# On-shell methods in gauge theories
# Part 3: a state of the art

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Scattering amplitudes in $\mathcal{N} = 4$ super-Yang-Mills
at weak and strong coupling

with: Z. Bern, L. Dixon, D. Kosower, M. Spradlin, C. Vergu, A. Volovich

## Outline

- Scattering and Wilson loops: a butchered summary and preview
- Scattering at weak coupling
  - structure of amplitudes
  - perturbation theory and unitarity
  - symmetries
  - cues from strong coupling; large number of legs
- 6-points at 2-loops
- BDS vs. Amplitude vs. DHKS/BHT and the remainder function
- Outlook

## Scattering amplitudes in CFT

- CFT:
  - interactions never turn off $\longrightarrow$ no free asymptotic states
  - IR divergences
- Regularization:
  - use dimensional regularization $d = 4 - 2\epsilon$
  - breaks conformal invariance
  - allows definition of asymptotic states
  - recovered as $\epsilon \to 0$
- Similarly to QCD:
  - on-shell gauge invariance
  - for "$\mathcal{N} = 4$ collider" observables, need to turn them into scattering of gauge singlets
  - finiteness exposes properties of amplitudes which hold in any gauge theory

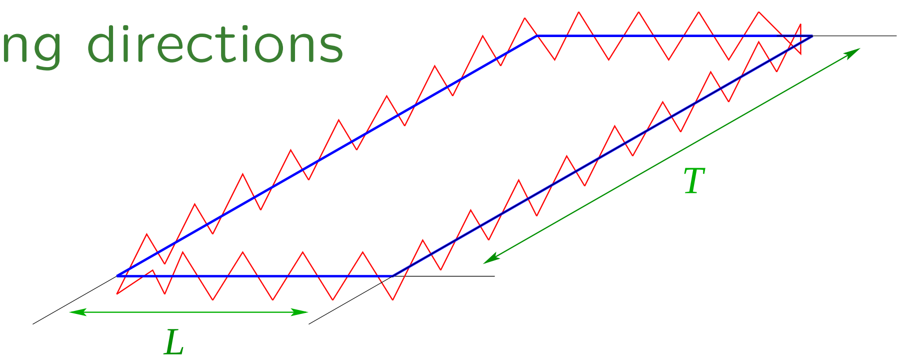
Scattering amplitudes in AdS/CFT: Alday's lectures: Alday, Maldacena

◇ mathematically – same calculation as for Wilson loops

- $\mapsto$ minimal surfaces with prescribed boundary conditions
- $\mapsto$ regularization is required
- $\mapsto$ WL = closed polygon w/ light-like edges

◇ In general – difficult problem simplifies for large number of legs

- many gluons moving in alternating directions
- Further approximation: $T \gg L$; find leading term in $T/L$
- Space-like rectangle




◇ Leading order: $q\bar{q}$ potential multiplied by distance

$$\ln\langle W\rangle = \sqrt{\lambda}\,\frac{4\pi^2}{\Gamma\left(\frac{1}{4}\right)^4}\,\frac{T}{L} \qquad \lambda \gg 1$$

- dependence on $T/L$ suggests that it arises from nontrivial function

Weak coupling implications: New conjecture

$$\frac{A_n^{\text{MHV}}}{A_n^{\text{tree, MHV}}} = \langle W_n \rangle$$

Drummond, Henn, Korchemsky, Sokatchev

order by order in perturbation theory

Brandhuber, Heslop, Travaglini

◇ Evidence:

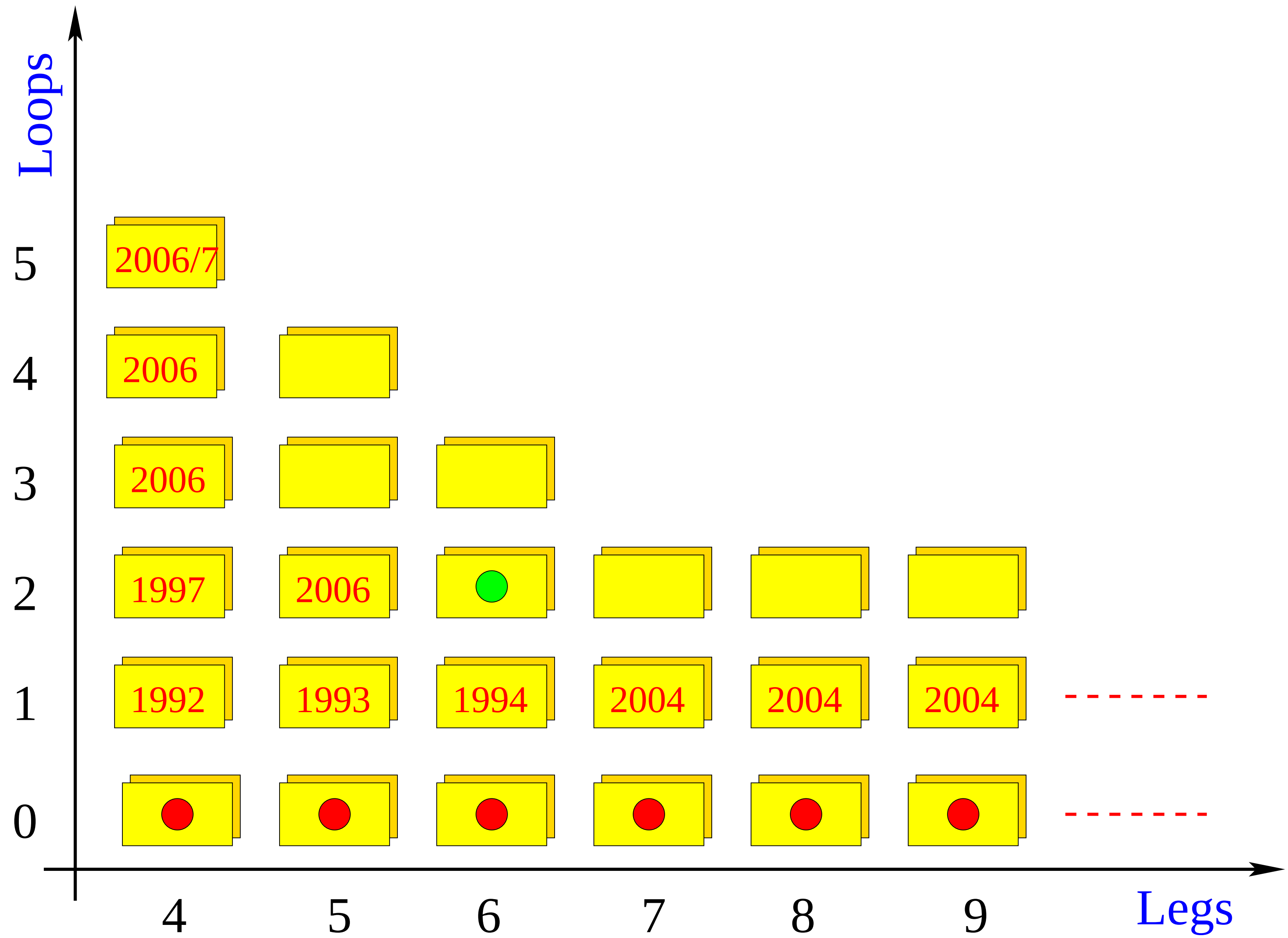
- 4-points at 1-loop — Drummond, Korchemsky, Sokatchev
- n-points at 1-loop — Brandhuber, Heslop, Travaglini

$\langle W \rangle_{1\ \text{loop}}$ is the same in any gauge theory!

