Generalized Integrability in Higher Dimensional Classical Theories

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- Integrability in classical models:
    - Zero curvature formulation
    - Infinitely many conserved quantities
- Exact methods:
    - Inverse scattering methods
    - Bäclund transformation, dressing methods
    - Lax pair formulation etc...

Theorem: The ZC in 1+1 dim $\equiv$ the condition for the path ordered integral

$$Pe^{\int_\Gamma dx^\mu A_\mu}$$

to be independent of the path Γ, for the fixed end points

Proof:

Def: W as

$$\frac{dW}{d\sigma} + A_\mu \frac{dx^\mu}{d\sigma} W = 0, \tag{1}$$

where Γ is parameterized by $\sigma \in [0, 2\pi]$ and $A_\mu \in \mathcal{G}$

How does W change under a fixed end-point deformation of Γ?

$$\frac{d\delta W}{d\sigma} + A_\mu \frac{dx^\mu}{d\sigma}\delta W + \delta\left(\frac{dx^\mu}{d\sigma}\right) W = 0$$

But

$$W^{-1}W = 1 \quad \Rightarrow \quad \frac{d\delta W^{-1}}{d\sigma} = W^{-1}A_\mu \frac{dx^\mu}{d\sigma}$$

$$\Downarrow$$

$$\frac{d}{d\sigma}\left(W^{-1}\delta W\right) = -W^{-1}\left(\partial_\lambda A_\mu \delta x^\lambda \frac{dx^\mu}{d\sigma} + A_\mu \frac{d\delta x^\mu}{d\sigma}\right) W$$

$$W^{-1}\delta W\Big|_0^{\sigma'} =$$

$$-\int_0^{\sigma'} d\sigma \ \left(W^{-1}\partial_\lambda A_\mu W \delta x^\lambda \tfrac{dx^\mu}{d\sigma} + W^{-1} A_\mu \left(\tfrac{d\delta x^\mu}{d\sigma}\right) W \right) =$$

$$-W^{-1}A_\mu W\delta x^\mu\Big|_0^{\sigma'} + \int_0^{\sigma'} d\sigma W^{-1} F_{\mu\nu} W \tfrac{dx^\mu}{d\sigma}\delta x^\nu$$

Fixed end-points $\delta x^\mu(0) = \delta x^\mu(2\pi) = 0$

$$W^{-1}\delta W(2\pi) = \int_0^{2\pi} d\sigma W^{-1} F_{\mu\nu} W \frac{dx^\mu}{d\sigma}\delta x^\nu, \qquad (2)$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$$

Consider:
Γ a closed loop $\quad x_0 = x^\mu(0) = x^\mu(2\pi)$
Σ a surface $\quad \partial\Sigma = \Gamma$

Scan Σ using loops parameterized by $\tau \in [0, 2\pi]$

$\tau = 0$ constant loop at x_0
$\tau = 2\pi$ Γ

Variation is the deformation of one loop into the other $\delta = \delta\tau \frac{d}{d\tau}$

Then

$$\frac{W}{d\tau} = W(2\pi) \int_0^{2\pi} d\sigma W^{-1} F_{\mu\nu} W \frac{dx^\mu}{d\sigma} \frac{dx^\nu}{d\tau} \tag{3}$$

Thus, W is defined by (1) and (3) $\Rightarrow$ non-Abelian Stokes theorem

$$P\ e^{\int_\Gamma dx^\mu A_\mu} = P_2\ \text{Exp}\left(\int_\Sigma d\sigma d\tau W^{-1} F_{\mu\nu} W \frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\tau}\right)$$

$$\Downarrow$$

$$F_{\mu\nu} = 0 \quad \Rightarrow \quad \text{l.h.s. is } \Gamma \text{ independent}$$

$\square$

Idea: The generalized ZC in 2+1 dim is the condition for the surface ordered integral of a rank 2 tensor

$$P_2 \, \mathrm{Exp} \left(\int_\Sigma d\sigma d\tau W^{-1} B_{\mu\nu} W \frac{dx^\mu}{d\sigma} \frac{dx^\nu}{d\tau} \right)$$

to be independent of the surface Σ

Def: Operator V

$$\frac{dV}{d\tau} - V \; T(B, A, \tau) = 0, \quad V(\tau = 0) = 1 \tag{4}$$

$$T \equiv \int_0^{2\pi} d\sigma W^{-1} B_{\mu\nu} W \frac{dx^\mu}{d\sigma} \frac{dx^\nu}{d\tau}$$

