Hydrodynamics and gauge/gravity duality

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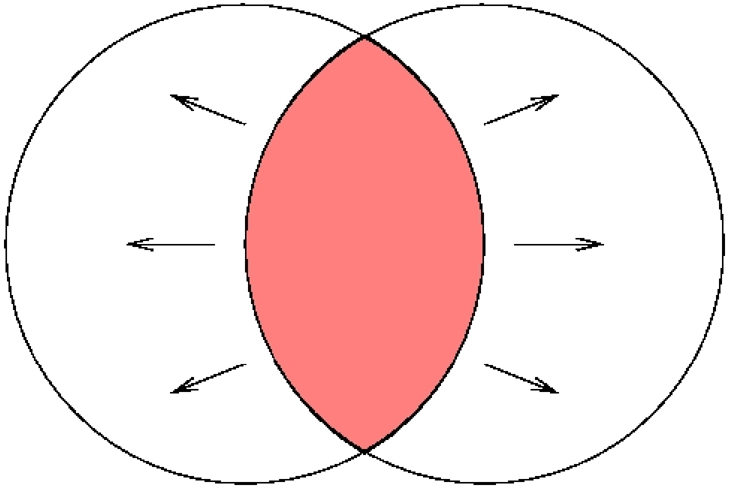
Topics of this talk

- Hydrodynamics
  - As efffective theory
  - First-order hydrodynamics
  - Second-order hydrodynamics
- Gauge/gravity duality
  - AdS/CFT prescription for real-time field theory
  - Transport coefficients from AdS/CFT

Motivation and Introduction

Motivation for studying hydrodynamics

- Applications, e.g., in heavy ion collisions



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- Elliptic flow: final particles have anisotropic momentum distribution
- is a collective effect
- Hydrodynamic models work well for elliptic flow.

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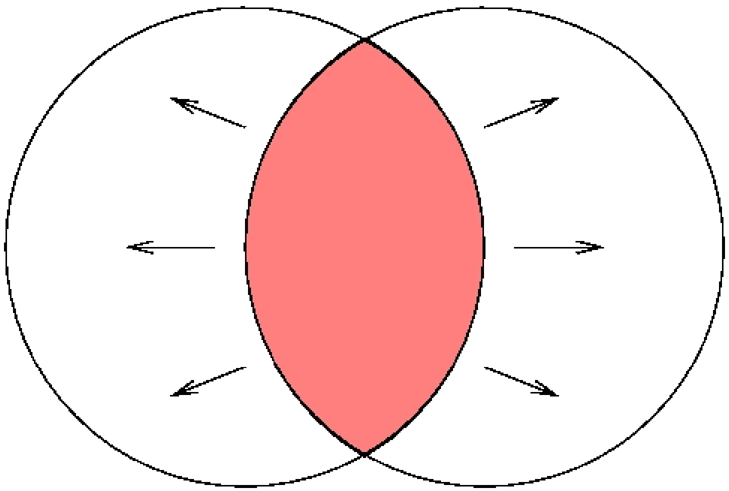

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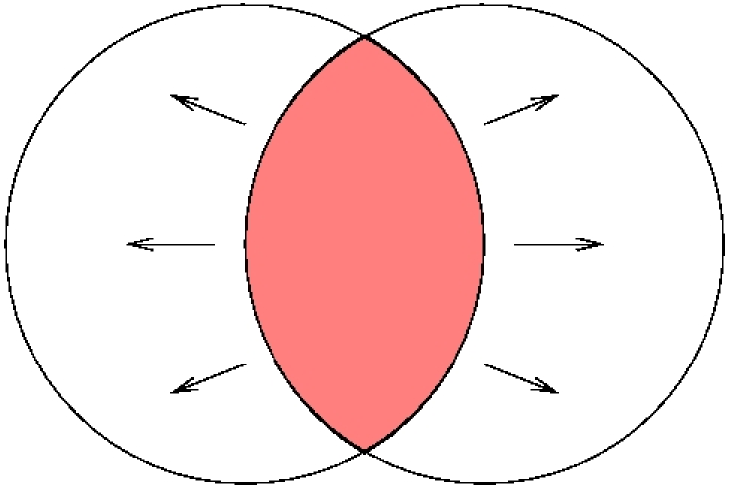

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- Conceptually a much simpler theory than QFT:
  - Few d.o.f.
  - Classical: bosonic modes at $\omega \ll T$

Why gauge/gravity duality

Practical consideration:

- Strong coupling, not treatable by other methods
- Simple calculations

Conceptual consideration:

- Deep connection between QFT and black-hole physics
- sharp contrast to weak coupling:
  weak coupling: QFT $\rightarrow$ kinetic theory $\rightarrow$ hydro
  strong coupling: QFT $\rightarrow$ hydro

Original AdS/CFT correspondence

Maldacena; Gubser, Klebanov, Polyakov; Witten

between $N = 4$ supersymmetric Yang-Mills theory
and type IIB string theory on AdS$_5\times$ S^5

$$ds^2 = \frac{R^2}{z^2}(d\vec{x}^2 + dz^2) + R^2 d\Omega_5^2$$

Large 't Hooft limit in gauge theory $\Leftrightarrow$ small curvature limit in string theory

$$g^2 N_c \gg 1 \Leftrightarrow R/l_s = \sqrt{\alpha'}\, R \gg 1$$

Correlation function are computable at large 't Hooft coupling, where string theory $\longrightarrow$ supergravity.