- 4- and 5-points at 2-loop — Drummond, Henn, Korchemsky, Sokatchev

• How far does it go? Why does it work at all?

Current perturbative analytic results in $\mathcal{N} = 4$ SYM



## Properties of scattering amplitudes

- Two types of IR divergences:

  - soft : $\int \frac{d\omega}{\omega^{1+\epsilon}} \propto \frac{1}{\epsilon}$
  - collinear : $\int \frac{dk_T}{k_T^{1+\epsilon}} \propto \frac{1}{\epsilon}$

  $\rightarrow$ the leading pole at L loops is $\frac{1}{\epsilon^{2L}}$

- IR pole structure is predictable due to soft/collinear factorization and exponentiation theorem

  - Extensive QCD literature: Akhoury (1979), Mueller (1979), Collins (1980), Sen (1981), Sterman (1987), Botts, Sterman (1989), Catani, Trentadue (1989), Korchemsky (1989), Magnea, Sterman (1990), Korchemsky, Marchesini (1992), Catani (1998), Sterman, Tejeda-Yeomans (2002)

- Simplifications at large $N$: IR in terms of $\beta(\lambda)$, cusp anomaly $\gamma_K(\lambda)$ or the large spin limit of the twist-2 anomalous dimension, "collinear" anomalous dimension $\mathcal{G}_0(\lambda)$

- $\mathcal{N}=4$ SYM – further simplifications $\beta=0$

## Soft/Collinear factorization

Magnea, Sterman
Sterman, Tejeda-Yeomans

◇ Rescaled amplitude factorizes in three parts:

$$\mathcal{M}_n = S(k, \frac{Q}{\mu}, \alpha_s(\mu), \epsilon) \times \left[ \prod_{i=1}^{n} J_i(\frac{Q}{\mu}, \alpha_s(\mu), \epsilon) \right] \times h_n(k, \frac{Q}{\mu}, \alpha_s(\mu), \epsilon)$$

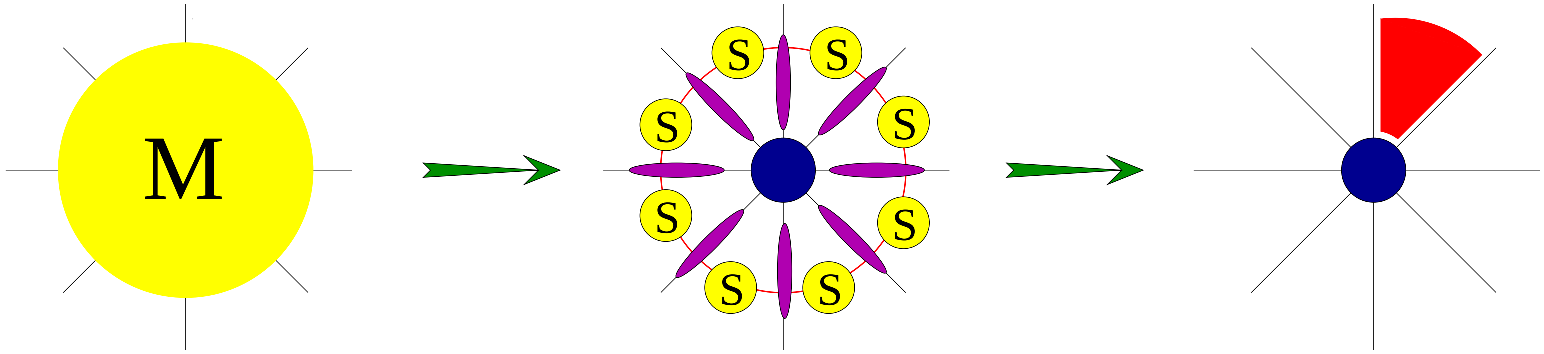
- $S(k, \mu, \alpha_s(\mu), \epsilon)$ soft function; captures the soft gluon radiation; defined up to overall function
- $J_i(k, \mu, \alpha_s(\mu), \epsilon)$ independent of color flow; all collinear dynamics
- $h_n(\mu, \alpha_s(\mu), \epsilon)$ is finite as $\epsilon \to 0$

◇ Independence of $Q$: factorization vs. evolution

- Consequences of the large $N$ limit: Bern, Dixon, Smirnov

  1) trivial color structure: $S$ can be absorbed in $J$

  2) planarity: gluon exchange is confined to neighboring legs




$$\mathcal{M}_n = \times \left[\prod_{i=1}^{n} \mathcal{M}^{[gg\to 1]}\left(\frac{s_{i,i+1}}{\mu}, \lambda, \epsilon\right)\right]^{1/2} \times h_n(k, \lambda, \epsilon)$$

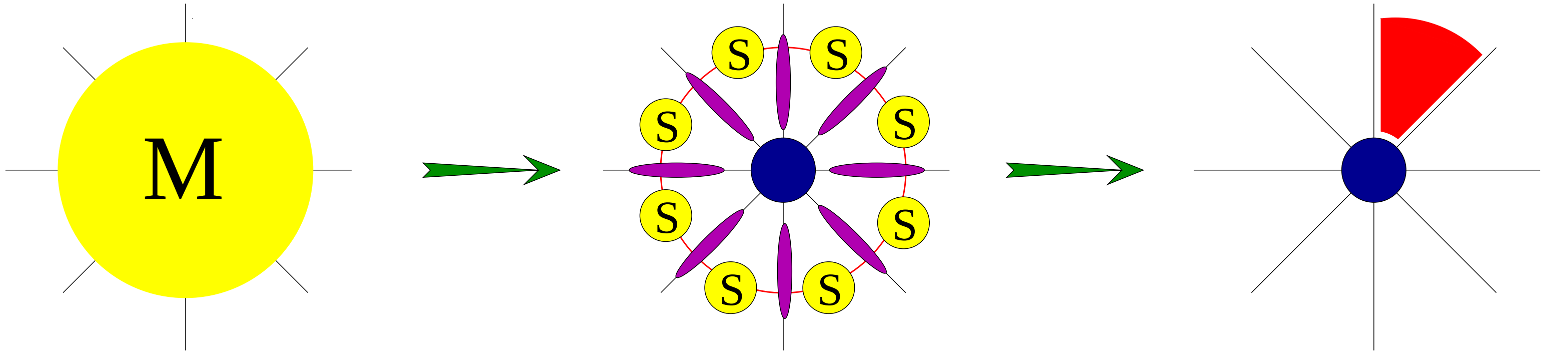
- Sudakov form factor: decay of a scalar into 2 gluons
- Factorization $\mapsto$ diferential (RG) equation for $\mathcal{M}^{[gg\to 1]}$

