Large N phase transitions under scaling and their uses.

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1. Abelian Wilson loop operators

Free abelian gauge theory

$$Z[J_\mu] = \int e^{-\frac{1}{4g^2}\int d^dx F_{\mu\nu}^2 + i\int d^dx J_\mu A_\mu}$$

$$\int e^{-\frac{1}{2g^2}\int \tilde{A}_\mu(k)[\delta_{\mu\nu}k^2 - k_\mu k_\nu]\tilde{A}_\nu(-k)} e^{i\int \tilde{J}_\mu(-k)\tilde{A}_\mu(k)}$$

Current conservation: s.p. cond. has solution

$$\partial_\mu F_{\mu\nu} = g^2 J_\nu \Rightarrow \partial_\mu J_\mu = 0$$

Decouple vector indices in mom. space

$$\tilde{A}_\mu(k) = \frac{k_\mu}{|k|} a_L(k) + \epsilon_\mu^i(k) a_\perp^i(k)$$

No a_L in action

a_L not coupled to a conserved current

Current associated with a closed contour

$$J_\mu(x) = \int_0^l d\tau \delta^d(x - z(\tau)) \frac{dz_\mu}{d\tau}$$

Closed contour $\Rightarrow$ current conserved

Fix parametrization by

$$\left(\frac{dz_\mu}{d\tau}\right)^2 = 1$$

$l =$ perimeter of loop

Wilson loop operator

$$W[A] = e^{i \int d^d x J_\mu A_\mu} = e^{i \oint dz_\mu A_\mu(z)}$$

Problems in evaluating $\langle W[A] \rangle$

- No weight for $\tilde{A}_\mu(0)$; $\tilde{J}_\mu(0) = 0$; Wilson loops appear to be “infrared safe”.

- $a_L(k)$ integral unbounded; fix by extra weight

$$e^{-\frac{1}{2a_0 g^2}\int \frac{d^dk}{(2\pi)^d} k^2 a_L(k) a_L(-k)}$$

Current conservation $\Rightarrow$ a_0-independence

- Product of integrals over modes diverges; solved by ultraviolet cutoff Λ with

$$k^2 < \Lambda^2$$

- $J_\mu(x)$ is a distribution and cannot be squared

$$e^{-\frac{1}{2}\int d^dx d^dy J_\mu(x) J_\nu(y) G_{\mu\nu}(x-y)}$$

Solved by setting

$$\tilde{J}_\mu(k) = 0 \quad \text{for} \quad k^2 > \Lambda^2$$

$$J_\mu^\Lambda(x) = \int_{k^2<\Lambda^2} \frac{d^dk}{(2\pi)^d} e^{-ikx} \int_0^l d\tau \frac{dz_\mu}{d\tau} e^{ikz(\tau)}$$

is conserved, but no longer localized.

Circular loop; $d = 4$

Exercise: Show

$$\langle W \rangle = e^{g^2 (\Lambda R)^2 \int_0^1 d\xi \log \xi \mathcal{J}_1^2(R\Lambda\sqrt{\xi})/2}$$

Exercise: Show

Exponent is linearly divergent

$$\sim c_0 \, (\Lambda R) + \text{lower orders}$$

Exercise: Show

Can make $\langle W \rangle$ finite by $J^{\Lambda} \to J^{\Lambda'}$ with Λ' kept finite as $\Lambda \to \infty$

In $d = 3$ only log divergence

In $d = 2$ no divergence

Holonomy

$$e^{a_\mu \partial_\mu^x} \psi(x) = \psi(x + a)$$

$\Rightarrow$ for a closed curve

$$e^{\oint dz_\mu \partial_\mu^x} \psi(x) = \psi(x)$$

Minimal substitution

$$e^{\oint dz_\mu [\partial_\mu^x - i A_\mu(z)]} \psi(x) = W^* \psi(x)$$

W is phase factor = holonomy

Small loops

$$W \approx 1 + i \delta\sigma_{\mu\nu} F_{\mu\nu}$$

Holonomy determines action.

