On-shell methods in gauge theories
Part 2: Loops

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The plan:

Yesterday: – tree amplitudes
– same for susy and non-susy theories

Today: – (mostly 1-)loop amplitudes
– focus on maximal supersymmetry

Main message: It pays to stay on-shell
Tree-level amplitudes determine “everything”

Structure of 1-loop amplitudes in any $D = 4$ theory

$$A_n^{(1)} = \sum_i d_i \,\text{Box}_i + \sum_i c_i \,\text{Triangle}_i + \sum_i b_i \,\text{Bubble}_i + \text{Rat}$$

◇ justified by integral reduction: given any Feynman diagram, one may bring it to this form by a sequence of transformations:

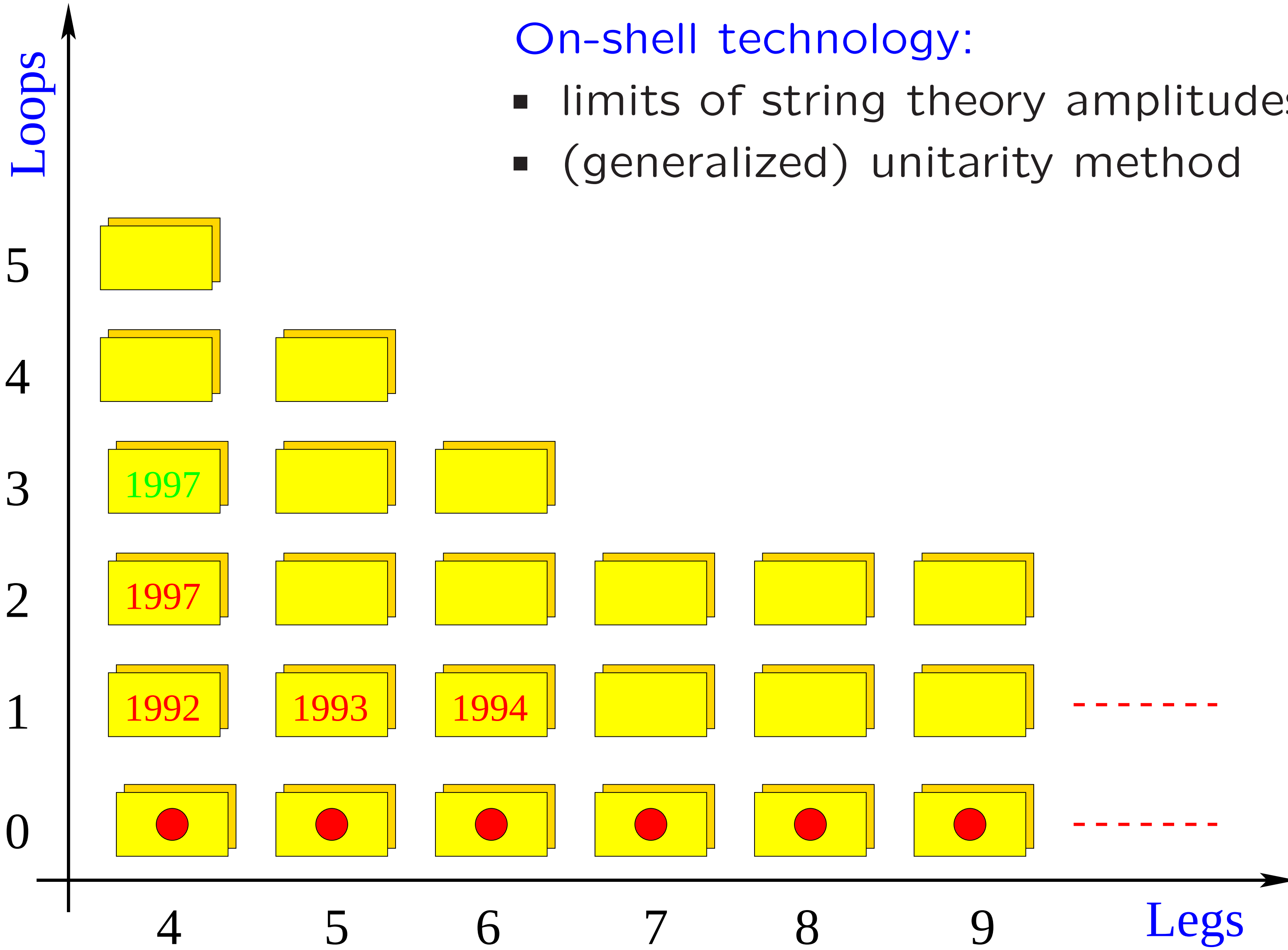
- decompose numerator tensors constructed from loop momenta in a basis of external momenta
- construct inverse propagators
- reduce scalar integrals using
$$1 = \frac{\sum_{i=1}^5 b_i(p_i^2 + m^2)}{\sum_{i=1}^5 b_i(p_i^2 + m^2)} = \frac{\sum_{i=1}^5 b_i(l + p_i)^2 + m^2}{\sum_{i=1}^5 b_i(p_i^2 + m^2)}$$
- reduce pentagons to boxes by other means
(van Neerven/Vermasseren)

