R^{\rho\sigma}{}_{\alpha\beta}R^{\alpha\beta}{}_{\mu\nu}$$

$$g(k) \approx ck^{-2}$$

Higher derivative gravity

$$\Gamma_k = \int d^4x \sqrt{g} \left[2Z\Lambda - ZR + \frac{1}{2\lambda} C^2 + \frac{1}{\xi} R^2 + \frac{1}{\rho} E \right]$$

$$Z = \frac{1}{16\pi G} \; ; \qquad \frac{1}{\xi} = -\frac{\omega}{3\lambda} \; ; \qquad \frac{1}{\rho} = \frac{\theta}{\lambda}$$

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Gauge fixing

$$S_{GF} = \int d^4x\sqrt{g}\,\chi_\mu Y^{\mu\nu}\chi_\nu$$

$$Y^{\mu\nu} = \frac{1}{\alpha}\left[g^{\mu\nu}\nabla^2 + \gamma\nabla^\mu\nabla^\nu - \delta\nabla^\nu\nabla^\mu\right]$$

$$\chi_\nu = \nabla^\mu h_{\mu\nu} + \beta\nabla_\nu h$$

$$S_c = \int d^4x\sqrt{g}\,\bar{c}_\nu(\Delta_{gh})^\nu_\mu c^\mu$$

$$(\Delta_{gh})^\nu_\mu = -\delta^\nu_\mu\Box - (1+2\beta)\nabla_\mu\nabla^\nu + R^\nu_\mu$$

$$S_b = \frac{1}{2}\int d^4x\sqrt{g}\,b_\mu Y^{\mu\nu}b_\nu$$

$$\left(\Gamma_k + S_{GF}\right)^{(2)} = \frac{1}{2}\int d^4x\sqrt{g}\;\delta g\mathbf{K}\boldsymbol{\Delta}^{(4)}\delta g$$

$$\boldsymbol{\Delta}^{(4)} = \mathbf{1}\Box^2 + \mathbf{V}^{\rho\lambda}\nabla_\rho\nabla_\lambda + \mathbf{U}$$

Beta functions I

$$\beta_\lambda = -\frac{1}{(4\pi)^2}\frac{133}{10}\lambda^2$$

$$\beta_\xi = -\frac{1}{(4\pi)^2}\left(10\lambda^2 - 5\lambda\xi + \frac{5}{36}\right)$$

$$\beta_\rho = \frac{1}{(4\pi)^2}\frac{196}{45}\rho^2\lambda$$

$$\lambda(k) = \frac{\lambda_0}{1 + \lambda_0 \frac{1}{(4\pi)^2}\frac{133}{10}\log\left(\frac{k}{k_0}\right)}$$

$$\omega(k) \to \omega_* \approx -0.0228$$

$$\theta(k) \to \theta_* \approx 0.327$$

Beta functions II

$$\beta_{\tilde{\Lambda}} = -2\tilde{\Lambda} + \frac{1}{(4\pi)^2}\left[\frac{1+20\omega^2}{256\pi\tilde{G}\omega^2}\lambda^2 + \frac{1+86\omega+40\omega^2}{12\omega}\lambda\tilde{\Lambda}\right]$$

$$-\frac{1+10\omega^2}{64\pi^2\omega}\lambda + \frac{2\tilde{G}}{\pi} - q(\omega)\tilde{G}\tilde{\Lambda}$$

$$\beta_{\tilde{G}} = 2\tilde{G} - \frac{1}{(4\pi)^2}\frac{3+26\omega-40\omega^2}{12\omega}\lambda\tilde{G} - q(\omega)\tilde{G}^2$$