2 Nonabelian holonomy

$\mathcal{G} = su(N)$; $A_\mu(x) \to A^j_\mu(x)$, $j = 1, ..., N^2 - 1$

R: Irreps. ; Generators $T^{(R)j}_{a,b}$, $a, b = 1, .., d_R$;

Tr $T^{(R)i} T^{(R)j} \propto \delta^{ij}$, $[T^{(R)i}, T^{(R)j}] = iC^{ijk} T^{(R)k}$

Covariant derivative acting on $\psi^{(R)}_a(x)$

$$[D_\mu \psi]_a(x) = [\partial^x_\mu \delta_{ab} - i T^{(R)j}_{ab} A^j_\mu(x)] \psi_b(x)$$

Parallel transport $\psi^{(R)}_a(x)$ round a closed curve

$$W_R(x) = P e^{i \oint dz_\mu A^j_\mu(z) T^{(R)j}}$$

Under gauge transformation $g(x)$

$$W_R(x) \to g^{(R)}(x) W_R(x) g^{(R)\dagger}(x)$$

Gauge invariant content of holonomy is x-indep.

$$\chi_R(W) = \text{Tr}\, W_R(x) \quad \forall R$$

Equivalently, eigenvalues of matrix $W_f(x)$

$$e^{i\lambda_a}, \;\; a = 0, ...N-1, \;\; \Im(\lambda_a) = 0, \;\; \prod_{a=0}^{N-1} e^{i\lambda_a} = 1$$

$W_f(x) = Pe^{i\oint dz_\mu A^j_\mu(z) T^{(f)j}}$, $(f) =$ fundamental

Wilson loop probability density $P(W)$

Action determined by $P_0(W)$, with

$$W_f \sim 1 + \delta\sigma_{\mu\nu} F^j_{\mu\nu} T^{(f)j}$$

Natural choice for $P_0(W)$: heat-kernel, $t \geq 0$

$$\frac{\partial}{\partial t} P_0(W;t) \propto \nabla^2_W P_0(W;t), \;\; P_0(W;0) = \delta_{\mathrm{Haar}}(W,1)$$

$$P_0(W;t) = \sum_R \chi_R(W) e^{-\frac{t}{2N} C_2(R)}$$

Product over one $P_0(W)$ for all little loops $\Rightarrow$ $P(W)$ for a big loop $\Rightarrow$ class function

$$P(W) = \sum_R \Upsilon_R \ \chi_R(W)$$

The Υ_R have all information determining

$$\langle \chi_{R_1}(W) \chi_{R_2}(W) .... \rangle$$

3 Two dimensions

Tile area $\mathcal{A}$ of loop by small loops, $\lambda = g^2 N \Rightarrow$

$$P(W) = P_0(W;\tau), \quad \tau = \lambda \mathcal{A} \left(1 + \frac{1}{N}\right)$$

Durhuus-Olesen non-analyticity at $N = \infty$: "Infinite N phase transition"

Exercise:

$$\langle \chi_R(W(\tau)) \rangle = d_R e^{-\frac{\tau}{2N} C_2(R)}$$

Exercise:

Generator of all antisymmetric irreps:

$$\psi^{(N)}(z,\tau) = \langle \det(z - W_f(\tau)) \rangle$$

Define

$$\phi^{(N)}(z,\tau) = \frac{i}{N} \frac{1}{\psi^{(N)}(z,\tau)} \left[z\frac{\partial}{\partial z} + \frac{N}{2} \right] \psi^{(N)}(z,\tau)$$

Define

$$\varphi^{(N)}(y,\tau) = \phi^{(N)}(-e^y,\tau), \;\; y \text{ real}$$

Exercise: Burgers' equation

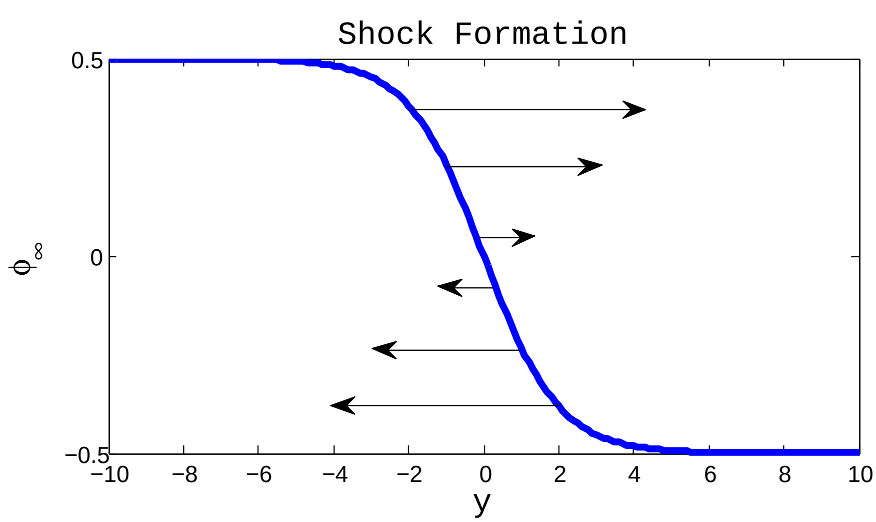
$$\frac{\partial\varphi^{(N)}(y,\tau)}{\partial\tau}+\varphi^{(N)}(y,\tau)\frac{\partial\varphi^{(N)}(y,\tau)}{\partial y}=\frac{1}{2N}\frac{\partial^2\varphi^{(N)}(y,\tau)}{\partial y^2}$$