The dictionary of gauge/gravity duality

gauge theory	gravity
operator $\hat{O}$ energy-momentum tensor $T_{\mu\nu}$ dimension of operator	field ϕ graviton $h_{\mu\nu}$ mass of field
globar symmetry conserved current anomaly ...	gauge symmetry gauge field Chern-Simon term ...

$$\int e^{iS_{4\mathrm{D}}+\phi_0 O} = \int e^{iS_{5\mathrm{D}}}$$

where $S_{5\mathrm{D}}$ is computed with nontrivial boundary condition

$$\lim_{z\to 0}\phi(\vec{x},z) = \phi_0(\vec{x})$$

Green's function from AdS/CFT

Let us compute the correlator of $O = -\mathcal{L}$, which corresponds to the dilaton Φ in supergravity.
First write down the field equation

$$\partial_\mu(\sqrt{-g}\, g^{\mu\nu}\partial_\nu\phi) = 0$$

Solution with boundary condition $\phi \to J$ at $z \to 0$:

$$\phi(z,p) = f_p(z)J(p), \qquad f_p(z) = \tfrac{1}{2}(pz)^2 K_2(pz)$$

Substituting to the action one finds

$$S_{\rm cl} = \int_p J(-p)\mathcal{F}(p,z)J(p)|_{z\to 0}, \qquad \mathcal{F}(p,z) = \frac{N^2}{16\pi^2} z^{-3} f_{-p}(z) f_p'(z)$$

Correlator is obtained by differentiating $S_{\rm cl}$ with respect to J:

$$\langle OO\rangle_p = -2\lim_{z\to 0}\mathcal{F}(p,z) = \frac{N^2}{64\pi^2} p^4 \ln(p^2)$$

Other correlators

Other correlators can be computed similarly:

- Correlators of R-charge currents: solve Maxwell equation

  $$D_\mu F^{\mu\nu} = 0$$

  with boundary condition

  $$\lim_{z\to 0} A_\mu = A_\mu^0$$

  and differentiate the 5D action with respect to A_μ^0

- Correlators of stress-energy tensor: solve the Einstein equation with boundary condition

  $$ds^2 = \frac{R^2}{z^2}(dz^2 + g_{\mu\nu}^0 dx^\mu dx^\nu)$$

  and then differentiate the gravitational action with respect to $g_{\mu\nu}^0$.

Finite-temperature AdS/CFT correspondence

Black 3-brane solution:

$$ds^2 = \frac{r^2}{R^2}[-f(r)dt^2 + d\vec{x}^2] + \frac{R^2}{r^2 f(r)}dr^2 + R^2 d\Omega_5^2, \qquad f(r) = 1 - \frac{r_0^4}{r^4}$$

- $r_0 = 0,\ f(r) = 1$: is $\text{AdS}_5 \times \text{S}^5$, $r = R^2/z$.
- $r_0 \neq 0$: corresponds to $\mathcal{N} = 4$ SYM at temperture

$$T = T_H = \frac{r_0}{\pi R^2}$$

Entropy $= A/4G$

$$S = \frac{\pi^2}{2} N_c^2 T^3 V_{\text{3D}}$$

This formula has the same N^2 behavior as at zero 't Hooft coupling $g^2 N_c = 0$ but the numerical coefficient is 3/4 times smaller.

Thermodynamics

$$S = f(g^2 N_c) \frac{2\pi^2}{3} N_c^2 T^3 V_{3D}$$

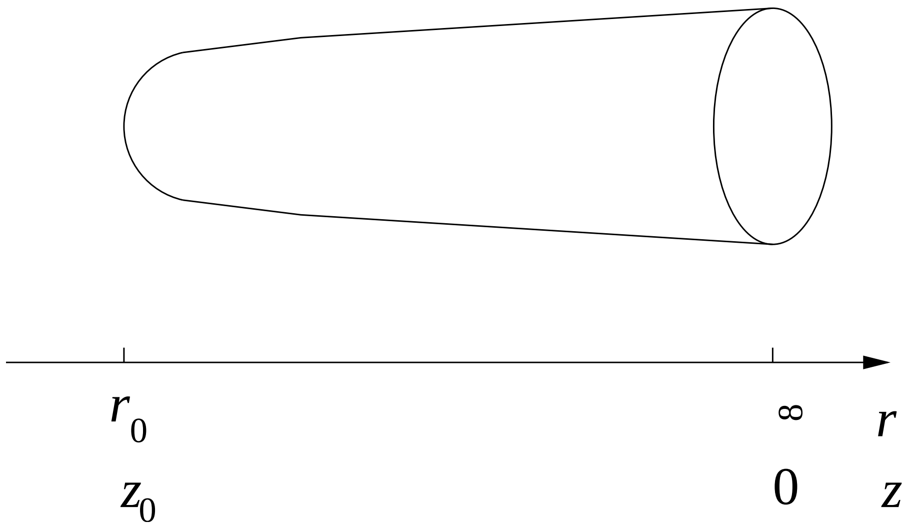
where the function f interpolates between weak-coupling and strong-coupling values, which differ by a factor of 3/4:

$$f(\lambda) = \begin{cases} 1 - \frac{3}{2\pi^2}\lambda + \frac{\sqrt{2}+3}{\pi^3}\lambda^{3/2} + \cdots, & \lambda \ll 1 \\ \frac{3}{4} + \frac{45\zeta(3)}{32\lambda^{3/2}} + \cdots, & \lambda \gg 1 \end{cases} \qquad (0)$$

Euclidean correlators

Correlation functions: can be obtained by a finite-temperature version of AdS/CFT:

$$Z_{4\mathrm{D}}[J] = e^{-S[\phi_{\mathrm{cl}}]}$$



Due to geometry, correlation functions are periodic in Euclidean time.