where $q(\omega) = (83 + 70\omega + 8\omega^2)/18\pi$

Flow in $\tilde{\Lambda}$–$\tilde{G}$ plane I

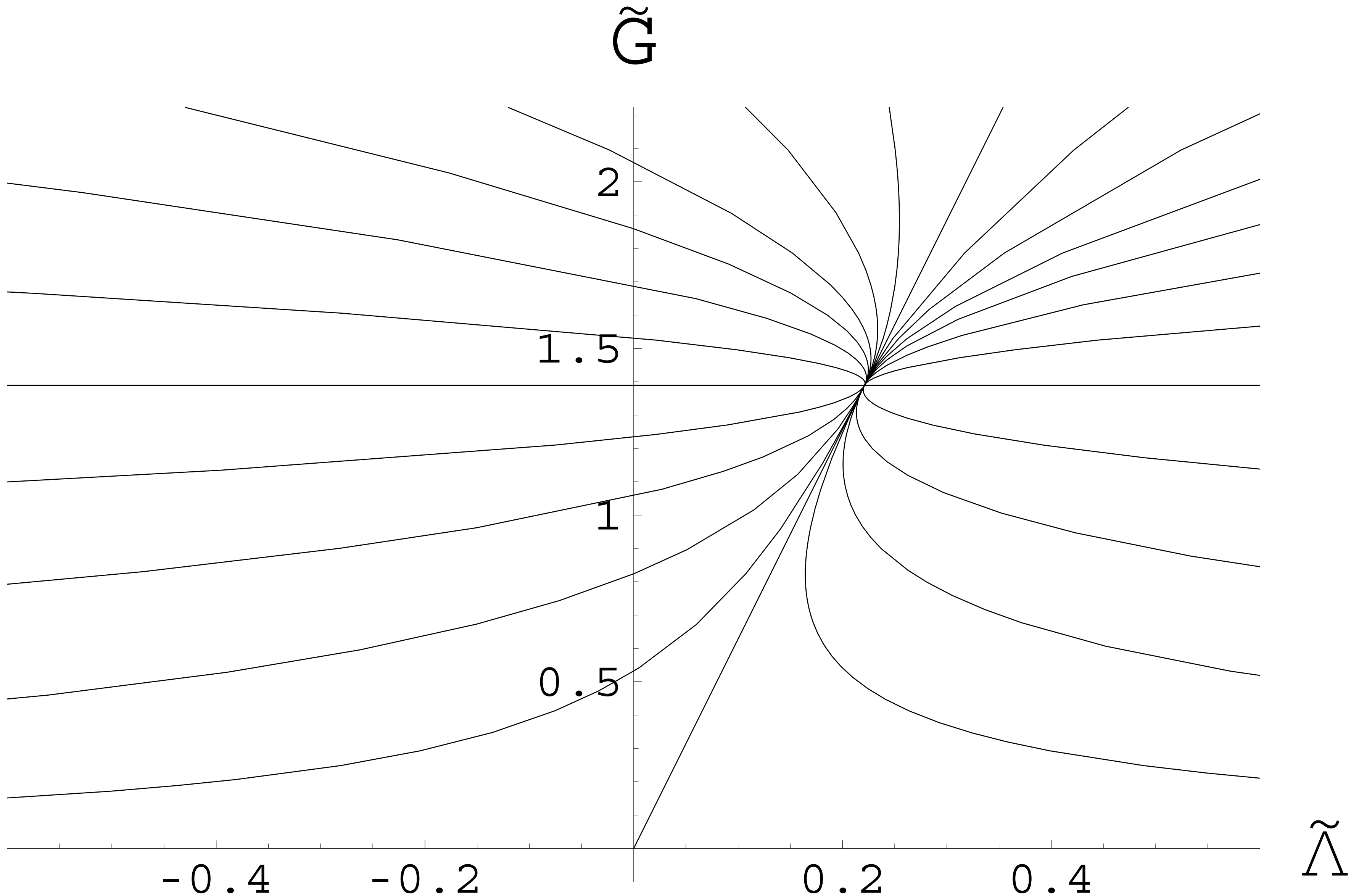
$$\beta_{\tilde{\Lambda}} = -2\tilde{\Lambda} + \frac{2\tilde{G}}{\pi} - q_*\tilde{G}\tilde{\Lambda}$$
$$\beta_{\tilde{G}} = 2\tilde{G} - q_*\tilde{G}^2$$

where $q_* = q(\omega_*) \approx 1.440$

$$\tilde{\Lambda}(t) = \frac{(2\pi\tilde{\Lambda}_0 - \tilde{G}_0(1 - e^{4t}))e^{-2t}}{\pi(2 - q_*\tilde{G}_0(1 - e^{2t}))} \; ; \qquad \tilde{G}(t) = \frac{2\tilde{G}_0 e^{2t}}{2 - q_*\tilde{G}_0(1 - e^{2t})}$$

$$\tilde{\Lambda}_* = \frac{1}{\pi q_*} \approx 0.221 \; , \qquad \tilde{G}_* = \frac{2}{q_*} \approx 1.389 \; .$$