◇ SUSY restricts coefficients

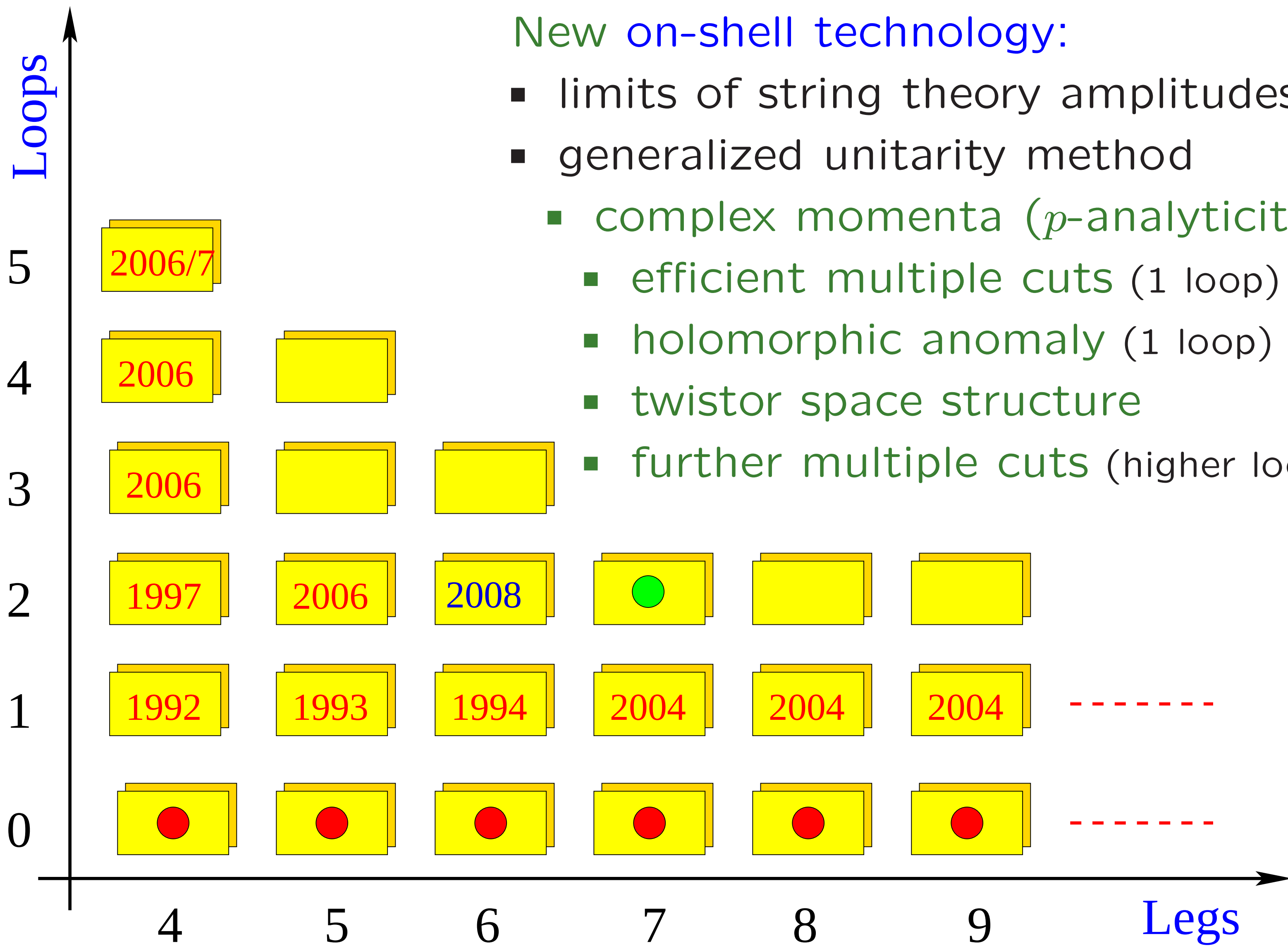
Pre-twistor perturbative analytic results in $\mathcal{N} = 4$ SYM

On-shell technology:

- limits of string theory amplitudes
- (generalized) unitarity method



Current perturbative analytic results in $\mathcal{N} = 4$ SYM



New on-shell technology:

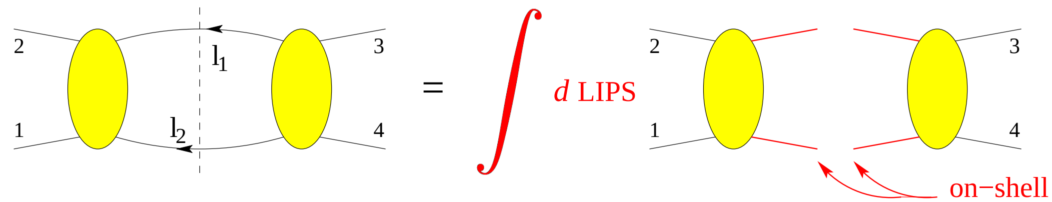
- limits of string theory amplitudes
- generalized unitarity method
  - complex momenta (p-analyticity)
    - efficient multiple cuts (1 loop)
    - holomorphic anomaly (1 loop)
    - twistor space structure
    - further multiple cuts (higher loops)

The "old" unitarity

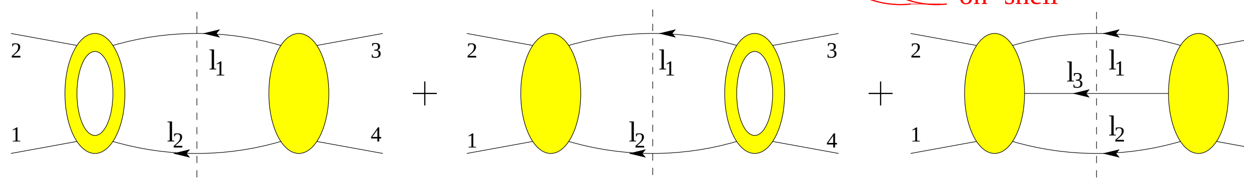
$\mathbb{1} = SS^\dagger \quad \Rightarrow \quad 2\Im T = TT^\dagger$

Unitarity: relation between discontinuity of amplitude at some loop order and lower loop amplitudes

1 loop : $2\Im T_4^{1\,\text{loop}} =$



2 loops: $2\Im T_4^{2\,\text{loops}} =$



- Knowing all cuts of an amplitude allows its reconstruction – up to rational functions of momenta

$$\Re T = \frac{1}{\pi} P \int_{-\infty}^{\infty} dw \frac{\Im T}{w-s} - C_\infty$$

◇ A question:

MHV vertices work at tree-level. Do they also work at loop level?
If so, is there a relation to existing methods?

◇ Various arguments for negative answers across the board

- MHV vertices are nonlocal; unitarity will be messed up
- Twistor string has extra states — Berkovits, Witten
- MHV vertices suggest localization on lines; Cachazo, Svrček and Witten studied the twistor space structure of available loop amplitudes and found deviations from such localization
- $i\epsilon$ prescription is unclear; needs to be "prescribed"
- twistor formulation is intrinsic to $d = 4$; regularization?
- off-shell spinors were "invented"; potential problems?