Initial condition

$$\varphi^{(N)}(y,0)=-\frac{1}{2}\tanh\frac{y}{2}$$

Exercise: Shock at $y=0$ when τ reaches 4

Exercise: Explain figure below



Let z_a be the zeros of $\psi^{(N)}(z,\tau)$

Exercise: Prove $|z_a(\tau)| = 1, \; a = 0, .., N-1$

Exercise: $z_a(\tau) = e^{i\theta_a(\tau)}$; Prove

$$\frac{d\theta_a}{d\tau} = \frac{1}{2N} \sum_{a \neq b} \cot \frac{\theta_a - \theta_b}{2}$$

Exercise: The $\theta_a(\tau)$ for $\tau \ll 1$ are given by

$$\theta_a(\tau) = 2\eta_a \sqrt{\frac{\tau}{N}}; \;\; H_N(\eta_a) = 0, \;\; a = 0, 1, ..., N-1$$

Exercise: $\theta_a(\tau)$ are paired in $\left[\frac{N}{2}\right]$ pairs of opposite signs and for odd N there is one $\theta \equiv 0$

Exercise: Show

$$\theta_a(\tau = \infty) = \frac{2\pi}{N}\left(a - \frac{N-1}{2}\right) \equiv \Theta_a$$

Exercise: The $\theta_a(\tau)$ for $\tau \gg 1$ are given by:

$$\delta\theta_a(\tau) \sim -2e^{-\frac{\tau}{2N}(N-1)} \sin \Theta_a$$

Exercise: $N \gg 1$. Show that the pair of zeros closest to -1 at $\tau = 4$ is

$$z_M \sim -\exp\left[\pm\frac{3.7i}{N^{\frac{3}{4}}}\right]$$

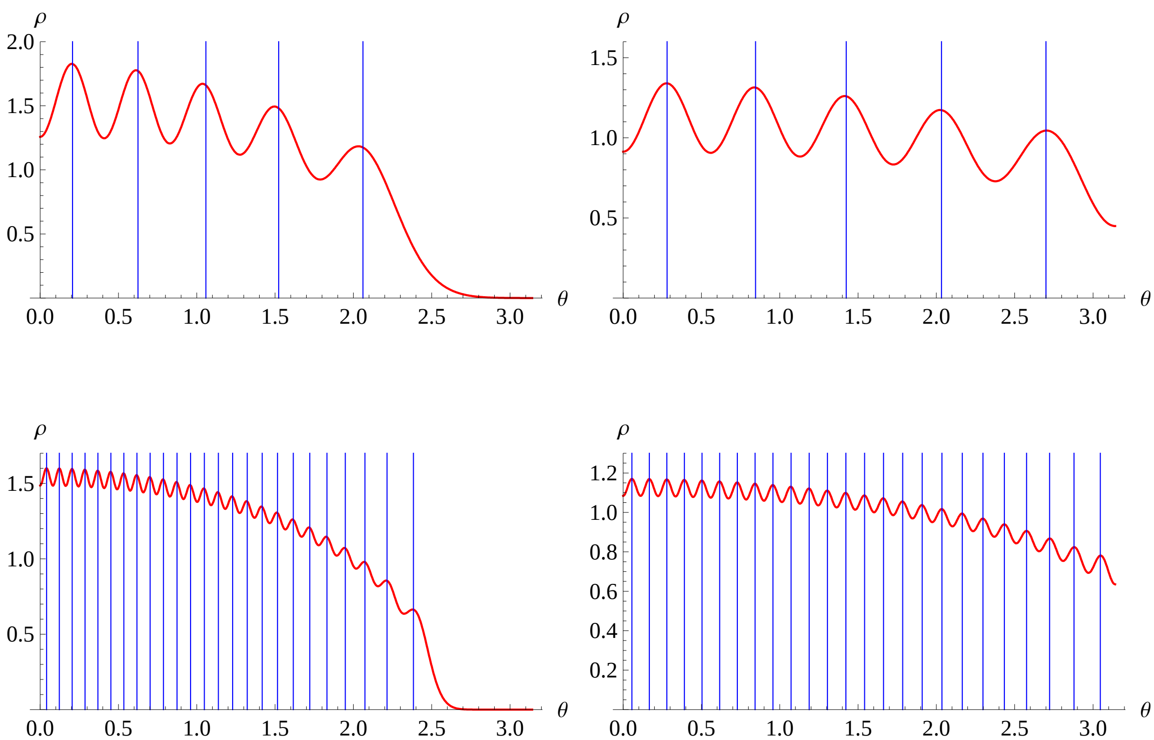
Exercise: Let $N \gg 1$. Let $\frac{\tau}{4} = 1 + \frac{\alpha}{N^\nu}$. Show that for $\nu = 1/2$ $z_M(\tau)$ is a finite nontrivial function of α at $N = \infty$.