Note: fixing the boundary condition at the boundary $r = \infty$ completely determines the solution. No separate boundary condition at $r = r_0$ is necessary

Hydrodynamics

Hydrodynamics

- is the effective theory describing the long-distance, low-frequency behavior of interacting finite-temperature systems. Hydrodynamic regime
- Valid at distances $\gg$ mean free path, time $\gg$ mean free time.
- At these length/time scales: local thermal equilibrium: T, μ vary slowly in space.
- Simplest example of a hydrodynamic theory: the Navier-Stokes equations
- The quark-gluon plasma can be described by a relativistic version of the Navier-Stokes equation.
- All microscopic physics reduces to a small number of *kinetic coefficients* (shear viscosity η, bulk viscosity, diffussion coeffecients).

Relativistic hydrodynamics

Consider a neutral plasma: no conserved charge, except energy and momentum.

Thermodynamics: one variable T

$$P = P(T), \quad s = \frac{\partial P}{\partial T}, \quad \epsilon = Ts - P$$

Ideal (zeroth order) hydrodynamics

$$\nabla_\mu T^{\mu\nu} = 0 \qquad T^{\mu\nu} = (\epsilon + P) u^\mu u^\nu + P g^{\mu\nu}$$

4 equations for 4 unknowns (T and u^μ, $u^2 = -1$).

Viscous hydrodynamics

$$T^{\mu\nu} = T^{\mu\nu}_{\text{ideal}} + \underbrace{\Pi^{\mu\nu}}_{\text{viscous stress}}$$

Ambiguity of defining u^μ beyond leading order: fixed by $u_\mu \Pi^{\mu\nu} = 0$ ("Landau-Lifshitz frame")

Physical interpretation: in the local rest frame momentum density is zero: $T^{0i} = 0$.

Shear and bulk viscosities

The most general form of the viscous stress is

$$\Pi^{\mu\nu} = -\eta \partial^{\langle\mu} u^{\nu\rangle} - \zeta P^{\mu\nu} (\partial \cdot u)$$

$$P^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu$$

$$A^{\langle\mu\nu\rangle} = \frac{1}{2} P^{\mu\alpha} P^{\nu\beta} (A_{\alpha\beta} + A_{\beta\alpha}) - \frac{1}{3} P^{\mu\nu} P^{\alpha\beta} A_{\alpha\beta}$$

Shear viscosity η and bulk viscosity ζ. Affect damping of shear and sound modes.

In theories with conformal invariance (such as $\mathcal{N} = 4$ SYM theory), $T^\mu_\mu = 0$ leads to

$$\epsilon = 3P, \qquad \zeta = 0$$

Linearized hydrodynamics

We linearize around the static solution:

$$\begin{aligned} \epsilon &= \epsilon_0 + \delta\epsilon \\ P &= P_0 + \delta P \\ u^0 &= 1 + O(\vec{u}^2) \\ \vec{u} &= \vec{u} \ll 1 \end{aligned}$$

Energy-momentum tensor:

$$\begin{aligned} T^{00} &= \epsilon_0 + \delta\epsilon \\ T^{0i} &= (\epsilon_0 + P_0)u^i \\ T^{ij} &= (P_0 + \delta P)\delta^{ij} - \eta(\partial_i u_j + \partial_j u_i) - (\zeta - \tfrac{2}{3}\eta)\delta^{ij}\partial_k u^k \end{aligned}$$

Linearized hydrodynamic equations:

$$\omega\delta\epsilon - (\epsilon_0 + P_0)q^i u^i = 0$$

$$[(\epsilon_0 + P_0)\omega + i\eta q^2]u^i - q^i\delta P + i(\zeta + \tfrac{1}{3}\eta)q_i(\vec{q}\cdot\vec{u}) = 0$$

Shear modes

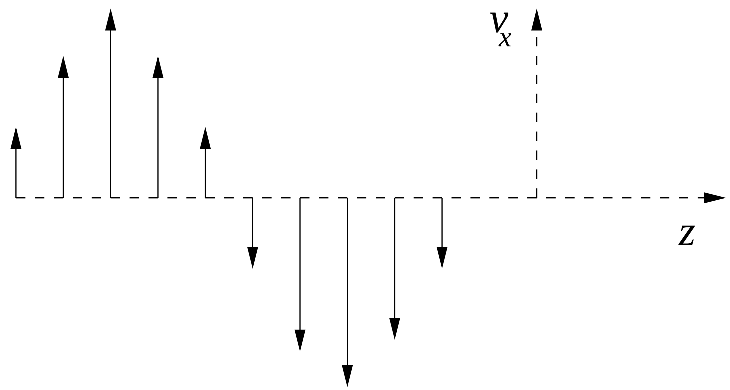
Decompse the velocity $\vec{u}$ into longitudinal and transverse parts:

$$\vec{u} = \vec{u}_\perp + \vec{u}_\parallel, \qquad \vec{q} \cdot \vec{u}_\perp = 0, \quad \vec{u}_\parallel \parallel \vec{q}$$

Equation for transverse modes:

$$[(\epsilon_0 + P_0) + i\eta q^2] u_\perp = 0$$

corresponds to an overdamped shear mode



with dispersion relation

$$\omega = -i\mathcal{D}q^2, \qquad \mathcal{D} = \frac{\eta}{\epsilon_0 + P_0}$$