◇ Post factum: MHV rules have Lagrangian origin

General prescription:

Brandhuber, Spence, Travaglini

- # of MHV vertices equals # of negative helicity gluons

- similarly to tree amplitudes, sum over possible assignments of external legs consistent with cyclic ordering; sum over internal helicity

- momentum integral

– on-shell and transverse w/ η; phase-space and dispersion integral

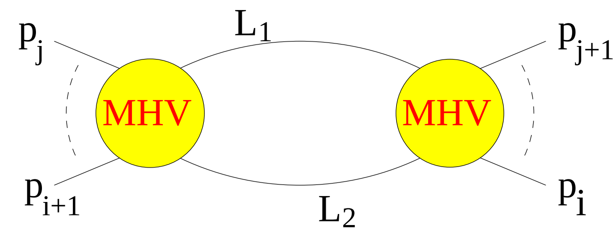
$$L_{a\dot{a}} = l_a \tilde{l}_{\dot{a}} + z\,\eta_a \tilde{\eta}_{\dot{a}} \;\rightarrow\; \frac{d^4 L}{L^2} = \frac{dz}{z}\left[\langle l dl\rangle d^2\tilde{l} - [\tilde{l} d\tilde{l}] d^2 l\right] = 4i\frac{dz}{z} d^4 l \delta^{(+)}(l^2)$$

– one dispersion int. per propagator; use l_a and $\tilde{l}_{\dot{a}}$ in MHV vertices

– dimensionally-regularize phase-space integral

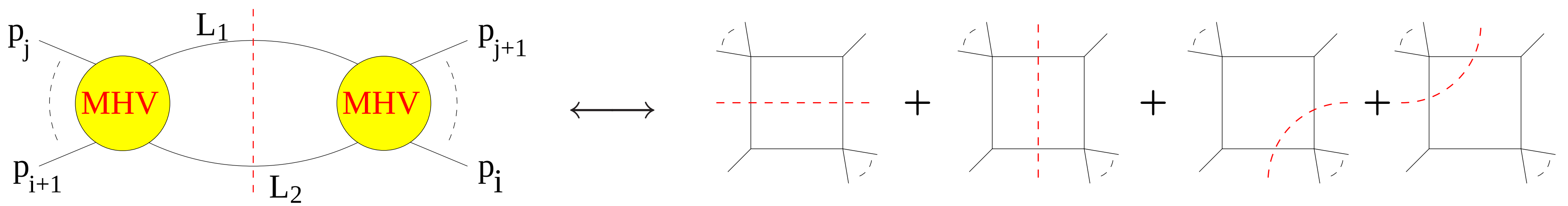
– $i\epsilon$ prescription: $\dfrac{dz}{z} \rightarrow \dfrac{dz}{z + i\epsilon}$

Typical contribution to 1-loop MHV ampl.:



$$A = \int \frac{d^4 L_1}{L_1^2} \frac{d^4 L_2}{L_2^2} \delta^4(L_1 + L_2 + p_{i+1,\ldots,j}) A_L(L_2, i+1, \ldots, j, L_1) A_R(-L_1, j+1, \ldots, i, -L_2)$$

- reorganization of cuts of box integrals



- Identify IR divergences: only if 4-point MHV vertices are present

Bena, Bern, Kosower, RR

- Transform to (λ, μ): exp. disconnected structure + subtleties
- Interesting open problem: derive 6-point NMHV using this method

The (generalized) unitarity-based method

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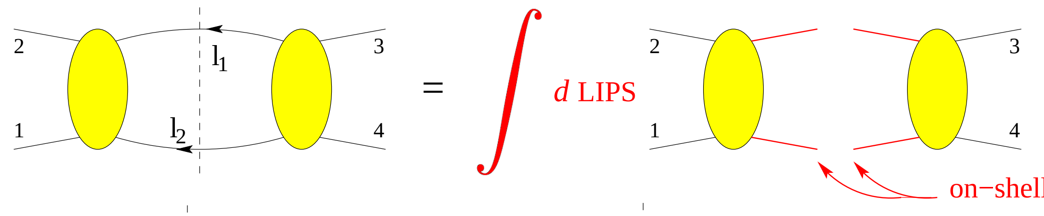
Unitarity: relation between discontinuity of amplitude at some loop order and lower loop amplitudes

1 loop : $2\Im T_4^{1\,\text{loop}} =$



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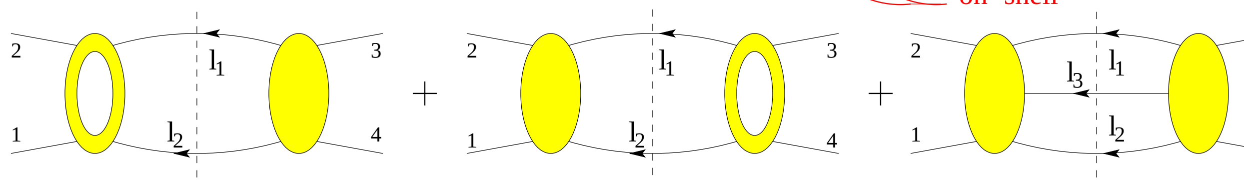


- Knowing all cuts of an amplitude allows its reconstruction – up to rational functions of momenta
- Another way: use unitarity in d dimensions
  - rational functions related to $(d-4) \times$ (multivalued functions)
- susy theories: rational and multivalued functions come together
- re-interpretation of the meaning of unitarity cut

Various statements and “theorems”:

Bern, Dixon, Dunbar, Kosower
Bern, Morgan

- Any amplitude in any massless theory is fully determined from D-dimensional tree amplitudes to all loop orders; no off-shell formulation is necessary

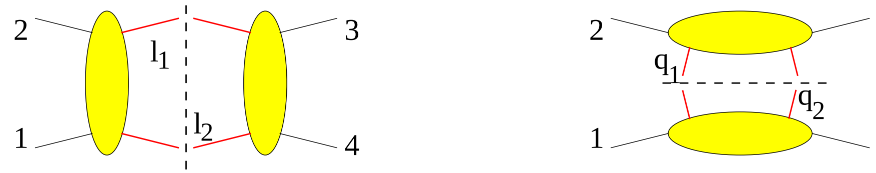
- At 1-loop, any amplitude in a massless supersymmetric field theory is fully constructible from 4-dimensional tree amplitudes, regardless potential UV and IR singularities