Observation: $F_{\mu\nu} = 0 \;\; \Rightarrow \;\; V$ does not depend upon the way we choose to scan Σ

Proof:

A flat $\Rightarrow$ W is path independent
$x \in \Sigma \;\; \Rightarrow \;\; \exists\, \gamma$ - a loop scanning Σ : $\;\; x \in \gamma \; \Rightarrow \; W$ defined by (1)
change the way we scan $\Sigma \;\; \Rightarrow \;\; W$ at x is constant
thus T is a local function on Σ

□

How does V change under a fixed boundary deformation of Σ?

$$\frac{d\delta V}{d\tau} - \delta V\ T - V\delta T = 0$$

But

$$V^{-1}V = 1 \ \Rightarrow\ \frac{dV^{-1}}{d\tau} = -TV^{-1} \ \Rightarrow\ \frac{d}{d\tau}\left(\delta V\ V^{-1}\right) = V(\delta T)V^{-1}$$

$$\delta V\ V^{-1} = \int d\tau V(\delta T)V^{-1}$$

$$\bullet A_\mu \text{ flat} \Rightarrow \quad A_\mu = -\partial_\mu W \cdot W^{-1}$$

$$\delta W = -A_\mu W \delta x^\mu, \quad \delta W^{-1} = W^{-1} A_\mu \delta x^\mu$$

$$\bullet \delta T = \int_0^{2\pi} d\sigma \left(\delta W^{-1} B_{\mu\nu} W + W^{-1} \delta B_{\mu\nu} W + W^{-1} B_{\mu\nu} \delta W \right) \frac{dx^\mu}{d\sigma} \frac{dx^\nu}{d\tau}$$

$$+ W^{-1} B_{\mu\nu} W \delta \left(\frac{dx^\mu}{d\sigma} \frac{dx^\nu}{d\tau} \right)$$

$$\Downarrow$$

$$\delta V \; V^{-1} = V(\tau)\left(\int_0^{2\pi} d\sigma W^{-1} B_{\mu\nu} W \frac{dx^\mu}{d\sigma}\delta x^\nu\right) V^{-1}(\tau) +$$

$$\int_0^\tau d\tau' V(\tau')\left[\int_0^{2\pi} W^{-1}\left(D_\lambda B_{\mu\nu} + D_\mu B_{\nu\lambda} + D_\nu B_{\lambda\mu}\right) W \frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\tau'}\delta x^\lambda\right] V^{-1}(\tau')$$

$$-\int_0^\tau d\tau' V(\tau')\left[T(B, A, \tau'), \int_0^{2\pi} W^{-1} B_{\mu\nu} W \frac{dx^\mu}{d\sigma}\delta x^\nu\right] V^{-1}(\tau')$$

Consider:

Σ a closed surface where the loop Γ collapses to x_0

Ω a volume $\quad \partial\Omega = \Sigma$

Scan Ω using closed surfaces parameterized by $\zeta \in [0, 2\pi]$

$$\zeta = 0 \quad \text{constant surface at } x_0$$
$$\zeta = 2\pi \quad \Sigma$$

Variation is the deformation of one closed surface into the other

$\delta = \delta\zeta \frac{d}{d\zeta}$

Then

$$\frac{dV}{d\tau} - \left(\int_0^{2\pi} d\zeta V^{-1} \mathcal{K} V \right) V = 0, \tag{5}$$

where

$$\mathcal{K} = \int_0^{2\pi} d\sigma W^{-1} \left(D_\lambda B_{\mu\nu} + D_\mu B_{\nu\lambda} + D_\nu B_{\lambda\mu} \right) W \frac{dx^\mu}{d\sigma} \frac{dx^\nu}{d\tau'} \frac{dx^\lambda}{d\zeta}$$

$$-[T(B, A, \tau), T(B, A, \zeta)]$$

Thus, V is defined by (4) and (5) $\Rightarrow$ a generalized non-Abelian Stokes theorem

$$P_2 \text{ Exp } \left(\int_\Sigma d\tau d\sigma W^{-1} B_{\mu\nu} W \frac{dx^\mu}{d\sigma} \frac{dx^\nu}{d\tau} \right) = P_3 \text{ Exp } \left(\int_\Omega d\zeta d\tau V \mathcal{K} V^{-1} \right)$$

$$\Downarrow$$

$$\mathcal{K} = 0 \quad \Rightarrow \quad \text{l.h.s. is } \Sigma \text{ independent}$$