Sound modes

Longitudinal modes: coupled system of equations for $\delta\epsilon$ and $u_{\|}$:

$$\omega\delta\epsilon - (\epsilon_0 + P_0)qu_{\|} = 0$$

$$-q\left(\frac{\partial P}{\partial \epsilon}\right)\delta\epsilon + [(\epsilon_0 + P_0)\omega + i(\zeta + \tfrac{4}{3}\eta)q^2]u_{\|} = 0$$

yields propagating sound modes

$$\omega = \pm c_s q - i\Gamma q^2, \qquad c_s = \left(\frac{\partial P}{\partial \epsilon}\right)^{1/2}, \quad \Gamma = \frac{1}{2}\frac{\zeta + \frac{4}{3}\eta}{\epsilon_0 + P_0}$$

Kubo's Formula

Kubo's formula: preliminaries

Viscosities can be expressed in terms of Green's functions

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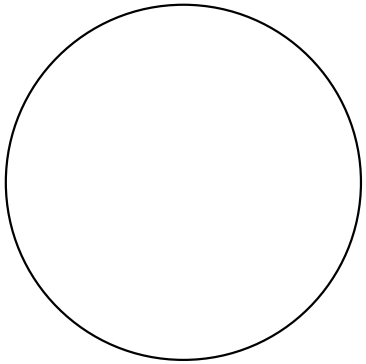

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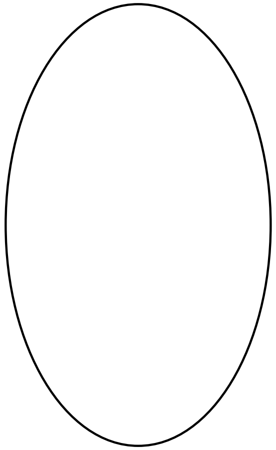

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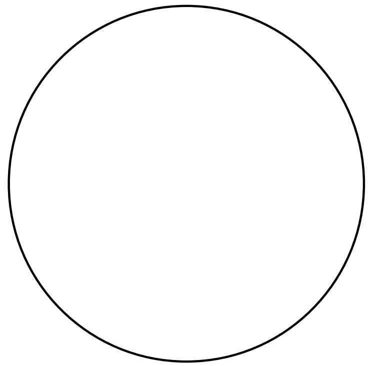

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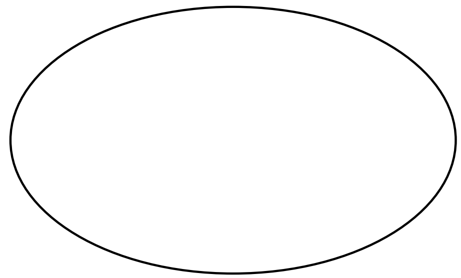

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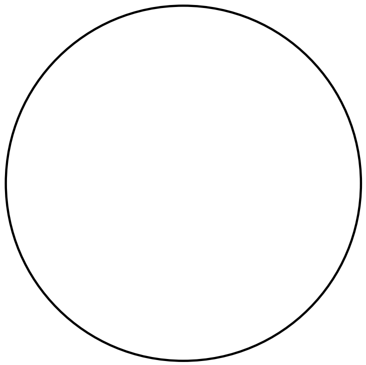

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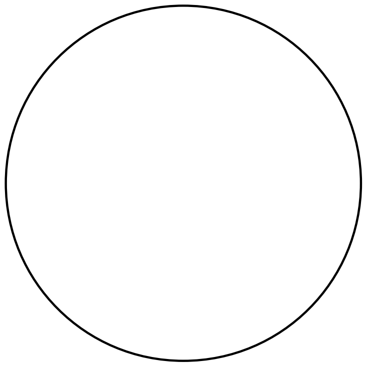


Long-wavelength ravitational waves induce hydrodynamic perturbations

Generalization to curved space

- To find the response of a hydrodynamic medium to external gravitational perturbations, one needs to generalize the hydrodynamic equations to curved spacetime.
- Replacing derivative by covariant derivative:

$$\nabla_\mu T^{\mu\nu} = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\,T^{\mu\nu}) + \Gamma^\nu_{\mu\lambda}T^{\mu\lambda} = 0$$

$$\tau^{\mu\nu} = -\eta P^{\mu\alpha}P^{\nu\beta}(\nabla_\alpha u_\beta + \nabla_\beta u_\alpha) + \cdots$$

Linear response theory

Consider a fluid initially in thermal equilibrium: $T = T_0$, $u^\mu = (1, \vec{0})$.
Let us probe the fluid by a weak metric perturbation:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

Linear response theory:

$$\langle \delta T^{\mu\nu}(x) \rangle = -\int dy \, G_R^{\mu\nu,\alpha\beta}(x-y) h_{\alpha\beta}(y)$$

where G_R is the retarded propagator of $T^{\mu\nu}$

We can use the hydrodynamic equation to find G_R at low momenta

Kubo's formula

For simplicity, consider perturbation spatially homogeneous, dependent on time only:

$$h_{xy} = h_{xy}(t)$$

all other components are zero

Spin-2 perturbation: does not excite motion of the fluid: $u^\mu = (1, \vec{0})$, $T = T_0$.

Nontrivial response from Christofell symbols:

$$\delta T^{xy} = -P h_{xy} - \eta(\nabla_x u_y + \nabla_y u_x)$$

but

$$\nabla_i u_j = \underbrace{\partial_i u_i}_{=0} - \Gamma^0_{ij} u_0 = \frac{1}{2}\partial_t h_{ij}$$

Therefore:

$$G^{xy,xy}(\omega, \vec{0}) = P - i\eta\omega$$

We find Kubo's formula relating shear viscosity with correlation function:

$$\eta = -\lim_{\omega\to 0}\frac{1}{\omega}\,\mathrm{Im}\, G_R^{xy,xy}(\omega, \vec{0})$$

Summary: two-point hydrodynamic correlators

Consider a momentum $(\omega, 0, 0, q)$.