- Amplitudes of $\mathcal{N} = 4$ super-Yang-Mills theory are simpler than they should be. Any 1-loop amplitude is a linear combination of box integrals

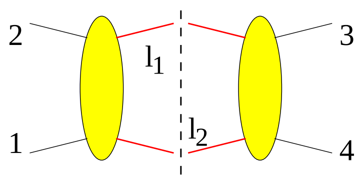
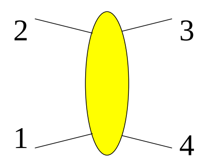
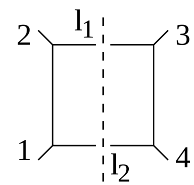
Example: 1-loop 4-point gluon amplitude in $\mathcal{N} = 4$ super-Yang-Mills

Bern, Dixon, Dunbar, Kosower

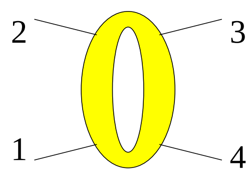
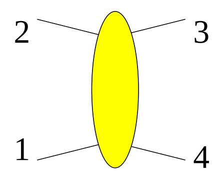
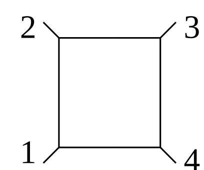
- two 2-particle cuts:



- at least one cut involves a nontrivial sum over all $\mathcal{N} = 4$ states

$\sum_{\mathcal{N}=4}$  $= -is_{12}s_{23}$   $= -i\, s_{12}s_{23} \dfrac{A_4^{\text{tree}}(1,2,3,4)}{(2l_1 \cdot k_2)(2l_2 \cdot k}$

- both cuts contain the same information – even though only one of them involves a nontrivial sum; consequence of susy

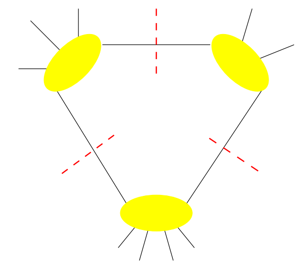
- collect all cuts:  = i s_{12} s_{23}   ← Box integral Φ^3 theory

$$A_4^{1\,\text{loop}}(1,2,3,4) = is_{12}s_{23}\, A_4^{\text{tree}}(1,2,3,4) \int \frac{d^dq}{q^2(q-k_1)^2(q-k_{12})^2(q+k_4)^2}$$

The generalized unitarity-based method

Bern, Dixon, Kosower

Britto, Cachazo, Feng



- cut more than 2 propagators

  Interpretation: cut propagators = not canceling

- In the past – primarily used at higher loops

- Use at 1-loop in conjunction with structure of amplitudes
  - $\mathcal{N} = 4$ SYM: amplitudes are sums of box integrals

    – e.g. these functions are sufficient to account for all factorization properties

The generalized unitarity-based method

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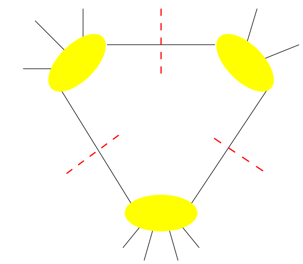

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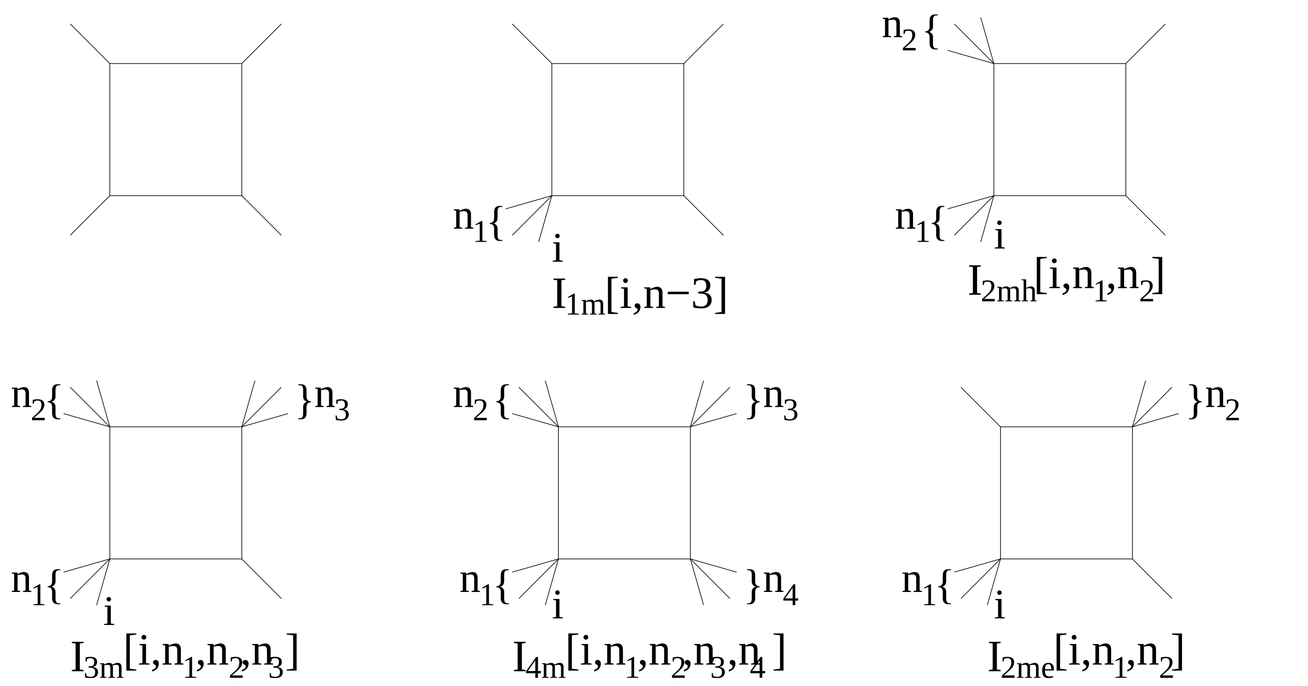