Mueller (1979); Collins (1980); Sen(1981); Korchemsky, Radyushkin (1987); Korchemsky (1989); Magnea, Sterman (1990)

$$\frac{d}{d\ln Q^2}\mathcal{M}^{[gg\to 1]}\left(\frac{Q^2}{\mu^2},\lambda,\epsilon\right) = \frac{1}{2}\left[K(\epsilon,\lambda) + G(\frac{Q^2}{\mu^2},\lambda,\epsilon)\right]\mathcal{M}^{[gg\to 1]}\left(\frac{Q^2}{\mu^2},\lambda,\epsilon\right)$$

$$\left(\frac{d}{d\ln\mu} + \beta(\lambda)\frac{d}{dg}\right)(K+G) = 0 \qquad \left(\frac{d}{d\ln\mu} + \beta(\lambda)\frac{d}{dg}\right)K(\epsilon,\lambda) = -\gamma_K(\lambda)$$

- Exact solution

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  Mueller (1979); Collins (1980); Sen(1981); Korchemsky, Radyushkin (1987); Korchemsky (1989); Magnea, Sterman (1990)
- Exact solution for $\mathcal{N} = 4$ SYM — Bern, Dixon, Smirnov

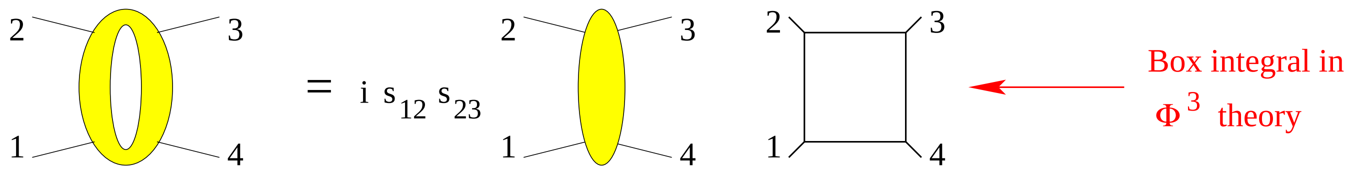
$$\mathcal{M}_n = \exp\left[ -\frac{1}{8} \sum_l a^l \left( \frac{\gamma_K^{(l)}}{(l\epsilon)^2} + \frac{2\mathcal{G}_0^{(l)}}{l\epsilon} \right) \sum_i \left( \frac{\mu^2}{-s_{i,i+1}} \right)^{l\epsilon} \right] \times \; h_n$$

$$f(\lambda) = \sum_l a^l \, \gamma_K^{(l)}$$ universal scaling function

# Technology of choice: generalized unitarity-based method

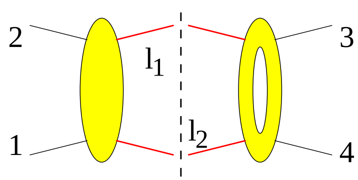
◇ 2-loop 4-point amplitudes: Bern, Rozowsky, Yan

Recall from earlier:

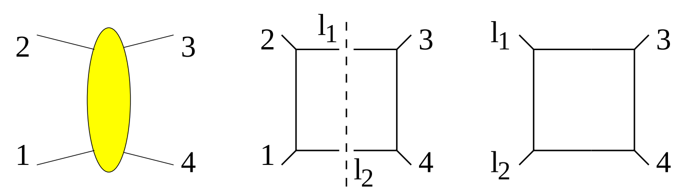



Green, Schwarz, Brink (1982)

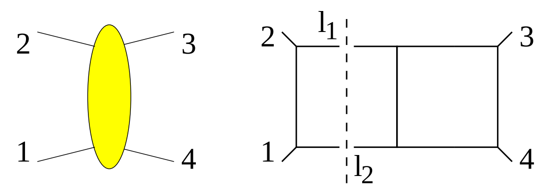
$$A_4^{\text{1 loop}}(1,2,3,4) = i s_{12} s_{23}\, A_4^{\text{tree}}(1,2,3,4) \int \frac{d^d q}{q^2 (q-k_1)^2 (q-k_{12})^2 (q+k_4)^2}$$

$\sum_{\mathcal{N}=4}$ 


$$= \; -i s_{12} s_{23} \times (-i s_{12} (l_1 \cdot k_3))$$




$$= \; -s_{12}^2 s_{23}$$




▪ similar in the $t$-channel

# Technology of choice: generalized unitarity-based method

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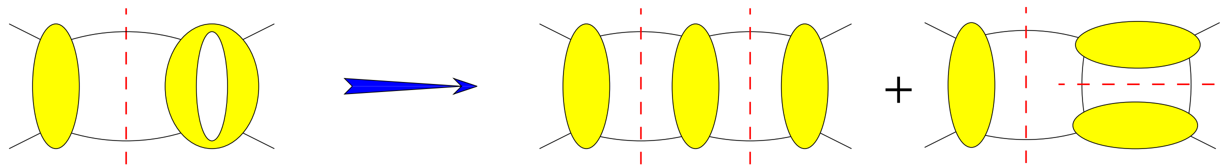
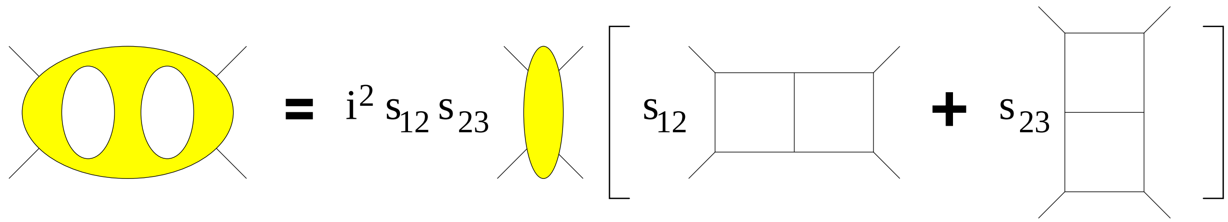