Components of $T^{\mu\nu}$ are classified by O(2) in xy directions

Expectation:

- Spin-2 components (e.g., T^{xy}): correlators do not show low-momentum singularity, but imaginary part is tied to shear viscosity through Kubo's formula
- Spin-1 componets (e.g., T^{0x}, T^{zx}): correlators show shear-mode pole $\omega = -iDq^2$.
- Spin-0 components (e.g., T^{00}): correlators have sound-wave pole.

Note: all correlators above are real-time correlators.

Real-Time Finite-Temperature AdS/CFT

Real-time correalation functions from AdS/CFT

Naive generalization of AdS/CFT correspondence runs into problem: solution is not uniquely fixed by the boundary condition at $z = 0$.

$$\partial_\mu(\sqrt{-g}\, g^{\mu\nu}\partial_\nu\phi) = 0$$

Two solutions near $z = z_0$:

$$f_\pm \sim (z - z_0)^{\pm i\omega/4\pi T}$$

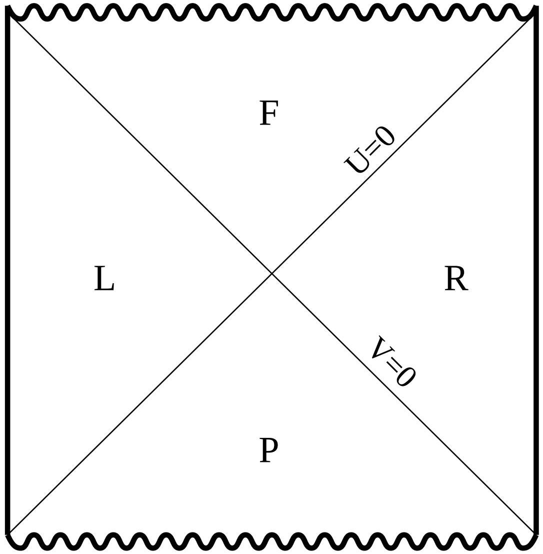
are both regular.

Correspond to incoming and outgoing waves.

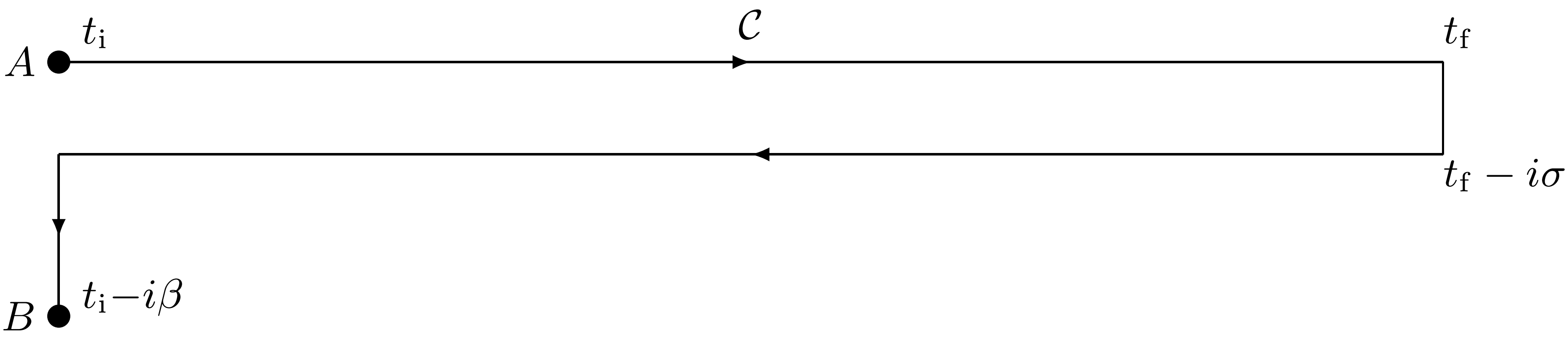
Picking only incoming wave solution provides partial relieve, but the complete formulation of real-time AdS/CFT requires a more radical approach.

Penrose diagram and Schwinger-Keldysh formalism

Extend the metric beyond the horizon: Penrose diagram



Two boundaries (left and right) correspond to two contours in the close-time-path formulation of real-time finite-temperature field theory with $\sigma = \beta/2$



Formulation of real-time AdS/CFT

$$Z_{4\mathrm{D}}[J_1, J_2] = e^{iS[\phi_{\mathrm{cl}}]}$$

where ϕ_{cl}

- satisfies field equation in the whole Penrose diagram
- approaches J_1 on right boundary and J_2 on left boundary
- satisfies certain boundary conditions at the horizon (readily formulated in Kruskal coordinates)

Diffentiating the partition function one can find any real-time Green's function. For example, the Schwinger-Keldysh propagators are

$$G_{ab} \sim \frac{\delta^2 S}{\delta J_a \delta J_b}, \qquad a, b = 1, 2.$$

Can be shown to satisfy all properties required for Schwinger-Keldysh propagator (fluctuation dissipation theorem etc.)

Retarded propagator from AdS/CFT

Retarded Greens function can be computed from the component of this matrix:

$$G_R = G_{11} - \underbrace{e^{-\beta\omega/2}}_{\sigma=\beta/2} G_{12}$$

But using the general formalism one can show that the following procedure works for 2-point retarded Green's function

- Forget about Penrose diagram
- Find the mode function $f_p(z)$ with incoming-wave boundary condition at the horizon
- Use the same formula as at zero temperature:

$$G_R \sim \lim_{z\to 0} z^{-3} f_{-p}(z) f'_p(z)|_{z\to 0}$$

Naively: reduce the action to boundary integrals and pick up only a horizon contribution.