Further restrictions on the amplitude:

$$A = \sum (\text{coefficients}) \times (\text{box integrals})$$

- IR equation – important guide – simplifies life

$$A_n^{1\,\text{loop}}\Big|_{\text{singular}} = \frac{\Gamma(1+\epsilon)\Gamma(1-\epsilon)^2}{\epsilon^2\,(4\pi)^{2-\epsilon}\Gamma(1-2\epsilon)} A_n^{\text{tree}}$$

  - relations between coefficients and tree-level amplitudes

- box integrals are multi-valued function; position of branch cuts depend on invariants; no overlap! e.g.

$$I_{2me}[i,n_1,n_2] \propto \cdots + \text{Li}_2\left(1 - \frac{t_i^{[n_1]}}{t_{i-1}^{[n_1+1]}}\right) + \text{Li}_2\left(1 - \frac{t_i^{[n_1]}}{t_i^{[n_1+1+1]}}\right) + \ldots$$

$$I_{2mh}[i,n_1,n_2] \propto \cdots + \frac{1}{2}\ln^2\left(\frac{t_{i-2}^{[2]}}{t_{i-1}^{[n_1+1]}}\right) + \text{Li}_2\left(1 - \frac{t_i^{[n_1]}}{t_{i-1}^{[n_1+1]}}\right) + \text{Li}_2\left(1 - \frac{t_{i-2}^{[n_1+2]}}{t_{i-1}^{[n_1+1]}}\right) + \ldots$$

More observations

- ∃! one box integral whose external momenta correspond to each decomposition of the ordered external legs in four groups

$(1,\ldots,n) \;\mapsto\; [(i,\ldots,i+n_1-1),\,(i+n_1\ldots i+n_{12}-1),\,(i+n_{12}\ldots i+n_{123}-1),\,\text{rest}]$

- Localization

on-shell condition on the 4 internal propagators of the box integral
$\mapsto$ freezes loop momentum up to (sum over) discrete choices

$$l^2 = 0 \qquad (l-K_1)^2 = 0 \qquad (l-K_1-K_2)^2 = 0 \qquad (l+K_4)^2 = 0$$

- Useful and forgotten:

- (very) complicated solutions; luckily, often not needed explicitly
- $1,2,3-$mass boxes – solution exists only for complex momenta

  $-p_{a\dot{a}} = \lambda_a \tilde{\lambda}_{\dot{a}}$ with $\tilde{\lambda}_{\dot{a}} \neq (\lambda_a)^*$ i.e. $A(++-) \neq 0$ and $A(++-) \neq 0$

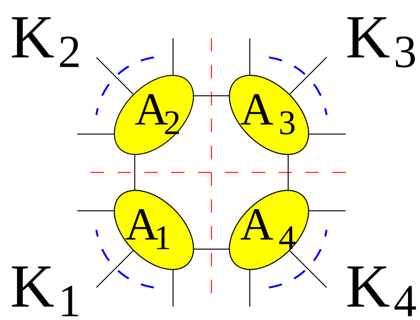
- Feynman diagrams underlie all amplitudes $\mapsto$ on each side of generalized cut there is a tree-level amplitude

The algorithm:

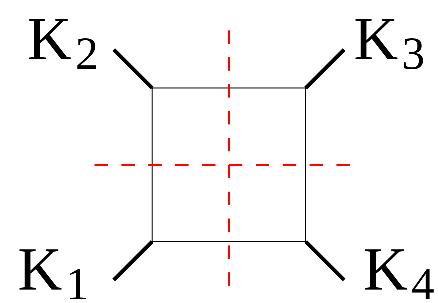
1) start with ansatz

$$A_n = \sum \Big[\quad c_{i,n-3} I_{1m}[i, n-3]$$
$$+ c^{e}_{i,n_1,n_2} I_{2me}[i, n_1, n_2] + c^{h}_{i,n_1,n_2} I_{2mh}[i, n_1, n_2]$$
$$+ c_{i,n_1,n_2,n_3} I_{3m}[i, n_1, n_2, n_3] + c_{i,n_1,n_2,n_3} I_{4m}[i, n_1, n_2, n_3, n_4]$$

2) Isolate one coefficient via the appropriate quadruple cut

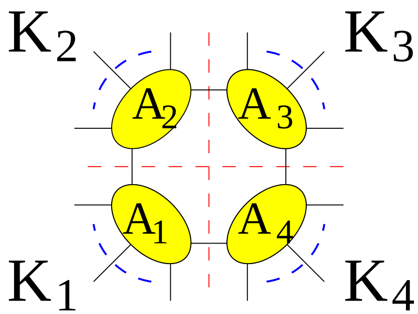


$$= c_{i,n_{K_1},n_{K_2},n_{K_3},n_{K_4}} \times$$

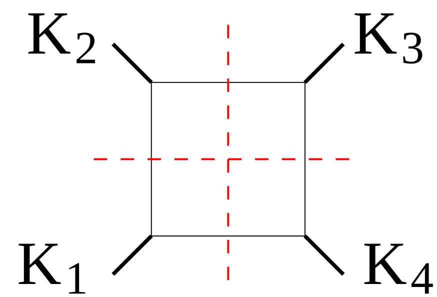


3) compute the coefficient by multiplying the appropriate tree amplitudes (complex momenta are implicitely used if one encounters 3-point tree amplitudes)

- sum over different allowed helicity assignments for internal lines



$= c_{i,n_{K_1},n_{K_2},n_{K_3},n_{K_4}} \times$



$$c_{i,n_{K_1},n_{K_2},n_{K_3},n_{K_4}} = \frac{1}{\#\mathrm{sol}} \sum_{\mathrm{hel's}} (A_1)(A_2)(A_3)(A_4)\Big|_{\mathrm{sol.\,to\,on-shell\,condition}}$$

- cancellation of Jacobian from integral of on-shell condition
- extract one coefficient at a time
- tree-level simplicity translates into 1-loop simplicity

(Not too many) tips for solving the on-shell condition

- if possible, find spinors
- solve conditions at 3-point corners (up to scale freedom)
- choose representation of tree amplitudes; expose loop momenta(?)
- search for inconsistencies implied by these solutions

  $\to$ vanishing contributions (or vanishing coefficients)
- turn holomorphic spinor into antiholomorphic spinor (or vice versa)

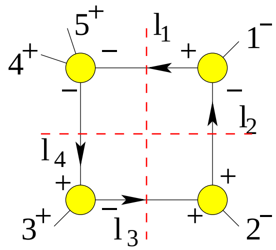
  $-\ \langle lX \rangle = \dfrac{[i|l|X\rangle}{[il]}$ for some external line i
- ratios of the type $\dfrac{\langle lX \rangle}{\langle lY \rangle}$ may sometimes be simplified

...