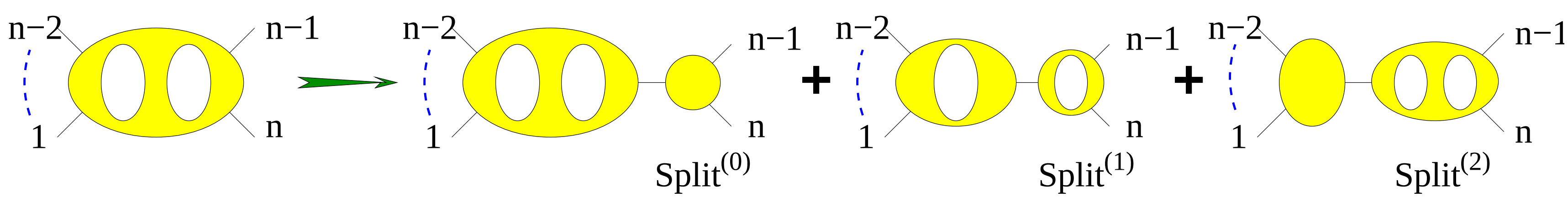
- double-two-particle cuts suffice to reconstruct the amplitude




◇ 2-loop splitting amplitude Bern, Dixon, Kosower

controls the behavior of amplitudes as two adjacent momenta become collinear

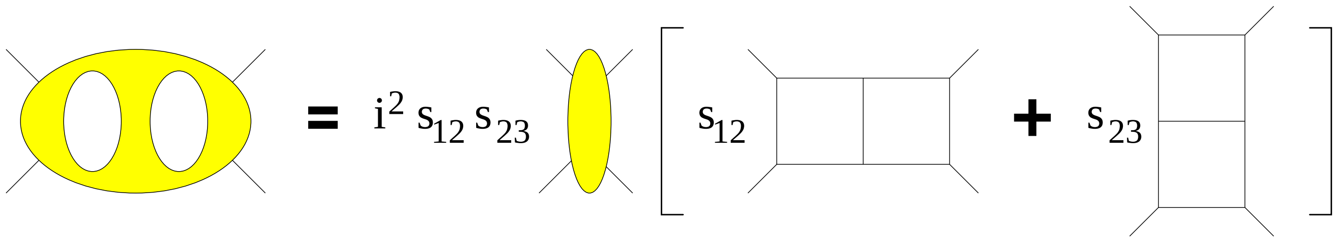
$$p_{n-1} = zP \qquad p_n = (1-z)P \qquad P^2 \to 0$$




$$\text{Split}^{(l)} = \text{Split}^{(0)}\, r_S^{(l)}(z, s_{n-1,n}, \epsilon) \qquad r_S^{(2)}(\epsilon) = \frac{1}{2}\left(r_S^{(1)}(\epsilon)\right)^2 + f^{(2)} r_S^{(1)}(2\epsilon) + \mathcal{O}(\epsilon)$$

# Technology of choice: generalized unitarity-based method

◇ 2-loop 4-point amplitudes: Bern, Rozowsky, Yan

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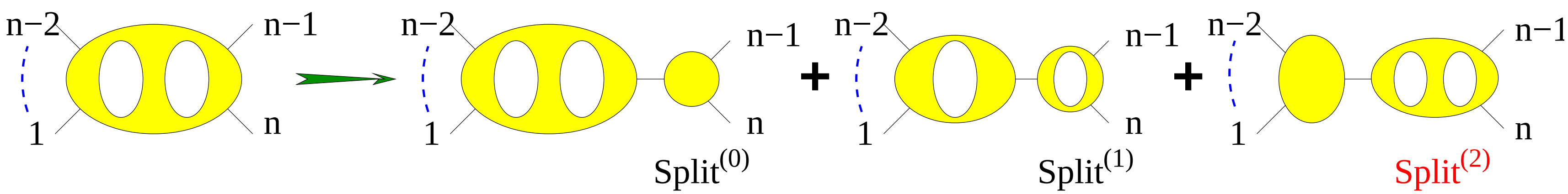


Splitting amplitudes may be computed directly!

◇ Both consistent with: Anastasiou, Bern, Dixon, Kosower

$$M_n^{(2)}(\epsilon) = \frac{1}{2}\Big(M_n^{(1)}(\epsilon)\Big)^2 + f^{(2)}(\epsilon)\, M_n^{(1)}(2\epsilon) + C^{(2)} + \mathcal{O}(\epsilon)$$

$\mapsto$ ABDK conjecture that it holds for any $n$ at 2-loops

◇ Striking resemblence with the general structure of IR poles

$$\mathcal{M}_n = \exp\left[ -\frac{1}{8} \sum_l a^l \left( \frac{\gamma_K^{(l)}}{(l\epsilon)^2} + \frac{2\mathcal{G}_0^{(l)}}{l\epsilon} \right) \sum_i \left( \frac{\mu^2}{-s_{i,i+1}} \right)^{l\epsilon} \right] \times h_n$$

$f(\lambda) = \sum_l a^l \gamma_K^{(l)}$ universal scaling function

- $\frac{1}{\epsilon^2} \sum_i \left( \frac{\mu^2}{-s_{i,i+1}} \right)^{\epsilon}$ – singular part of the 1-loop n-point amplitude

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◇ $h_n \to \exp\left(\sum_l a^l h_n^{(l)}\right)$ and combine with the pole part

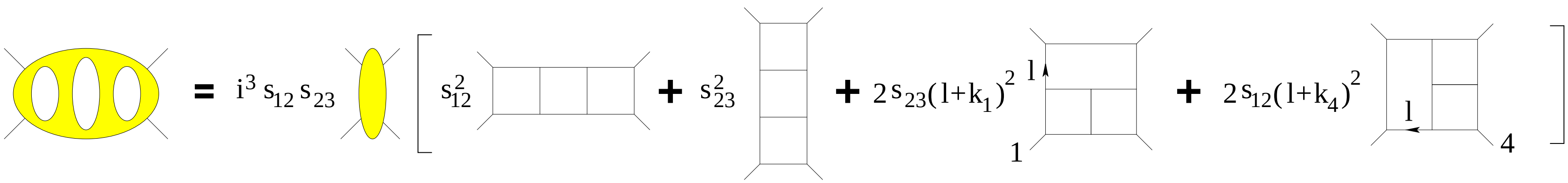
Bern, Dixon, Smirnov

$$\mathcal{M}_n = \exp\left[-\frac{1}{8}\sum_l a^l f^{(l)}(\epsilon) \mathcal{M}_n^{(1)}(l\epsilon) + C^{(l)}\right] \quad f^{(l)}(\epsilon) = f_0^{(l)} + \epsilon f_1^{(l)} + \epsilon^2 f_2^{(l)}$$

Captures correctly collinear limit: $M_n^{(1)} \mapsto M_{n-1}^{(1)} + r_S^{(1)}$