Critical exp. governing $N \to \infty$: 1/2 and 3/4.

Zeros $z_a(\tau) \sim$ peaks of ev density of W

$$\rho_N(\theta; W) = \frac{1}{N}\sum_a \langle \delta_{2\pi}(\theta - \gamma_a(W))\rangle$$

Exercise: Compute $\rho_N(\theta : W)$. Hint: start by expanding $\det(1 + uW_f)/\det(1 - vW_f)$ in characters, then take the average, and next study the limit $u \to -v$. Result can be expressed as a double sum or a double integral



The density $\rho_N(\theta)$ (oscillatory red curve) and the positions of the phases of the zeros θ_a (vertical blue lines) for $\tau < 4$ (left) and $\tau > 4$ (right), $N = 10$ (top), and $N = 50$ (bottom).

4 $D > 2$: Hypothesis: same large-N singularity

Need nonperturbative method. Use numerical lattice simulations.

Need to define Wilson loops outside perturbation theory, so that they have a finite limit.

Smearing:

Introduce extra dimension $\rho \geq 0$

$$\frac{\partial A^j_\nu(x,\rho)}{\partial\rho} = [D_\mu F_{\mu\nu}(x,\rho)]^j \, ; \quad A^j_\mu(x,0) = A^j_\mu(x)$$

where D_μ is with respect to $A^j_\mu(x,\rho)$ and $F^j_{\mu\nu}(x,\rho) =$

$$\partial_\mu A^j_\nu(x,\rho) - \partial_\nu A^j_\mu(x,\rho) + C^{ikj} A^i_\mu(x,\rho) A^k_\nu(x,\rho)$$

Smeared Wilson loops are

$$W_R = P\,\text{exp}[\oint dz_\mu A^j_\mu(z,\rho) T^{(R)j}]$$

Smearing to lowest order

$$\frac{\partial \tilde{A}^j_\mu(k,\rho)}{\partial \rho} = -(k^2\delta_{\mu\nu} - k_\mu k_\nu) A^j_\nu(k,\rho)$$

with

$$A^j_\mu(k,0) = A^j_\mu(k)$$

a^j_L remains ρ independent and

$$a_\perp(k,\rho) = e^{-\rho k^2} a_\perp(k)$$

Curve not resolved beyond $\sqrt{\rho}$.

Choose

$$\rho = \frac{l^2}{[l\Lambda']^2 + c}; \quad c \sim 20$$

Continuum limit: For large L

$$\log\langle W_f(L)\rangle \sim -\Sigma(b,N)L^2$$

$$\Sigma(b,N)L^2 = [La(b)]^2 \left[\frac{\sqrt{\Sigma(b,N)}}{a(b)}\right]^2$$

Keep both [] factors finite as $b \to \infty$.