Flow in $\tilde{\Lambda}$–$\tilde{G}$ plane II



WITH MATTER

n_S scalars, n_D Dirac, n_M Maxwell fields, minimally coupled

$$\Delta\beta_\lambda = -2a_\lambda^{(4)}\lambda^2$$

$$\Delta\beta_\xi = -a_\xi^{(4)}\xi^2$$

$$\Delta\beta_\rho = -a_\rho^{(4)}\rho^2$$

$$\Delta\beta_{\tilde G} = 32\pi a^{(2)}\tilde G^2$$

$$\Delta\beta_{\tilde\Lambda} = 8\pi a^{(0)}\tilde G + 32\pi a^{(2)}\tilde\Lambda\tilde G$$

FP still exists

Coefficients

$$a^{(0)} = \frac{1}{32\pi^2}\left(n_S - 4n_D + 2n_M\right)$$

$$a^{(2)} = \frac{1}{96\pi^2}\left(n_S + 2n_D - 4n_M\right)$$

$$a_\lambda^{(4)} = \frac{1}{2880\pi^2}\left(\frac{3}{2}n_S + 9n_D + 18n_M\right)$$

$$a_\xi^{(4)} = \frac{1}{2880\pi^2}\left(-\frac{1}{2}n_S - \frac{11}{4}n_D - 31n_M\right)$$

$$a_\rho^{(4)} = \frac{1}{2880\pi^2}\frac{5}{2}n_S$$

Beyond 1 Loop

$$\tilde{\Lambda}_* = 0.11, \quad \tilde{G}_* = 0.865, \quad \lambda_* = -44.86, \quad \omega_* = 0.932\,.$$

$$\theta = 2.33 \pm 0.6i\,; \quad \theta_2 = 13.72\,; \quad \theta_3 = -7.00\,.$$

[D. Benedetti, P.F. Machado, F. Saueressig, arXiv:0901.2984]

$f(R)$ GRAVITY

$$\Gamma_k(g_{\mu\nu}) = \int d^4x \sqrt{g} f(R)$$

$$f(R) = \sum_{i=0}^{n} g_i(k) R^i$$

- workable for $n \leq 8$

[A. Codello, R.P. and C. Rahmede Int. J. Mod. Phys. A23:143-150 e-Print:0705.1769 [hep-th]; Annals Phys. 324 414-469 (2009) e-Print: arXiv:0805.2909]

[P.F. Machado, F. Saueressig, Phys. Rev. **D** e-Print: arXiv:0712.0445 [hep-th]]