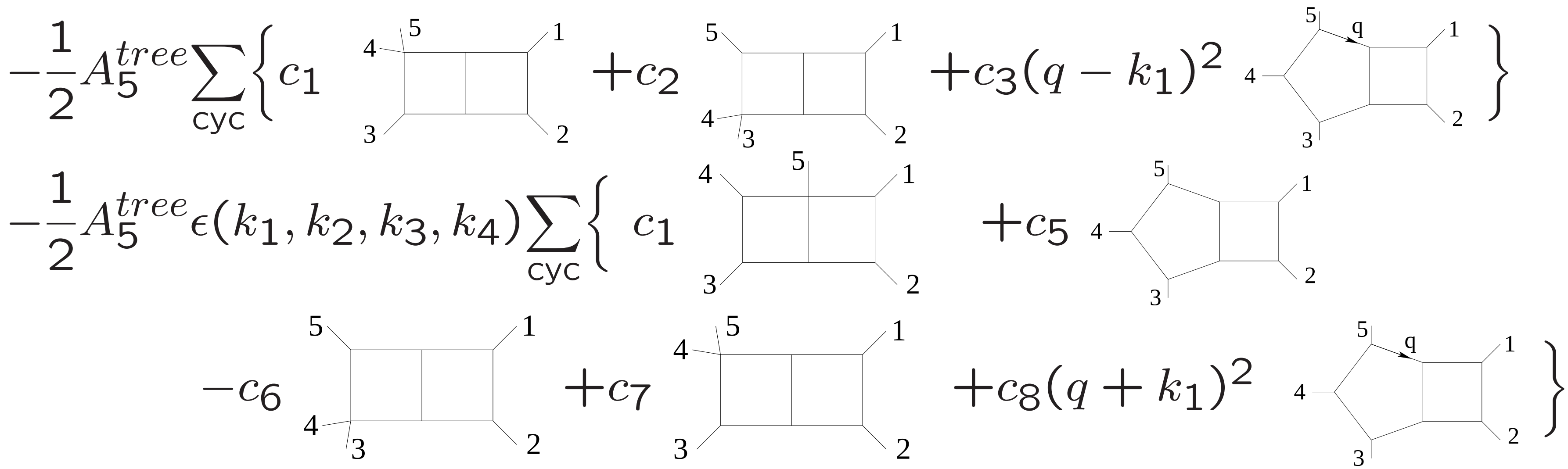
Nevertheless... not a proof. There is evidence for the conjecture:

- 3-loop 4-points — Bern, Dixon, Smirnov




- Strong coupling limit of the 4-gluon amplitude — Alday, Maldacena

- 2-loop 5-points — Cachazo, Spradlin, Volovich; Bern, Czakon, Dixon, Kosower, RR, Smirnov

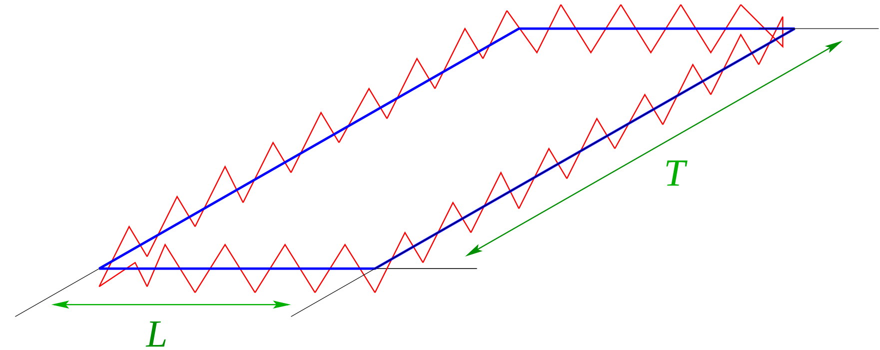


## The (anti-cronologically) emerging puzzle

- Large coupling limit of $n$-gluon amplitude in the large $n$ limit and a special kinematic configuration

Alday, Maldacena

$$\ln\langle W_{n\to\infty}\rangle = \frac{\sqrt{\lambda}}{4}\,\frac{16\pi^2}{\Gamma\left(\frac{1}{4}\right)^4}\,\frac{T}{L} + \ldots$$




$$\sum_l a^l f^{(l)}(\epsilon)\mathcal{M}_n^{(1)}(l\epsilon) = \sum_l a^l f^{(l)}(\epsilon)\langle W_n\rangle^{(1)}(l\epsilon) \xrightarrow{n\to\infty} \frac{\sqrt{\lambda}}{4}\,\frac{T}{L} + \ldots$$

- Hexagon Wilson loop to 2-loops differs from BDS ansatz by non-trivial function of momenta

Drummond, Henn, Korchemsky, Sokatchev

- Difficulties with multi-Regge limits

# Dual conformal invariance

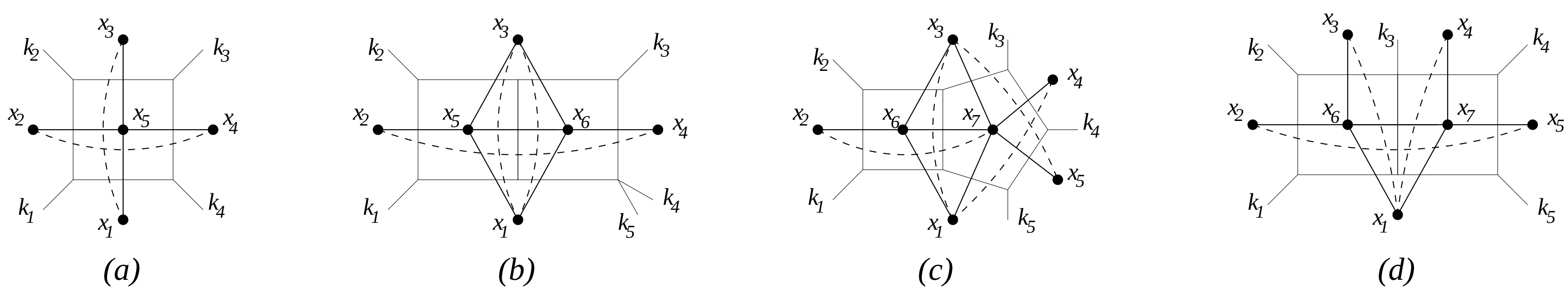
Drummond, Henn, Sokatchev, Smirnov

4-point amplitudes: sum of integrals with definite properties under dual conformal transformations if regularized by staying in $d = 4$ and continuing external momenta off-shell

Solve momentum conservation: $k_i = x_i - x_{i+1}$ (constraint $\to$ invariance)

Can define inversion: $\mathbf{I} : x_i^\mu \mapsto \frac{x_i^\mu}{x_i^2}$; $\mathbf{I} : x_{ij}^2 \mapsto \frac{x_{ij}^2}{x_i^2 x_j^2}$, $\mathbf{I} : d^d x_i \mapsto \frac{d^d x_i}{(x_i^2)^d}$

- Graphical representation of transformations




- solid line: denominator dashed line: numerator

Invariance under inversion: result is function of $u_{ijkl} = \frac{x_{ij}^2 x_{kl}^2}{x_{ik}^2 x_{jl}^2}$

- Dimensional regularization breaks dual conformal invariance
- Dimensional regularization only finite rations may appear
- Potential anomalies
  - Wilson loop expectation value obeys an anomalous Ward identity for finite part, determined by IR poles

Drummond, Henn, Korchemsky Sokatchev

$$\mathbb{K}^\mu \ln F_n^W = \frac{1}{2} f(\lambda) \sum_{i=1}^{n} x_{i,i+1}^\mu \ln \frac{x_{i,i+2}^2}{x_{i-1,i+1}^2}$$

  - Subtracted BDS ansatz obeys Ward identity!

$$\ln F_n^{BDS} = \frac{1}{4} f(\lambda)\, F_n^{(1)}(0)$$

- BDS $\neq \langle W_6 \rangle$; What about amplitudes?