- coffee might help
- If nothing works, reconstruct loop momenta; use explicit sol.

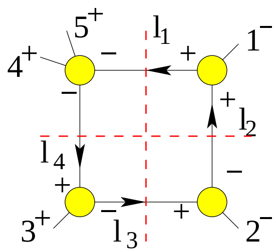
Example: 5-points MHV amplitude: five I_{1m} (incoming momenta)

a)



$$\left(\frac{\langle 1 l_2\rangle^3}{\langle l_2 l_1\rangle\langle l_1 1\rangle}\right)\left(\frac{[l_3 l_2]^3}{[l_2 2][2 l_3]}\right)\left(\frac{[3 l_4]^3}{[l_4 l_3][l_3 3]}\right)\left(\frac{\langle l_1 l_4\rangle^3}{\langle l_4 4\rangle\langle 45\rangle\langle 5 l_1\rangle}\right)$$

b)



$$\left(\frac{[l_2 l_1]^3}{[1 l_2][l_1 1]}\right)\left(\frac{\langle 2 l_2\rangle^3}{\langle 2 l_3\rangle\langle l_3 l_2\rangle}\right)\left(\frac{[3 l_4]^3}{[l_4 l_3][l_3 3]}\right)\left(\frac{\langle l_1 l_4\rangle^3}{\langle l_4 4\rangle\langle 45\rangle\langle 5 l_1\rangle}\right)$$

a)
$$[l_1 l_2] = [l_1 1] = [l_2 1] = 0 \qquad \tilde\lambda_{l_1} \propto \tilde\lambda_1 \ ; \ \tilde\lambda_{l_2} \propto \tilde\lambda_1$$
$$\langle l_2 l_3\rangle = \langle l_2 2\rangle = \langle 2 l_3\rangle = 0 \qquad \lambda_{l_2} \propto \lambda_2 \ ; \ \lambda_{l_3} \propto \lambda_2$$
$$\langle l_3 l_4\rangle = \langle l_3 3\rangle = \langle 3 l_4\rangle = 0 \qquad \lambda_{l_3} \propto \lambda_3 \ ; \ \lambda_{l_4} \propto \lambda_3$$

inconsistent

b)
$$\langle l_1 l_2\rangle = \langle l_1 1\rangle = \langle l_2 1\rangle = 0 \qquad \lambda_{l_1} \propto \lambda_1 \ ; \ \lambda_{l_2} \propto \lambda_1$$
$$[l_2 l_3] = [l_2 2] = [2 l_3] = 0 \qquad \tilde\lambda_{l_2} \propto \tilde\lambda_2 \ ; \ \tilde\lambda_{l_3} \propto \tilde\lambda_2$$
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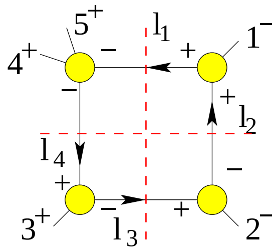
proceed

Reorganize factors using momentum conservation:

b)



$$\left(\frac{[l_2 l_1]^3}{[1 l_2][l_1 1]}\right)\left(\frac{\langle 2 l_2\rangle^3}{\langle 2 l_3\rangle\langle l_3 l_2\rangle}\right)\left(\frac{[3 l_4]^3}{[l_4 l_3][l_3 3]}\right)\left(\frac{\langle l_1 l_4\rangle^3}{\langle l_4 4\rangle\langle 45\rangle\langle 5 l_1\rangle}\right)$$

$$\langle 2 l_2\rangle[l_2 l_1] = \langle 2|l_1 + 1|l_1] = \langle 21\rangle[1 l_1]$$

$$\langle l_3 l_2\rangle[l_2 1] = \langle l_3|l_3 + 2|1] = \langle l_3 2\rangle[21]$$

$$\langle l_1 l_4\rangle[l_4 3] = \langle l_1|l_1 - 4 - 5|3] = -\langle l_1|(4+5)|3]$$

$$\langle 4 l_4\rangle[l_4 l_3] = \langle 4|l_3 + 3|l_3] = -\langle 43\rangle[3 l_3]$$

$$c_{123(45)} = \frac{1}{2}\frac{\langle 12\rangle^3[1 l_1]^2\langle l_1|(4+5)|3]^3}{[12]\langle 34\rangle\langle 45\rangle\langle 2 l_3\rangle^2[3 l_3]^2\langle 5 l_1\rangle}$$

Last: leftover spinors $\leftrightarrow$ momentum conservation and constraints

$$\lambda_{l_1} = \alpha_1\lambda_1 \quad \tilde\lambda_{l_3} = \beta_2\tilde\lambda_2$$

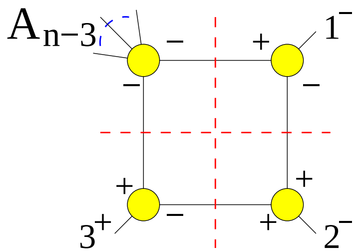
$$\lambda_{l_1}\tilde\lambda_{l_1} = \lambda_1\tilde\lambda_1 + \lambda_{l_2}\tilde\lambda_{l_2} \qquad \alpha_1\tilde\lambda_{l_1} = \tilde\lambda_1 + \beta_1\tilde\lambda_{l_2}$$

$$\lambda_{l_3}\tilde\lambda_{l_3} = \lambda_2\tilde\lambda_2 + \lambda_{l_2}\tilde\lambda_{l_2} \qquad \beta_2\lambda_{l_3} = \lambda_2 + \alpha_2\lambda_{l_2} = \lambda_2 + \alpha_2\beta_1\lambda_1$$