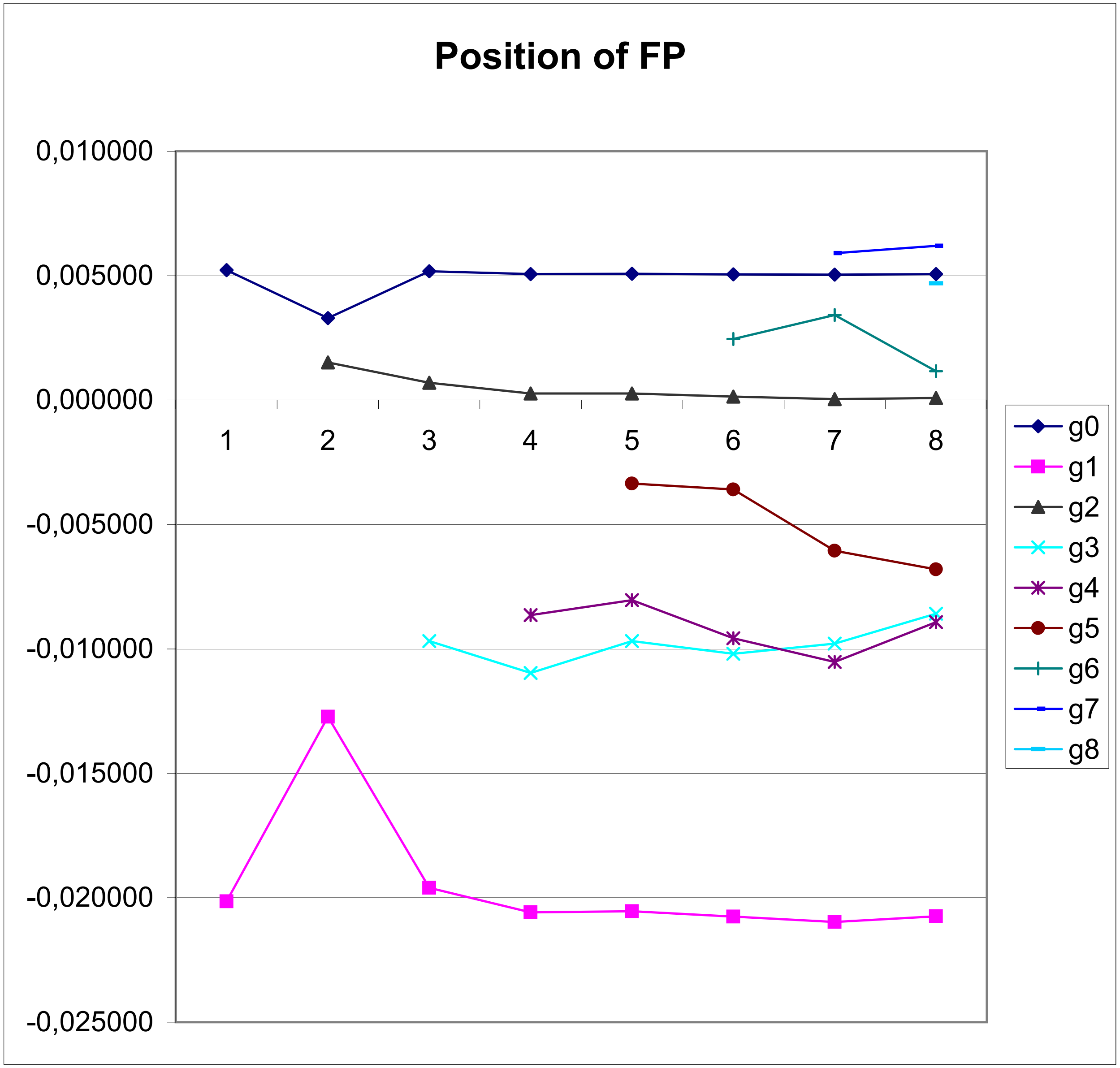
$f(R)$ GRAVITY II

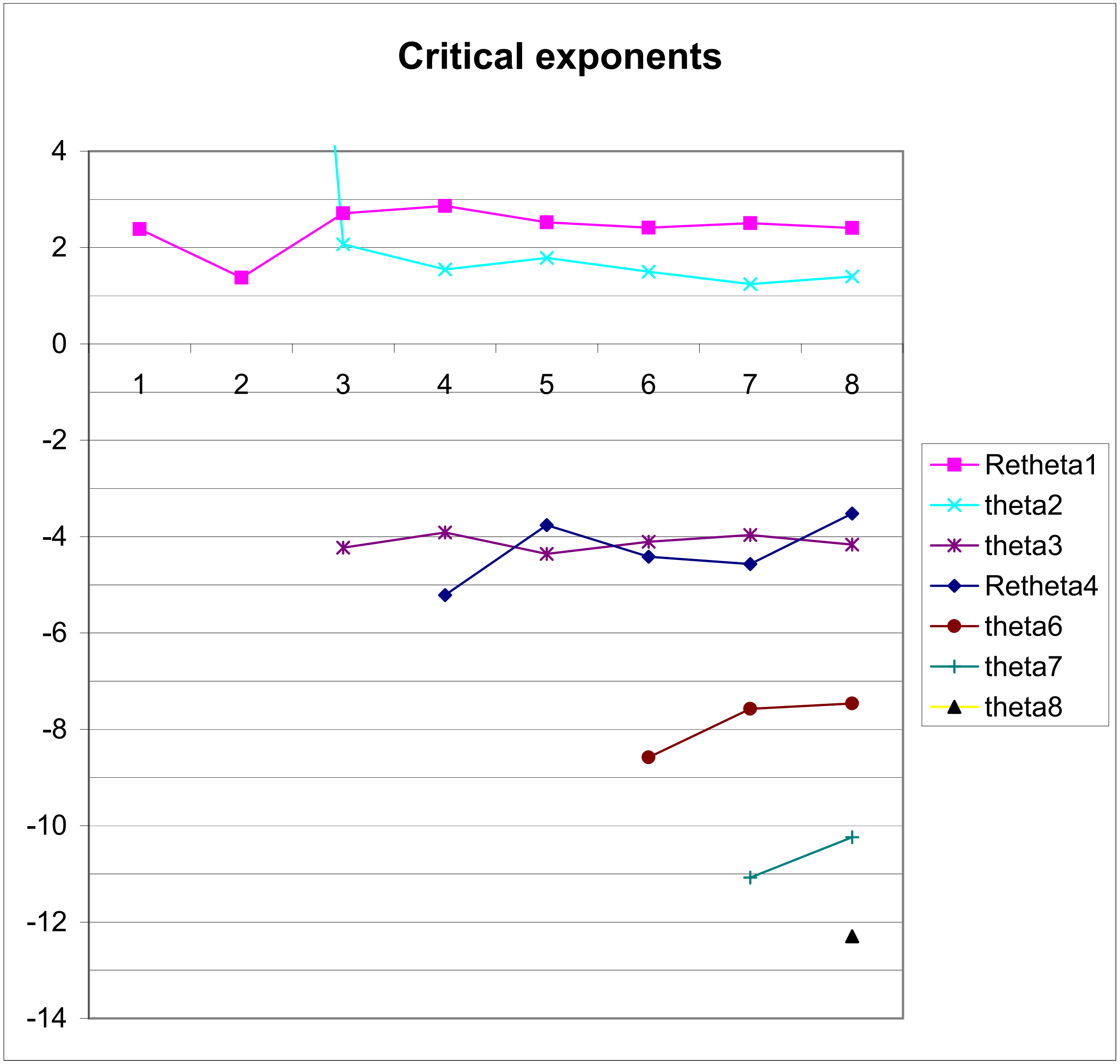
Position of FixedPoint ($\times 10^{-3}$)

n	$\tilde{g}_{0*}$	$\tilde{g}_{1*}$	$\tilde{g}_{2*}$	$\tilde{g}_{3*}$	$\tilde{g}_{4*}$	$\tilde{g}_{5*}$	$\tilde{g}_{6*}$	$\tilde{g}_{7*}$	$\tilde{g}_{8*}$
1	5.23	-20.1							
2	3.29	-12.7	1.51						
3	5.18	-19.6	0.70	-9.7					
4	5.06	-20.6	0.27	-11.0	-8.65				
5	5.07	-20.5	0.27	-9.7	-8.03	-3.35			
6	5.05	-20.8	0.14	-10.2	-9.57	-3.59	2.46		
7	5.04	-20.8	0.03	-9.78	-10.5	-6.05	3.42	5.91	
8	5.07	-20.7	0.09	-8.58	-8.93	-6.81	1.17	6.20	4.70

Critical exponents

n	$Re\vartheta_1$	$Im\vartheta_1$	ϑ_2	ϑ_3	$Re\vartheta_4$	$Im\vartheta_4$	ϑ_6	ϑ_7	ϑ_8
1	2.38	2.17							
2	1.38	2.32	26.9						
3	2.71	2.27	2.07	-4.23					
4	2.86	2.45	1.55	-3.91	-5.22				
5	2.53	2.69	1.78	-4.36	-3.76	-4.88			
6	2.41	2.42	1.50	-4.11	-4.42	-5.98	-8.58		
7	2.51	2.44	1.24	-3.97	-4.57	-4.93	-7.57	-11.1	
8	2.41	2.54	1.40	-4.17	-3.52	-5.15	-7.46	-10.2	-12.3