Define 2-loop "remainder":

$$R_n^{(2)} \equiv \lim_{\epsilon\to 0}\left[M_n^{(2)}(\epsilon) - \left(\frac{1}{2}\left(M_n^{(1)}(\epsilon)\right)^2 + f^{(2)}(\epsilon)\, M_n^{(1)}(2\epsilon) + C^{(2)}\right)\right]$$

- 6-pt kinematics – first homogeneous solution of W.I.

$\mapsto$ first potential departure from BDS=WL

If believe that dual conformal symmetry holds to all loops –

$\mapsto$ $R_6^{(2)}$ is the first potentially-nonzero remainder

- expressed in terms of

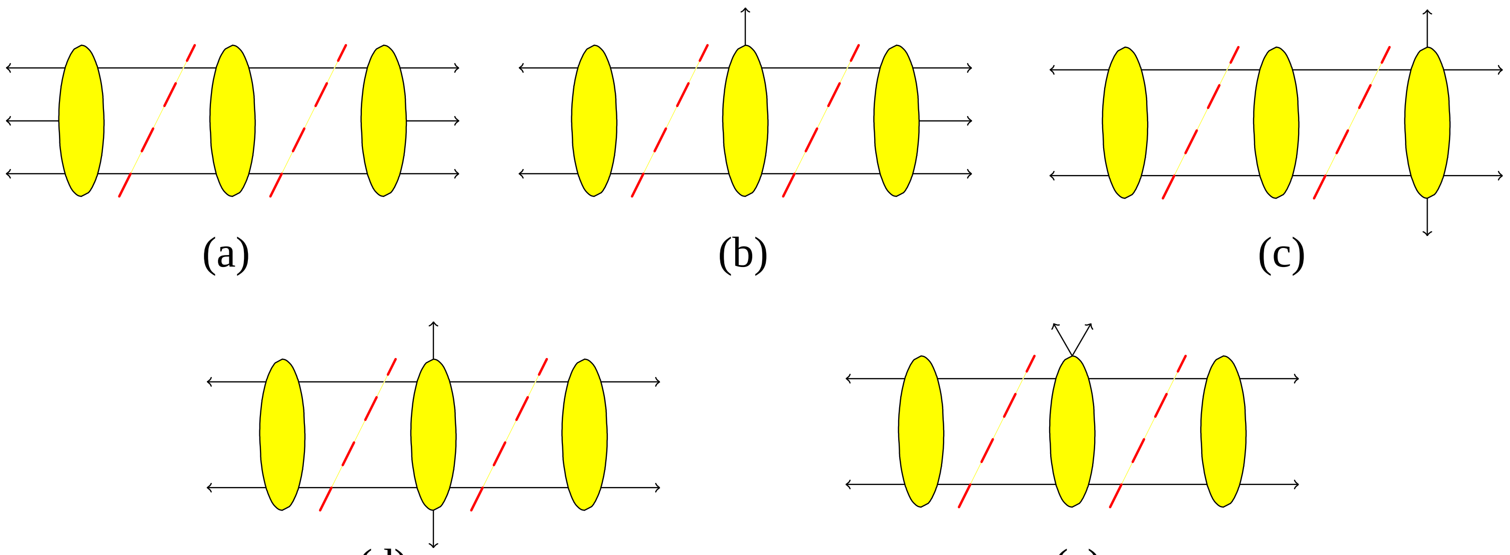
$$u_1 = \frac{x_{13}^2 x_{46}^2}{x_{14}^2 x_{36}^2} = \frac{s_{12}s_{45}}{s_{123}s_{345}} \quad , \quad u_2 = \frac{x_{24}^2 x_{51}^2}{x_{25}^2 x_{41}^2} = \frac{s_{23}s_{56}}{s_{234}s_{123}} \quad , \quad u_3 = \frac{x_{35}^2 x_{62}^2}{x_{36}^2 x_{52}^2} = \frac{s_{34}s_{61}}{s_{345}s_{234}}$$

- finite
- trivial collinear limits

## 6-point amplitude at 2-loops Bern, Dixon, Kosower, RR, Spradlin, Vergu, Volovich

- use genrealized unitarity
- no-triangle constraint $\mapsto$ double-two-particle cuts suffice