Hydrodynamics from Gauge/Gravity Duality

Calculating η from AdS/CFT

First write down equation for $\phi = h_y^x$,

$$\phi_p'' - \frac{1+u^2}{uf}\phi_p' + \frac{w^2 - k^2 f}{uf^2}\phi_p = 0, \qquad u = z^2/z_0^2$$

where $w = \omega/2\pi T$, $k = q/2\pi T$.

Solution for small w, q:

$$f_p = (1-u^2)^{-iw/2} + \cdots$$

Applying general formulas for the retarded correlators:

$$G_R^{xy,xy} = \# u^{-1} f_{-p} f_p' |_{u\to 0}$$

The coefficient $\#$ is fixed by the normalization of Hilbert-Einstein action. We find

$$G_R^{xy,xy}(\omega) = -i \underbrace{\frac{\pi}{8} N^2 T^3}_{\eta} \cdot \omega$$

Hydrodynamic poles

One can find poles in the Green's function that correspond to the shear ($\omega \sim -iq^2$) and sound modes.
Shear: start with the unperturbed metric

$$ds^2 = \frac{(\pi T R)^2}{u}(-f(u)dt^2 + d\vec{x}^2) + \frac{R^2}{4u^2 f(u)}du^2, \quad f(u) = 1 - u^2$$

Assume nonvanishing h_{tx}, h_{zx}; momentum along z direction In terms of

$$H_t = \frac{uh_{tx}}{(\pi T R)^2}\,, \qquad H_z = \frac{uh_{zx}}{(\pi T R)^2}$$

the field equations are

$$H_t' + \frac{kf}{w}H_z' = 0$$
$$H_t'' - \frac{1}{u}H_t' - \frac{wk}{uf}H_z - \frac{k^2}{uf}H_t = 0$$

where $k = q/(2\pi T)$ and $w = \omega/(2\pi T)$.

Shear pole

Can be converted to one 2nd order equation for H_t':

$$H_t''' - \frac{2u}{f} H_t'' + \frac{2uf - k^2 f + w^2}{uf^2} H_t' = 0$$

Boundary condition at $u = 1$: $H_t'(u) \sim (1-u)^{-iw/2}$ (incoming waves)
The equations can be solved at small ω, q
Boundary action:

$$S_{\rm boundary} \sim \frac{1}{u} (H_t(u) H_t'(u) - H_z(u) H_z'(u)) \Big|_{u=0}$$

differentiating which one finds the corerlators, e.g.,

$$G^{tx,tx}(\omega, q) = \frac{\pi N^2 T^3}{8} \frac{q^2}{i\omega - \mathcal{D} q^2}$$

where

$$\mathcal{D} = \frac{1}{4\pi T}$$

Hydrodynamic modes (continued)

The value of $\mathcal{D}$ extracted from the pole is consistent with

$$\mathcal{D} = \frac{\eta}{\epsilon + P}$$

as required by hydrodynamic equations.

Moreover: the correlator $\langle T^{00} T^{00} \rangle$ computed from AdS/CFT has a pole corresponding to sound wave:

$$\omega = \frac{q}{\sqrt{3}} - i\Gamma q^2$$

The sound damping rate is also consistent with the calculated η and $\zeta = 0$.

$$\Gamma = \frac{1}{6\pi T} = \frac{2}{3}\frac{\eta}{\epsilon + P}$$

Viscosity entropy ratio

One can consider other theories with gravity duals

It seems that for each theory one has to compute η again.

However, it turns out to be unnecessary, since the ration η/s can be shown to be constant across all theories with gravity duals

$$\frac{\eta}{s} = \frac{1}{4\pi}$$

Kovtun, Son, Starinets; Buchel, Liu

One method is to

- Use AdS/CFT and Kubo's formula to map viscosity into of graviton absorption cross section (by black hole)
- Using Einstein equation, show that the absorption cross section is equal to the area of the horizon
- Use Bekenstein's formula for the entropy $S = A/(4G)$ to show the constancy of η/s.

Viscosity/entropy ratio and uncertainty principle

Estimate of viscosity from kinetic theory

$$\eta \sim \rho v \ell, \qquad s \sim n = \frac{\rho}{m}$$

$$\frac{\eta}{s} \sim m v \ell \sim \hbar \frac{\text{mean free path}}{\text{de Broglie wavelength}}$$

Quasiparticles: de Broglie wavelength $\lesssim$ mean free path

Therefore $\eta/s \gtrsim \hbar$

- Weakly interacting systems have $\eta/s \gg \hbar$.
- Theories with gravity duals have universal η/s, but we don't know how to derive the constancy of η/s without AdS/CFT.