Homogeneity (no α and β): $c_{123(45)} = -\frac{1}{2}s_{12}s_{23}\left[\frac{\langle 12\rangle^3}{\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle}\right]$

A general vanishing result:

- The coefficient of a 1-mass box integral vanishes if there are two adjacent corners with the same helicity configuration



$$\left(\frac{\langle 1 l_2\rangle^3}{\langle l_2 l_1\rangle\langle l_1 1\rangle}\right)\left(\frac{[l_3 l_2]^3}{[l_2 2][2 l_3]}\right)\left(\frac{[3 l_4]^3}{[l_4 l_3][l_3 3]}\right) A_{n-3}$$

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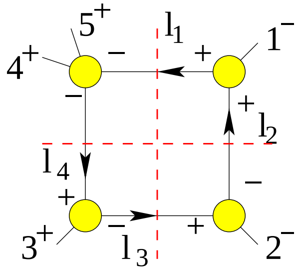
$$\langle l_2 l_3\rangle = \langle l_2 2\rangle = \langle 2 l_3\rangle = 0 \qquad \lambda_{l_2} \propto \lambda_2 \; ; \; \lambda_{l_3} \propto \lambda_2$$

$$\langle l_3 l_4\rangle = \langle l_3 3\rangle = \langle 3 l_4\rangle = 0 \qquad \lambda_{l_3} \propto \lambda_3 \; ; \; \lambda_{l_4} \propto \lambda_3$$

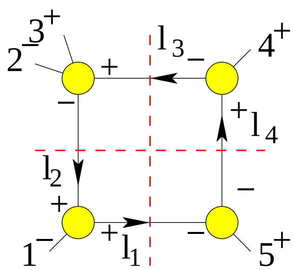
◇ cannot solve on-shell conditions for generic external momenta



$$c_{123(45)} = -\frac{1}{2} s_{12} s_{23} \left[\frac{\langle 12 \rangle^3}{\langle 23 \rangle \langle 34 \rangle \langle 45 \rangle \langle 51 \rangle} \right]$$

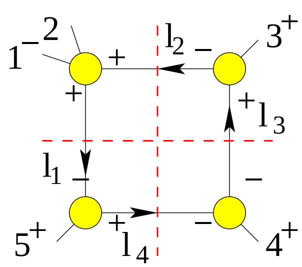
- $(12345) \longrightarrow (51234)$

Remaining nonvanishing coefficients



$$c_{451(23)} = -\frac{1}{2} s_{45} s_{51} \left[\frac{\langle 12 \rangle^3}{\langle 23 \rangle \langle 34 \rangle \langle 45 \rangle \langle 51 \rangle} \right]$$

- $(12345) \longrightarrow (45123)$



$$c_{345(12)} = -\frac{1}{2} s_{34} s_{45} \left[\frac{\langle 12 \rangle^3}{\langle 23 \rangle \langle 34 \rangle \langle 45 \rangle \langle 51 \rangle} \right]$$

Comments

- Algorithmic; yields any 1-loop amplitude in $\mathcal{N}=4$ SYM
- Simplicity due to new structures: $[a|b\ldots c|d\rangle$
- IR equations feed this simplicity back to trees $\rightarrow$ rec. rel.
- Existing explicit results:
  - all MHV amplitudes — Bern, Dixon, Dunbar, Kosower
  - all $\leq$ 7-point amplitudes — Britto, Cachazo, Feng; Bern, del Duca, Dixon, Kosower
  - all split-helicity NMHV amplitudes — Bern, Dixon, Kosower
- other fields of $\mathcal{N}=4$ SYM on external lines
- Extension to reduced susy and no susy; phenomenology

On to higher loops...

Brief comparison – or "Why are higher-loop calculations hard?"

1-loop	higher loops
technology: quadruple cuts freeze integral	no general analog; too few propagators
very efficient complete basis	over-complete or under-complete basis; naive guess insufficient
same basis for any number of external legs	limited experience; 4pt$\longrightarrow$ 5pt new integrals
	new structures: $\mathcal{O}(\epsilon)$ and $\mathcal{O}(\epsilon^2)$ in 1-loop amplitude are relevant
cuts easy to disentangle (even without quadruple cuts)	(very) nontrivial zeroes; many ways to reorganize cuts; hard to choose; some more useful than others
$4d$ algebra suffices	unclear; safety requires $D-$dim.

Generalized unitarity continues to work; strategy:

– cut at least $L+1$ and at most $4L-1$ propagators

– recognize integral functions containing the cut propagators

– test all relevant cuts for missed terms

- Cross-check against all sources of information

– (partial) localization (Buchbinder, Cachazo)

– IR properties (Sterman, Magnea; Catani)

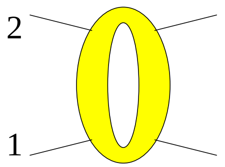
– collinear, soft and multi-particle factorization

- declare victory (invoking D-dimensional nature of calculation)

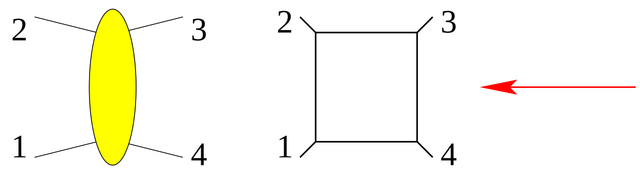
Case by case discussion: 4-point amplitudes seem the simplest

- work toward a effective rules capturing cut calculation

Recall from earlier:



$= i\, s_{12}\, s_{23}$



Box integral in Φ^3 theory

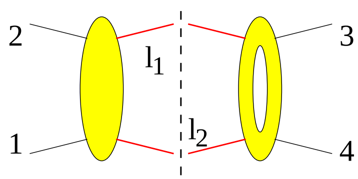
(Green, Schwarz, Brink (1982))

$$A_4^{1\,\text{loop}}(1,2,3,4) = i s_{12} s_{23}\, A_4^{\text{tree}}(1,2,3,4) \int \frac{d^d q}{q^2 (q-k_1)^2 (q-k_{12})^2 (q+k_4)^2}$$