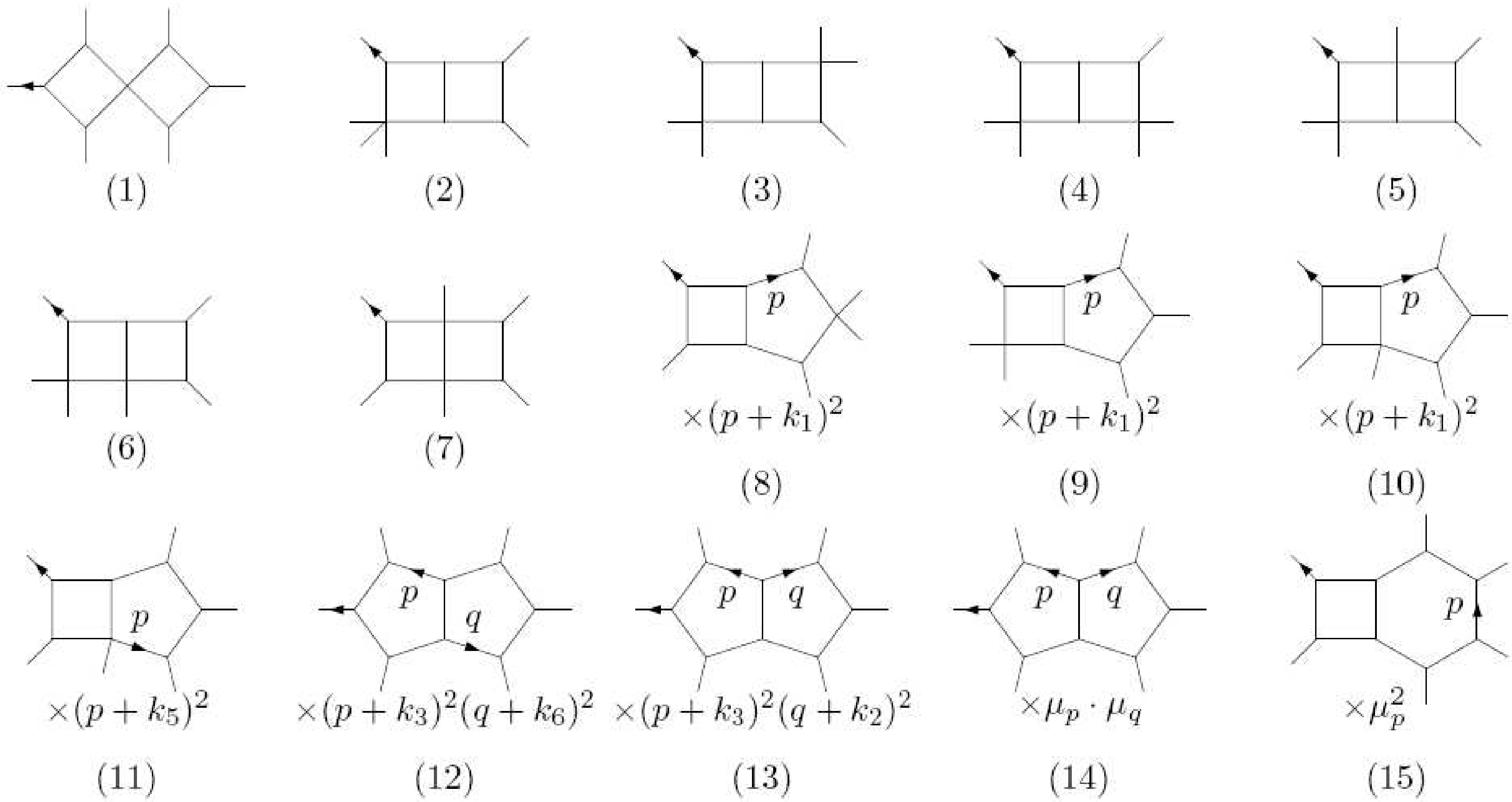
### Relevant generalized cuts:



(a) (b) (c) (d) (e)

- Main advantage: for $d = 4$ cuts each tree is MHV
- no guarantee that $d = 4$ suffice
- Split in $d = 4$ cuts and $d \neq 4$ cuts $\quad M_6^{(2)}(\epsilon) = M_6^{(2),D=4}(\epsilon) + M_6^{(2),\mu}(\epsilon)$

6-point amplitude at 2-loops: use dual conformal symmetry to organize integrals




- 26 possible dual conformal integrals

## The integrand:

$$M_6^{(2),D=4-2\epsilon}(\epsilon) = M_6^{(2),D=4}(\epsilon) + M_6^{(2),\mu}(\epsilon)$$

$$\begin{aligned} M_6^{(2),D=4}(\epsilon) \;=\; \frac{1}{16}\sum_{12\text{ perms.}} \Bigg[ & \frac{1}{4}c_1 I^{(1)}(\epsilon) + c_2 I^{(2)}(\epsilon) + \frac{1}{2}c_3 I^{(3)}(\epsilon) + \frac{1}{2}c_4 I^{(4)}(\epsilon) + c_5 I^{(5)}(\epsilon) \\ & + c_6 I^{(6)}(\epsilon) + \frac{1}{4}c_7 I^{(7)}(\epsilon) + \frac{1}{2}c_8 I^{(8)}(\epsilon) + c_9 I^{(9)}(\epsilon) \\ & + c_{10} I^{(10)}(\epsilon) + c_{11} I^{(11)}(\epsilon) + \frac{1}{2}c_{12} I^{(12)}(\epsilon) + \frac{1}{2}c_{13} I^{(13)}(\epsilon) \Bigg] \end{aligned}$$

$$M_6^{(2),\mu}(\epsilon) \;=\; \frac{1}{16}\sum_{12\text{ perms.}} \left[ \frac{1}{4}c_{14} I^{(14)}(\epsilon) + \frac{1}{2}c_{15} I^{(15)}(\epsilon) \right]$$

- Strategy: multiply trees; reorganize to expose propagators; identify integrals; alternatively, match numerically onto target expression
- $c_{15}$ may be obtained from a partial $d = 4$ cut

- The coefficients:

$$
\begin{aligned}
c_1 &= s_{16}s_{34}s_{123}s_{345} + s_{12}s_{45}s_{234}s_{345} + s_{345}^2(s_{23}s_{56} - s_{123}s_{234}) \\
c_2 &= 2\, s_{12}s_{23}^2 \\
c_3 &= s_{234}(s_{123}s_{234} - s_{23}s_{56}) \\
c_4 &= s_{12}s_{234}^2 \\
c_5 &= s_{34}(s_{123}s_{234} - 2s_{23}s_{56}) \\
c_6 &= -s_{12}s_{23}s_{234} \\
c_7 &= 2\, s_{123}s_{234}s_{345} - 4s_{16}s_{34}s_{123} - s_{12}s_{45}s_{234} - s_{23}s_{56}s_{345} \\
c_8 &= 2\, s_{16}(s_{234}s_{345} - s_{16}s_{34}) \\
c_9 &= s_{23}s_{34}s_{234} \\
c_{10} &= s_{23}(2s_{16}s_{34} - s_{234}s_{345}) \\
c_{11} &= s_{12}s_{23}s_{234} \\
c_{12} &= s_{345}(s_{234}s_{345} - s_{16}s_{34}) \\
c_{13} &= -s_{345}^2 s_{56} \\
c_{14} &= -2\, s_{126}(s_{123}s_{234}s_{345} - s_{16}s_{34}s_{123} - s_{12}s_{45}s_{234} - s_{23}s_{56}s_{345}) \\
c_{15} &= 2\, s_{16}\, (s_{123}s_{234}s_{345} - s_{16}s_{34}s_{123} - s_{12}s_{45}s_{234} - s_{23}s_{56}s_{345})
\end{aligned}
$$

- Some comments
  - $M_6^{(2),\mu}$ nonvanishing while integrand vanishes in $D = 4$
  - $M_6^{(2),D=4}$ is constructed out of pseudo-conformal integrals
  - Unlike n=4 and n=5, relative weights are not $0, \pm 1$
- Features of the result
  - IR divergences should be the same as in BDS
  - subtract BDS $\longrightarrow$ find conformal invariance?

$$u_1 = \frac{s_{12}s_{45}}{s_{123}s_{345}} \qquad u_2 = \frac{s_{23}s_{56}}{s_{234}s_{123}} \qquad u_3 = \frac{s_{34}s_{61}}{s_{345}s_{234}}$$

  - Evaluate numerically
  - How important is $D = 4$? ($\det k_i \cdot k_j = 0,\ i,j = 1,\cdots,5$)?
  - Comparison with expectation value of hexagon Wilson loop?