Corrections to η/s computed Buchel, Liu, Starinets; Myers, Paulos, Sinha

$$\frac{\eta}{s} = \frac{1}{4\pi}\left(1 + \frac{15\zeta(3)}{\lambda^{3/2}} + \frac{5\lambda^{1/2}}{16N^2} + \cdots\right)$$

Gravity/hydrodynamics correspondence

- Starting from a black-brane solution, i.e.,

$$ds^2 = \frac{r^2}{R^2}(-f dt^2 + d\vec{x}^2) + \frac{R^2}{r^2 f} dr^2, \qquad f = 1 - \frac{r^4}{r_0^4}$$

- Construct a family of configurations by changing $T \sim r_0/R^2$ and boosting along $\vec{x}$ directions by velocity $\vec{u}$

$$g_{\mu\nu} = g_{\mu\nu}(z; T, u^\mu)$$

- Promote T and $\vec{u}$ into fields.
- Require regularity away from $r = 0 \Rightarrow$ hydrodynamic equations

Bhattacharyya, Hubeny, Minwala, Rangamani

Second-Order Hydrodyamics

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- The hydrodynamic loops are suppressed in the large N limit Kovtun, Yaffe

Corrections to hydrodynamics

- Can one systematically go beyond the first-order hydrodynamics? Second-order hydrodynamics?
- Sharpen the question: formulate an effective theory which captures one more order in derivative expansion, i.e.,

$$G^{xy,xy}(\omega) = P - i\eta\omega + \#\omega^2$$

- Howerver it turns out that the correction after η is nonanalytic:

$$G^{xy,xy}(\omega) = P - i\eta\omega + \#\omega^{3/2}$$

  Origin: hydrodynamic loops, similar to chiral logarithms in chiral dynamics

- The hydrodynamic loops are suppressed in the large N limit   Kovtun, Yaffe
- In the large $N \to \infty$ limit (fix momenta) no nonanalytic behavior: second-order hydrodynamics

Second-order hydrodynamics

$$T^{\mu\nu} = (\epsilon + P)u^\mu u^\nu + P g^{\mu\nu} - \eta \nabla^{\langle\mu} u^{\nu\rangle} + \text{terms with two derivatives}$$

There are 16 possible terms with two derivatives.
Assume fundamental theory is a CFT,

$$T^\mu{}_\mu = 0 \qquad \text{in flat space}$$

In curved space: Weyl anomaly

$$g_{\mu\nu} T^{\mu\nu} \sim R^2_{\mu\nu\alpha\beta} \qquad \text{in curved space}$$

But $R \sim \partial^2 g_{\mu\nu}$: Weyl anomaly reproduced in hydrodynamics only at fourth order in derivatives.

$\Rightarrow g_{\mu\nu} T^{\mu\nu} = 0$ for our purposes

First order: $\zeta = 0$,

Second order: tracelessness of $T^{\mu\nu}$ reduces to 8 the number of possible structures in $\Pi^{\mu\nu}$

Conformal invariance

Further constraint: $T^{\mu\nu}$ transforms simply under Weyl transformation

$$g_{\mu\nu} \to e^{2\omega} g_{\mu\nu}, \qquad T_{\mu\nu} \to e^{6\omega} T_{\mu\nu}$$

8 $\to$ 5 possible structures in $\Pi^{\mu\nu}$

$$\Pi^{\mu\nu}_{\text{2nd order}} = \eta\tau_\pi \left[{}^{\langle} D\sigma^{\mu\nu\rangle} + \frac{1}{3}\sigma^{\mu\nu}(\nabla\cdot u) \right] + \kappa \left[R^{\langle\mu\nu\rangle} - 2u_\alpha R^{\alpha\langle\mu\nu\rangle\beta} u_\beta \right] + \lambda_1 \sigma^{\langle\mu}{}_\lambda \sigma^{\nu\rangle\lambda} + \lambda_2 \sigma^{\langle\mu}{}_\lambda \Omega^{\nu\rangle\lambda} + \lambda_3 \Omega^{\langle\mu}{}_\lambda \Omega^{\nu\rangle\lambda}$$

$$D \equiv u^\mu \nabla_\mu$$

$$\sigma^{\mu\nu} = 2\nabla^{\langle\mu} u^{\nu\rangle}$$

$$\Omega^{\mu\nu} = \frac{1}{2}(\nabla^{\langle\mu} u^{\nu\rangle} - \nabla^{\langle\nu} u^{\mu\rangle})$$ vorticity

κ only in curved space, but affects 2-point function of $T^{\mu\nu}$

λ_i nonlinear response

Second-order transport coefficients from AdS/CFT

τ_π and κ can be found similarly to η: using a Kubo's like formula

- Within hydro: compute some $\langle T^{\mu\nu}T^{\alpha\beta}\rangle$ from linear response theory: response to gravitational perturbations $g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$
- Compare with AdS/CFT calculations

Example: for momentum $q = (\omega, 0, 0, k)$ hydrodynamics predicts

$$\langle T^{xy}T^{xy}\rangle(\omega, k) = P - i\eta\omega + \eta\tau_\pi\omega^2 - \frac{\kappa}{2}(\omega^2 + k^2)$$

Matching with AdS/CFT calcualation, yields

$$\tau_\pi = \frac{2 - \ln 2}{2\pi T}, \qquad \kappa = \frac{\eta}{\pi T}$$

One can match to the soud-wave dispersion:

$$\omega = c_s q - i\Gamma q + \frac{\Gamma}{c_s}(c_s^2\tau_\pi - \tfrac{1}{2}\Gamma)q^3$$

to find the same value for τ_π.