Procedure:

1. Select l and N.
2. Select pairs $b, L(b)$'s at $l = L(b)a(b)$ held fixed
3. For $(b, L(b))$ find $\rho_N\left(\theta; W_f[L(b), \rho(L(b), b)]\right)$
4. Extrapolate $b \to \infty \Rightarrow \rho_N(\theta, l)$.
5. Repeat at same l, increasing N
6. Take $N = \infty \Rightarrow \rho_\infty(\theta, l)$.
7. Repeat varying $l \Rightarrow l_c$

$l_c \sim$ finite $\Rightarrow$ DO transition $\exists$ for $d > 2$

Want to test also for DO universality for $d > 2$

On the lattice:

$$O_N(b, L, \rho(L, b)) = \langle \det \left(e^{\frac{y}{2}} + e^{-\frac{y}{2}} W_f(L) \right) \rangle$$

Focus on: $y \sim 0, N \gg 1, l \sim l_c$; Assume:

can reverse order of $b \to \infty, N \to \infty$.

$O_N(y, b) = C_0(b, N) + C_1(b, N) y^2 + C_2(b, N) y^4 + \ldots$
$\Omega = \frac{C_2 C_0}{C_1^2}$. Vary b at fixed L, N to find $b_c(L)$

$$\Omega(b_c(L, N), N) = \frac{\Gamma(\frac{5}{4})\Gamma(\frac{1}{4})}{6\Gamma^2(\frac{3}{4})} = \frac{\Gamma^4(\frac{1}{4})}{48\pi^2} = 0.36474$$

Exercise: Calculate $\Omega(b = b_c, \infty)$ in $d = 2$

Invert $b_c(L)$ to $L_c(b)$

Same l_c as in orig $b \to \infty, N \to \infty$ limit order

$$l_c = \lim_{b \to \infty} L_c(b) a(b)$$

Define

$$\hat{y} = \left(\frac{4}{3N^3}\right)^{\frac{1}{4}} \frac{\xi}{a_1(L)} \quad \hat{b} = b_c(L)\left[1 + \frac{\alpha}{\sqrt{3N} a_2(L)}\right]$$

Universality: $\lim_{N \to \infty} \mathcal{N}(b, N, L) O_N(\hat{y}, \hat{b}) = \zeta(\xi, \alpha)$

$$\zeta(\xi, \alpha) = \int_{-\infty}^{\infty} du e^{-u^4 - \alpha u^2 + \xi u}$$

Exercise: Verify the above in $d = 2$.

Exercise: Calculate $\Omega(\alpha)$ in $d = 2$.

Exercise: Show

$$\left.\frac{d\Omega(b,N)}{d\alpha}\right|_{\alpha=0} = \frac{\Gamma^2(\frac{1}{4})}{6\sqrt{2}\pi}\left(\frac{\Gamma^4(\frac{1}{4})}{16\pi^2} - 1\right) = 0.04646$$

Verification of critical large N exponents

Define $a_2(L,N)$ by

$$\left.\frac{d\Omega(b,N)}{d\alpha}\right|_{\alpha=0} = \frac{1}{a_2(L,N)\sqrt{3N}}\left.\frac{d\Omega}{db}\right|_{b=b_c(L,N)}$$

Exponent 1/2 $\Rightarrow \exists$ limit $a_2(L,\infty) = a_2(L)$

Exercise: Show that in $d = 2$

$$\sqrt{\frac{4}{3N^3}}\frac{1}{a_1^2(L,N)}\frac{C_1(b_c(L,N),N)}{C_0(b_c(L,N),N)} = \frac{\pi}{\sqrt{2}\Gamma^2(\frac{1}{4})} = 0.169$$

Define $a_1(L,N)$ from above formula in $d > 2$.

Exponent 3/4 $\Rightarrow \exists$ limit $a_1(L,\infty) = a_1(L)$

Finally: Check $\exists \lim_{b\to\infty} a_{1,2}(L_c(b)) = a_{1,2}$

What we know now:

- $d = 3$ works
- $d = 4$ l_c works, large N univ not done yet

This checks consistency of critical exponents

To determine numerically exponents $\sim 1/2, 3/4$ is hard – need $N \sim 100$ – not done yet

Better analysis methods now available from new exact results in 2d

Obligatory conclusion:

NEED MORE COMPUTER POWER

5 The bigger picture

Objective: Analytically calculate string tension for $N \gg 1$ in terms of $\Lambda_{SU(N)}$

- Use perturbation theory for small loops
- Use large N universality to parametrize loops with $l \sim l_c$
- Use an effective string theory to parametrize large loops
- Sew together the three regimes by asymptotic matching

STILL A DREAM