Position of FP
0,010000
0,005000
0,000000
-0,005000
-0,010000
-0,015000
-0,020000
-0,025000
1
2
3
4
5
6
7
8
g0
g1
g2
g3
g4
g5
g6
g7
g8


Critical exponents
4
2
0
-2
-4
-6
-8
-10
-12
-14
1
2
3
4
5
6
7
8
Retheta1
theta2
theta3
Retheta4
theta6
theta7
theta8

$f(R)$ GRAVITY III

Critical surface:

$$\tilde{g}_3 = 0.00061243 + 0.06817374\,\tilde{g}_0 + 0.46351960\,\tilde{g}_1 + 0.89500872\,\tilde{g}_2$$

$$\tilde{g}_4 = -0.00916502 - 0.83651466\,\tilde{g}_0 - 0.20894019\,\tilde{g}_1 + 1.62075130\,\tilde{g}_2$$

$$\tilde{g}_5 = -0.01569175 - 1.23487788\,\tilde{g}_0 - 0.72544946\,\tilde{g}_1 + 1.01749695\,\tilde{g}_2$$

$$\tilde{g}_6 = -0.01271954 - 0.62264827\,\tilde{g}_0 - 0.82401181\,\tilde{g}_1 - 0.64680416\,\tilde{g}_2$$

$$\tilde{g}_7 = -0.00083040 + 0.81387198\,\tilde{g}_0 - 0.14843134\,\tilde{g}_1 - 2.01811163\,\tilde{g}_2$$

$$\tilde{g}_8 = 0.00905830 + 1.25429854\,\tilde{g}_0 + 0.50854002\,\tilde{g}_1 - 1.90116584\,\tilde{g}_2$$

Large N expansion I

$$n_S \approx n_D \approx n_M \approx N \to \infty$$

$$\Delta^{(S)} = -\nabla^2$$

$$\Delta^{(D)} = -\nabla^2 + \frac{R}{4}$$

$$\Delta^{(M)} = -\nabla^2 \delta^\mu_\nu + R^\mu{}_\nu$$

$$\partial_t \Gamma_k = \frac{n_S}{2} \mathrm{Tr}_{(S)} \left(\frac{\partial_t P_k(\Delta^{(S)})}{P_k(\Delta^{(S)})} \right) - \frac{n_D}{2} \mathrm{Tr}_{(D)} \left(\frac{\partial_t P_k(\Delta^{(D)})}{P_k(\Delta^{(D)})} \right) + \frac{n_M}{2} \mathrm{Tr}_{(M)} \left(\frac{\partial_t P_k(\Delta^{(M)})}{P_k(\Delta^{(M)})} \right) - n_M \mathrm{Tr}_{(gh)} \left(\frac{\partial_t P_k(\Delta^{(gh)})}{P_k(\Delta^{(gh)})} \right)$$

LARGE N EXPANSION II

$$\Gamma_k = \sum_{n=0}^{\infty} \sum_i g_i^{(n)} \mathcal{O}_i^{(n)}$$

$$\partial_t \tilde{g}_i^{(n)} = (2n-4)\tilde{g}_i^{(n)} + a_i^{(n)}$$

For all $n \neq 2$: $\tilde{g}_{i*}^{(n)} = \frac{1}{4-2n} a_i^{(n)}$

For $n = 2$: $g_i^{(2)}(k) = g_i^{(2)}(k_0) + a_i^{(2)} \ln(k/k_0)$

If $R_k(z) = (k^2 - z)\theta(k^2 - z)$, $\tilde{g}_{i*}^{(n)} = 0$ for $n \geq 3$.

[R. P., Phys. Rev. D **73**, 041501 (2006),]

Matter

- FP generically exists also in presence of minimally coupled matter fields
- Conjecture (Fradkin & Tseytlin): all matter interactions are asymptotically free in the presence of gravity.

Summary and Conclusions

- Asymptotic Safety of gravity plausible
- Bottom up approach
- Not exclusive

FAQs and Bibliography

nttp://www.percacci.it/roberto/physics/as/