When $\mathcal{K} = 0$?

$$D_\lambda B_{\mu\nu} + D_\mu B_{\nu\lambda} + D_\nu B_{\lambda\mu} = 0, \quad (a) \quad [T(B, A, \tau), T(B, A, \zeta)] = 0 \quad (b)$$

Type I

$$F_{\mu\nu} = 0, \quad D_\lambda B_{\mu\nu} = 0, \ [B^{(0)}_{\mu\nu}, B^{(0)}_{\rho\sigma}] = 0$$

where

$$B_{\mu\nu}(x) = W(x) B^{(0)}_{\mu\nu} W^{-1}(x)$$

Examples:

- BF theory theory without kinetic term, if $[B^{(0)}_{\mu\nu}, B^{(0)}_{\rho\sigma}] = 0$
- Chern-Simons theory
- $2+1$ gravity

Type II

$A_\mu \in \mathcal{G}$ - Lie algebra
$B_{\mu\nu} \in \mathcal{P}$ - abelian ideal $\Rightarrow$ $W^{-1}B_{\mu\nu}W \in \mathcal{P}$ $\Rightarrow$ (b)

Conditions:

$$F_{\mu\nu} = 0, \quad D_\mu \tilde{B}^\mu = 0 \tag{6}$$

- Conserved currents

$$j_\mu = W^{-1}\tilde{B}_\mu W$$

Number of currents = dim $\mathcal{P}$

Def: Model is integrable $\Leftrightarrow$ dim $\mathcal{P} = \infty$

- Gauge transformation

i)

$$A_\mu \to g A_\mu g^{-1} - \partial_\mu g g^{-1},$$

$$B_{\mu\nu} \to g B_{\mu\nu} g^{-1}, \qquad g \in \text{ Exp } \mathcal{G}$$

ii)

$$A_\mu \to A_\mu$$

$$B_{\mu\nu} \to B_{\mu\nu} + D_\mu \alpha_\nu - D_\nu \alpha_\mu$$

$\Rightarrow$ constructing solutions (dressing method)

In higher dim

Hypersurface independence of the hypersurface ordered operator $\mathcal{V}$

$$\mathcal{V} = \mathcal{P}\ \mathrm{Exp}\left(\int_{\Sigma_{d-1}} d\sigma^1 ... d\sigma^{d-1} W^{-1} B_{\mu_1 ... \mu_{d-1}} W \frac{dx^{\mu_1}}{d\sigma^1} \cdots \frac{dx^{\mu_{d-1}}}{d\sigma^{d-1}} \right)$$

$$\Uparrow$$

sufficient, local conditions

$$F_{\mu\nu} = 0, \quad D_\mu \tilde{B}^\mu = 0, \quad \tilde{B}^\mu = \frac{1}{(d-1)!} \epsilon^{\mu\nu_1 ... \nu_{d-1}} B_{\nu_1 ... \nu_{d-1}} \qquad (7)$$

• $\mathcal{G}$ Lie algebra of $G = SU(2)$ Lie group restricted to the equator of $SU(2)$
• $\mathcal{P} = \{\text{reps } R_{lm} \text{ of } su(2),\ m = \pm 1,\ l = 1 \ldots \infty\}$

Here spin-j representation
$[T_3, T_\pm] = \pm T_\pm,\ \ [T_+, T_-] = 2T_3$
$[T_3, P_m^{(j)}] = mP_m^{(j)}$
$[T_\pm, P_m^{(j)}] = \sqrt{j(j+1) - m(m \pm 1)}P_{m\pm 1}^{(j)}$
$[P_m^{(j)}, P_m^{(j')}] = 0$

Element of $G = SU(2)/U(1)$

$$W = \frac{1}{\sqrt{1+|u|^2}} \begin{pmatrix} 1 & iu \\ i\bar{u} & 1 \end{pmatrix}$$