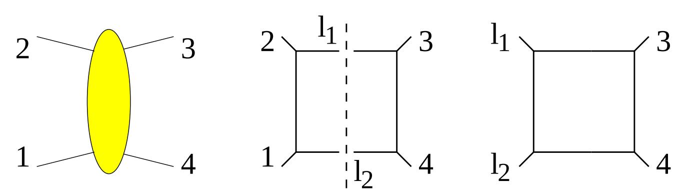
An observation: for 4-particle amplitudes, 2-particle cuts iterate to all orders

(Bern, Rozowsky, Yan)

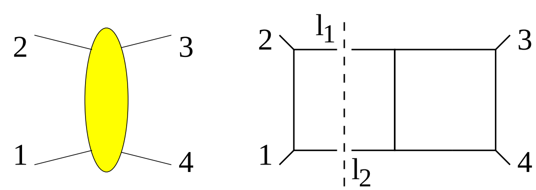
$\sum_{\mathcal{N}=4}$



$= \; - i s_{12} s_{23} \times (-i s_{12} (l_1 \cdot k_3))$

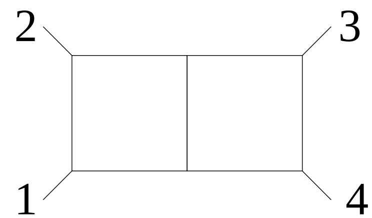


$= \; - s_{12}^2 s_{23}$

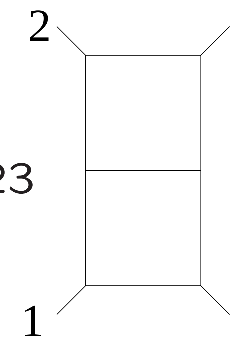


- similar in the t-channel

$A_4^{2-\text{loop}} = -s_{12}s_{23}A_4^{\text{tree}}\Big\{\ s_{12}$



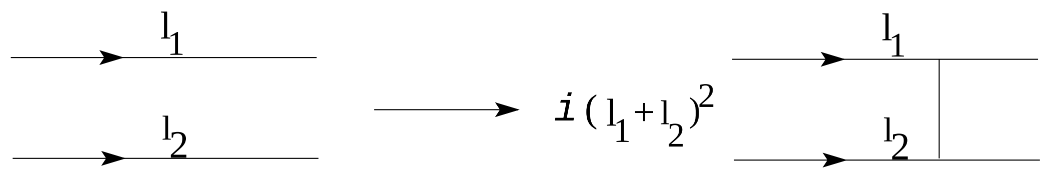
$+\ s_{23}$



$\Big\}$

(Bern, Rozowsky, Yan)

2-particle cuts $\leftrightarrow$ add all possible rungs and corresponding numerator factor (total momentum crossing the rung) (Bern, Rozowsky, Yan)



- at 3-loops gives the complete answer (6 diagrams plus cyclic)
- similar methods yield the 5-point 2-loop amplitude

Comments and summary:

- 1-loop S-matrix is analytically known in any massless gauge theory

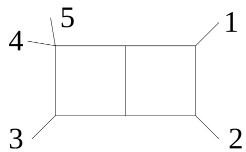
- Unitarity-based calculations can produce analytic expressions for the integrands of higher loop scattering amplitudes

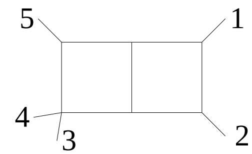
- Overcomplete basis; Care is needed to avoid overcounting

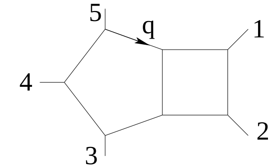
Tomorrow: Some of the implications of these results
the state of the art for multi-leg and multi-loop calcu

Summary: 5-gluon 2-loop amplitude in $\mathcal{N}=4$ SYM

$$A_5^{2\,\text{loops;even}} = -\frac{1}{2} A_5^{tree} \sum_{\text{cyc}}$$

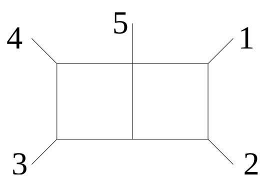
$$\Big\{ s_{12}^2 s_{23}$$ 

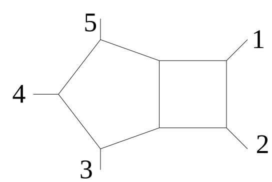
$$+ s_{12}^2 s_{15}$$ 

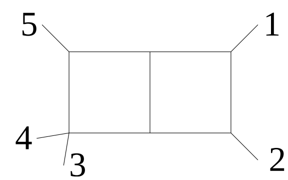
$$+ s_{12} s_{34} s_{45} (q - k_1)^2$$ 

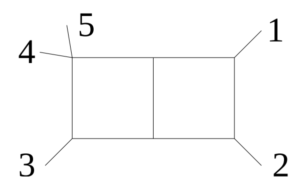
$$\Big\}$$

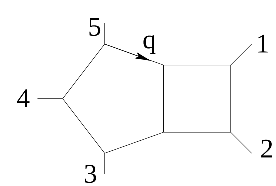
$$A_5^{2\,\text{loops;odd}} = \frac{1}{32} A_5^{\text{tree}} \text{Tr}\,(\gamma_5 \not{k}_1 \not{k}_2 \not{k}_3 \not{k}_4) \frac{s_{12} s_{23} s_{34} s_{45} s_{51}}{G(1,2,3,4)} \sum_{\text{cyc}}$$

$$\times \Big\{$$ 

$$+ 2 s_{12}$$ 

$$- \frac{s_{12}(s_{12}s_{15} - s_{12}s_{23} + s_{23}s_{34} - s_{15}s_{45} + s_{34}s_{45})}{s_{23}s_{34}s_{45}}$$ 

$$+ \frac{s_{12}(-s_{12}s_{51} + s_{12}s_{23} - s_{23}s_{34} + s_{45}s_{51} + s_{34}s_{45})}{s_{34}s_{45}s_{51}}$$ 

$$+ \frac{(s_{12}s_{51} + s_{12}s_{23} - s_{23}s_{34} + s_{45}s_{34} - s_{45}s_{51})}{s_{23}s_{51}} (q + k_1)^2$$ 

$$\Big\}$$