◇ "Direct" integration $\mapsto$ departure from BDS ansatz

$$R_A \equiv \mathcal{M}_6^{(2)} - \mathcal{M}_6^{(2)BDS}$$

$$R_A^0 \equiv R_A(K^{(0)}) = 1.0937 \pm 0.0057$$

| kinematics | $(u_1, u_2, u_3)$ | $R_A - R_A^0$ |
|---|---|---|
| $K^{(1)}$ | $(1/4, 1/4, 1/4)$ | $-0.018 \pm 0.023$ |
| $K^{(2)}$ | (0.547253, 0.203822, 0.881270) | $-2.753 \pm 0.015$ |
| $K^{(3)}$ | $(28/17, 16/5, 112/85)$ | $-4.7445 \pm 0.0075$ |
| $K^{(4)}$ | $(1/9, 1/9, 1/9)$ | $4.12 \pm 0.10$ |
| $K^{(5)}$ | $(4/81, 4/81, 4/81)$ | $10.00 \pm 0.50$ |

◇ Comparison with hexagon Wilson loop

$$R_A \equiv \mathcal{M}_6^{(2)} - \mathcal{M}_6^{(2)BDS}$$

$$R_A^0 \equiv R_A(K^{(0)}) = 1.0937 \pm 0.0057 \qquad R_W^0 = 13.26530$$

| kinematics | $(u_1, u_2, u_3)$ | $R_A - R_A^0$ | $R_W - R_W^0$ |
|---|---|---|---|
| $K^{(1)}$ | $(1/4, 1/4, 1/4)$ | $-0.018 \pm 0.023$ | $< 10^{-5}$ |
| $K^{(2)}$ | (0.547253, 0.203822, 0.881270) | $-2.753 \pm 0.015$ | $-2.7553$ |
| $K^{(3)}$ | $(28/17, 16/5, 112/85)$ | $-4.7445 \pm 0.0075$ | $-4.7446$ |
| $K^{(4)}$ | $(1/9, 1/9, 1/9)$ | $4.12 \pm 0.10$ | $4.0914$ |
| $K^{(5)}$ | $(4/81, 4/81, 4/81)$ | $10.00 \pm 0.50$ | $9.7255$ |

- Agreement within errors!
- $\mathcal{M}_n = \langle W_n \rangle \neq \mathcal{M}_n^{\rm BDS}$

Where else does $R_6^{(2)}$ crop up?

Does it have another (more physical) interpretation?

Where else does $R_6^{(2)}$ crop up? More physical interpretation?

Yes; a triple-collinear splitting amplitude

- triple-collinear limit:

$$k_a = z_1 P \qquad k_b = z_2 P \qquad k_c = z_3 P \qquad P^2 \to 0$$

$$z_1+z_2+z_3=1\,, \qquad 0\le z_i\le 1$$

$$A_n^{(l)}(k_1,\ldots,k_{n-2},k_{n-1},k_n) \mapsto \sum_{\lambda=\pm}\sum_{s=0}^{l} A_n^{(l-s)}(k_1,\ldots,P^\lambda)\ \text{Split}_{-\lambda}^{(s)}(k_{n-2}k_{n-1}k_n;P)$$

$s$-loop triple-collinear splitting amplitude

- MHV amplitudes $\mapsto$ four triple-collinear splitting amplitudes

$$\text{Split}_+(k_a^+k_b^+k_c^+;P) = 0$$

$$\text{Split}_-(k_a^+k_b^+k_c^+;P)\ ;\ \ \text{Split}_+(k_a^-k_b^+k_c^+;P)\ ;\ \ \text{Split}_+(k_a^+k_b^-k_c^+;P)$$

- $$\frac{\text{Split}_-(k_a^+k_b^+k_c^+;P)}{\text{Split}_-^{\text{tree}}(k_a^+k_b^+k_c^+;P)} = r_S\left(\frac{s_{ab}}{s_{abc}},\frac{s_{bc}}{s_{abc}},z_1,z_3\right)$$

- first time for 6-point kinematics $k_a = z_1 P,\ k_b = z_2 P,\ k_c = z_3 P$

◇ cross ratios are arbitrary! $\to$ $R$ survives

$$\bar{u}_1 = \frac{s_{45}}{s_{456}} \frac{1}{1 - z_3} \qquad \bar{u}_2 = \frac{s_{56}}{s_{456}} \frac{1}{1 - z_1} \qquad \bar{u}_3 = \frac{z_1 z_3}{(1 - z_1)(1 - z_3)}$$

- triple-collinear factorization vs. corrected BDS at 2-loops

$$\begin{aligned} M_6^{(2)} &\mapsto M_4^{(2)} + M_4^{(1)} r_S^{(1)} + r_S^{(2)} \\ &\mapsto M_4^{(2)BDS} + M_4^{(1)\mathrm{BDS}} r_S^{(1)\mathrm{BDS}} + r_S^{(2)\mathrm{BDS}} + R_6^{(2)}(\bar{u}) \end{aligned}$$

◇ remainder function $\leftrightarrow$ triple-collinear splitting amplitude

$$R_6^{(2)}(\bar{u}) = r_S^{(2)}\left(\frac{s_{ab}}{s_{abc}}, \frac{s_{bc}}{s_{abc}}, z_1, z_3, \epsilon\right) - r_S^{(2)\mathrm{BDS}}\left(\frac{s_{ab}}{s_{abc}}, \frac{s_{bc}}{s_{abc}}, z_1, z_3, \epsilon\right)$$

◇ Advantage: potentially simpler integrals

◇ All-loop remainder function vs. triple-collinear splitting amplitude

$$R_6^{(l)}(\bar{u}_1, \bar{u}_2, \bar{u}_3) = \sum_{s=2}^{l} M_4^{(l-s)} \left[ r_S^{(s)}\left(\frac{s_{45}}{s_{456}}, \frac{s_{56}}{s_{456}}, z_1, z_3, \epsilon\right) - r_S^{(s)\,\mathrm{BDS}}\left(\frac{s_{45}}{s_{456}}, \frac{s_{56}}{s_{456}}, z_1, z_3, \epsilon\right) \right]$$

- resummable

## Summary

- rescaled 6pt MHV amplitude=Wilson loop; BDS needs correction
- correction visible in triple-collinear limit; equals splitting amplitude

## Summary and open questions

- rescaled 6pt MHV amplitude=Wilson loop; BDS needs correction
- correction visible in triple-collinear limit; equals splitting amplitude

- Is it possible to find the analytic form of the remainder?
- Why are MHV amplitudes related to null Wilson loops? What about non-MHV amplitudes?
- Who ordered dual conformal invariance? What are the allowed types of contributions that break it?
- What other implications does it have? Is it relevant for non-MHV amplitudes and in what sense?
- Is dual conformal invariance restricted to the planar theory? What theories exhibit it? Does it have any relation to integrability of the dilatation operator?
- ...