- Triplet representation

$$A_\mu = -\partial_\mu W W^{-1} = \frac{1}{1+|u|^2}\left(-i\partial_\mu u T_+ - i\partial_\mu \bar{u} T_- + (u\partial_\mu \bar{u} - \bar{u}\partial_\mu u) T_3\right)$$

$$\tilde{B}_\mu = \frac{1}{1+|u|^2}\left(\mathcal{K}_\mu P_1^{(1)} - \bar{\mathcal{K}}_\mu P_{-1}^{(1)}\right)$$

GZC $\Rightarrow$ $(1+|u|^2)\,\partial^\mu \mathcal{K}_\mu - 2\bar{u}\,\mathcal{K}_\mu \partial^\mu u = 0$

Currents $J_\mu^{(1)} = \sum_{m=-1}^{1} J_\mu^{(1,m)} P_m^{(1)}$

$$J_\mu^{(1,1)} = \frac{1}{(1+|u|^2)^2}(\mathcal{K}_\mu + \bar{\mathcal{K}}_\mu u^2)$$
$$J_\mu^{(1,0)} = \frac{i\sqrt{2}}{(1+|u|^2)^2}(\bar{\mathcal{K}}_\mu u - \mathcal{K}_\mu \bar{u})$$

- Higher spin representation

GZC $\Rightarrow$ $(1+|u|^2)\,\partial^\mu \mathcal{K}_\mu - 2\bar{u}\,\mathcal{K}_\mu \partial^\mu u = 0$

Constrain $\mathcal{K}_\mu \partial^\mu u = 0$

- Infinitely many conserved currents = integrable system

$$J_\mu = \mathcal{K}_\mu \frac{\partial G}{\partial u} - \bar{\mathcal{K}}_\mu \frac{\partial G}{\partial \bar{u}}$$

Example: knotted solitons in Aratyn-Ferreira-Zimerman model

$$L = \left(H_{\mu\nu}^2\right)^{\frac{3}{4}}, \quad H_{\mu\nu} = \vec{n} \cdot (\partial_\mu \vec{n} \times \partial_\nu \vec{n})$$

where

$$\vec{n} = (n^1, n^2, n^3), \quad \vec{n}^2 = 1$$

Topological charge - Hopf index $Q_H \in \pi_3(S^2)$

$$\lim_{|\vec{x}| \to \infty} \vec{n} = \vec{n}_0 \ \text{ then } \ \vec{n} : R^3 \cup \{\infty\} \cong S^3 \to S^2$$

Stereographic projection

$$\vec{n} = \frac{1}{1+|u|^2}\left(u+\bar{u}, -i(u-\bar{u}), |u|^2-1\right)$$

$$L = 2^{3/2}\frac{(K_\mu \partial^\mu \bar{u})^{\frac{3}{4}}}{(1+|u|^2)^3}, \quad K_\mu = (\partial_\nu \bar{u} \partial^\nu u)\partial_\mu u - (\partial_\nu u)^2 \partial_\mu \bar{u}$$

Equation of motion

$$\partial_\mu \mathcal{K}^\mu = 0, \quad \mathcal{K}_\mu \equiv \frac{1}{1+|u|^2}(K_\mu \partial^\mu \bar{u})^{-1/4} K_\mu$$

Integrable model $\qquad \mathcal{K}_\mu \partial^\mu u \equiv 0$

Exact solitons
toroidal coordinates (η, ξ, ϕ)

$$x = q^{-1} \sinh \eta \cos \phi, \quad y = q^{-1} \sinh \eta \sin \phi$$

$$z = q^{-1} \sin \xi, \quad q = \cosh \eta - \cos \xi$$

solutions

$$u = \frac{\cosh \eta - \sqrt{n^2/m^2 + \sinh^2 \eta}}{\sqrt{1 + m^2/n^2 \sinh^2 \eta} - \cosh \eta} e^{i(m\xi + n\phi)}$$

topological charge $Q_H = -mn$

- S^3 Lagrangian

$$\mathcal{L} = \omega(u\bar{u}, \xi) H^q$$

where

$$H \equiv h_{\mu\nu\rho} u_\mu \bar{u}_\nu \xi_\rho,$$

$$h_{\mu\nu\rho} = u_\mu \bar{u}_\nu \xi_\rho + u_\rho \bar{u}_\mu \xi_\nu + u_\nu \bar{u}_\rho \xi_\mu - u_\nu \bar{u}_\mu \xi_\rho - u_\rho \bar{u}_\nu \xi_\mu - u_\mu \bar{u}_\rho \xi_\nu$$