Nonlinear coefficients $\lambda_{1,2,3}$

One needs to look beyond small perturbations around thermal equilibrium.
λ_1: can be found from long-time tail of a boost-invariant solution
Janik, Peschanski, Heller

$$\epsilon(\tau) \sim \frac{1}{\tau^{4/3}} - \frac{2\eta}{\tau^2} + \frac{\#}{\tau^{8/3}} \tag{-6}$$

Maching the coefficient of $\tau^{-8/3}$ term:

$$\lambda_1 = \frac{\eta}{2\pi T}$$

Bhattacharyya et al. also found

$$\lambda_1 = \frac{\eta}{2\pi T}\,, \qquad \lambda_2 = -\frac{2\ln 2}{2\pi T}\eta, \qquad \lambda_3 = 0$$

Israel-Stewart theory

- In the literature, variations of the Israel-Stewart theory are used
- Modified relatiship between $\Pi^{\mu\nu}$ and $\nabla^\mu u^\nu$

$$(\tau_\pi u^\lambda \nabla_\lambda + 1)\Pi^{\mu\nu} = -\eta\sigma^{\mu\nu}$$

- Frequently terms required by Weyl invariance are thrown away,

$$^{\langle} D\Pi^{\mu\nu\rangle} + \frac{4}{3}\Pi^{\mu\nu}(\nabla\cdot u)$$

  (but are kept in some papers, e.g., Romatschke & Romatschke). Such terms may be numerically important.
- In addition, $\lambda_1 = \lambda_3 = 0$ in IS theory; in $\mathcal{N} = 4$ SYM $\lambda_1 \neq 0$ (but $\lambda_3 = 0$).
- Additional terms nonlinear: not important for sound wave propagation, but important for Bjorken expansion

Other transport coefficients

Bulk viscosity: see also talk by Rocha

- Bulk viscosity ζ nonvanishing in theories with broken conformal symmetry
- In theories with gravity duals, it seems that c_s^2 is always less than $\frac{1}{3}$
- Parametrically

$$\frac{\zeta}{\eta} \sim (\frac{1}{3} - c_s^2)$$ Buchel

Diffusion coefficients:

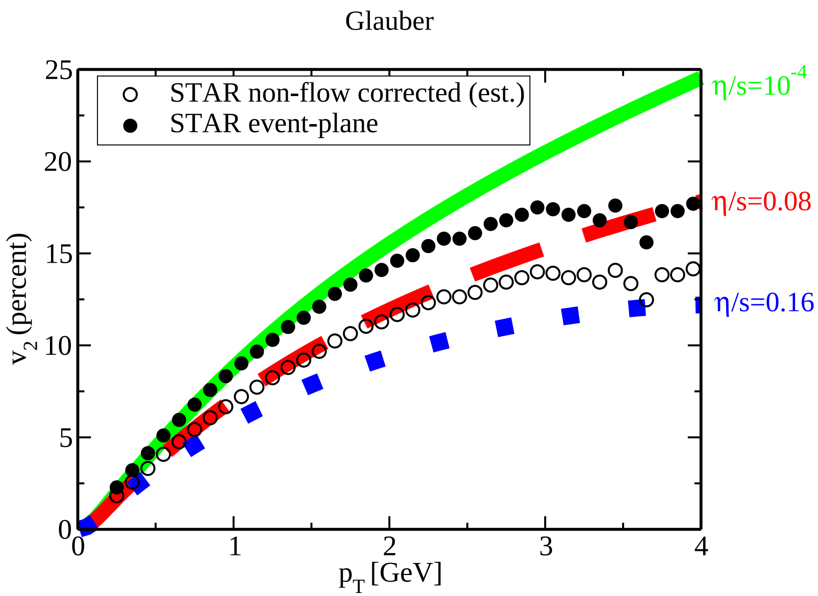
- Conserved charges in a plasma diffuse:

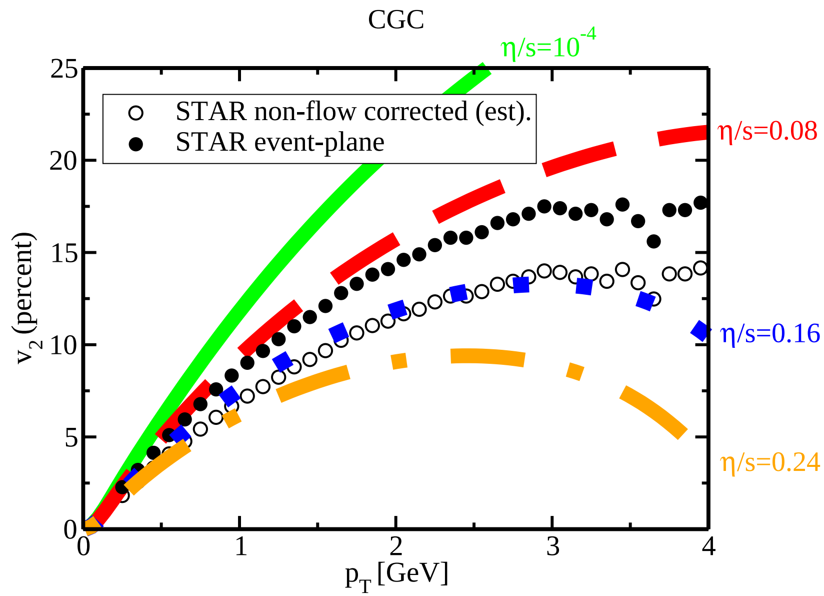
$$\partial_t \rho = D \nabla^2 \rho$$

- Diffusion coefficients D can be found by calculating current-current correlators, which have $\omega = -iDq^2$ poles (and also from a Kubo's formula)
- For R-charge in $\mathcal{N} = 4$ SYM plasma

$$D = \frac{1}{2\pi T}$$

Measuring η/s at RHIC





(from Luzum and Romatschke, 0804.4015)

Conclusion

- AdS/CFT correspondence can be generalized to finite temperature, real-time
- Reveals deep connection between thermal field theory, hydrodynamics and black hole physics
- η/s constant in all theories with Einstein gravity duals
- How relevant it is to QCD plasma ?
- At least we now have examples of strongly coupled plasmas that can be studied analytically