◦ Infinitely many conservation laws
◦ Exact solution with nontrivial topological charge

$$\pi_4(S^3) \cong Z_2 \quad d = 4+1, q = 2/3$$
$$\pi_5(S^3) \cong Z_2 \quad d = 5+1, q = 5/6$$
$$\pi_6(S^3) \cong Z_{12} \quad d = 6+1, q = 1$$

- $\mathcal{G}$ Lie algebra of $G = SU(2)$ Lie group
- $\mathcal{P} = \{$reps R_{lm} of $su(2)$, $m = \pm 1$, $l = 1 \ldots \infty\}$

Skyrme model

$$L = \frac{f_\pi^2}{4} Tr\left(U^\dagger \partial_\mu U U^\dagger \partial^\mu U\right) - \frac{1}{32e^2} Tr\left[U^\dagger \partial_\mu U, U^\dagger \partial_\mu U\right]^2,$$

where

$$U = e^{i\xi_i \tau^i} = e^{i\xi T}, \quad \xi = \sqrt{\xi_1^2 + \xi_2^2 + \xi_3^2}$$

and

$$T = \frac{1}{1+|u|^2}\begin{pmatrix} |u|^2 - 1 & -2iu \\ 2i\bar{u} & 1-|u|^2 \end{pmatrix}$$

- Triplet representation

$$A_\mu = \frac{1}{1+|u|^2}\left(-i\partial_\mu u T_+ - i\partial_\mu \bar{u} T_- + \frac{1}{2}(u\partial_\mu \bar{u} - \bar{u}\partial_\mu u) T_3\right)$$

$$B_\mu = -iR_\mu P_3 + \frac{2\sin\xi}{1+|u|^2}\left(e^{i\xi} S_\mu P_+ - e^{-i\xi}\bar{S}_\mu P_-\right)$$

$$R_\mu = \partial_\mu \xi - 8\lambda \frac{\sin^2\xi}{(1+|u|^2)^2}\left(N_\mu + \bar{N}_\mu\right)$$

$$S_\mu = \partial_\mu u + 4\lambda\left(M_\mu - \frac{2\sin^2\xi}{(1+|u|^2)^2} K_\mu\right)$$

$$M_\mu = (\partial_\nu u \partial^\nu \xi)\partial_\mu \xi - (\partial_\nu \xi)^2 \partial_\mu u, \quad N_\mu = (\partial_\nu u \partial^\nu \bar{u})\partial_\mu \xi - (\partial^\nu u \partial_\nu \xi)\partial_\mu \bar{u}$$

GZC $\Rightarrow$ Skyrme e.o.m.

- Higher spin representation

GZC $\Rightarrow$ Skyrme e.o.m.

$$\text{Constrains} \quad \mathcal{S}_\mu \partial^\mu u = 0 \text{ and } \quad \mathcal{S}_\mu \partial^\mu \xi = 0$$

- Integrable submodel

$$\partial_\mu S^\mu = 0, \quad \partial_\mu R^\mu = \frac{4 \sin\xi \cos\xi}{(1+|u|^2)^2} (S_\mu \partial^\mu \bar{u})$$

Infinitely many conserved currents

$$J^G_\mu = \mathcal{S}_\mu \frac{\partial G}{\partial u} - \bar{\mathcal{S}}_\mu \frac{\partial G}{\partial \bar{u}}$$

$$J^{H_1,H_2}_\mu = 4 \sin\xi \cos\xi \left(H_1 S_\mu + H_2 S^*_\mu\right) - (1+|u|^2)^2 \left(\frac{\partial H_1}{\partial \bar{u}} + \frac{\partial H_2}{\partial u}\right)$$

The simplest skyrmion with $Q = \pm 1$ is a solution of the subsystem

$$\frac{\vec{x}}{r} \equiv \frac{1}{1+|z|^2}\left(-i(z-\bar{z}), z+\bar{z}, |z|^2-1\right)$$

$$u = z, \quad \xi = \xi(r)$$

The constrains are

$$(\partial_\mu u)^2 = 0, \quad \partial_\mu u \partial^\mu \xi = 0$$

- New criterion for $d > 2$ integrability
- New integrable models
  - Exact solutions
  - Nontrivial topological charge
  - Infinitely many conserved quantities
- Integrable sectors of non-integrable models