On-shell methods in gauge theories
Part 1: Trees

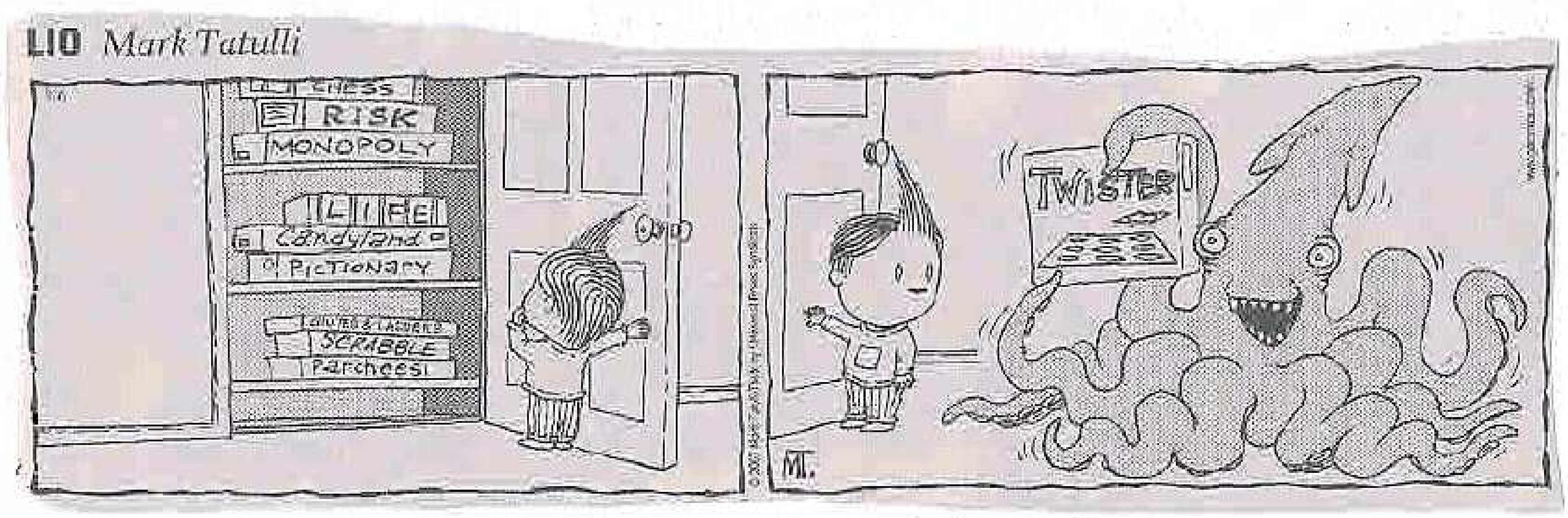
Radu Roiban
Pennsylvania State University

My goals for the three lectures:

- describe technology for tree-level and 1-loop higher point analytic calculations
- describe technology for higher-loop calculations and field theory conjectures for the resummation of certain amplitudes
- compare the state-of-the art calculation in $\mathcal{N} = 4$ SYM with a string theory inspired conjecture

– describe technology for tree-level and 1-loop higher point analytic calculations

Initially developed for maximally supersymmetric YM theory in four dimensions, the methods been improved and generalized to theories with reduced supersymmetry as well as QCD



courtesy of L. Dixon

The main messages: it pays to stay on-shell

everything is built from tree amplitudes

simplicity must start at tree-level

Off-shell questions:

- Green's functions?
- Effective actions?
- Renormalization of composite operators? γ_O
- Recycle

On-shell questions:

- (massless) Scattering amplitudes?
- Effective actions (up to field redefinitions)?
- Renormalization of composite operators? γ_O

Advantages of staying on-shell:

- cancellations due to on-shell condition
- cancellations due to gauge invariance
- use techniques not indigenous to field theories

Source of recent inspiration:

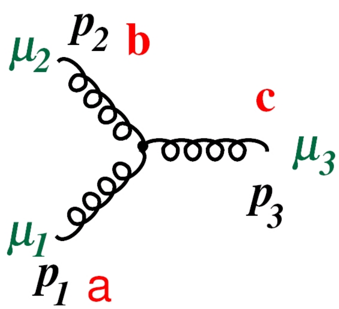
Witten's proposed relation between a certain type of string theory and the maximally supersymetric Yang-Mills theory in $d = 4$

- Compute *on-shell* scattering amplitudes as the scattering amplitudes of a topological string field theory in twistor space

- Formal expression of string scattering amplitudes

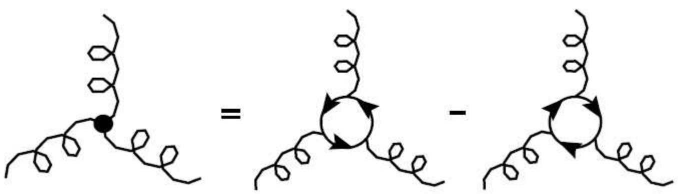
$$\mathcal{A}_{tree} = \int [d\mu_{d,\,g}] \int \prod_{i=1}^{n} \tilde{\Phi}(Z(\sigma_i))\ d\sigma_i\ \langle J(\sigma_1) \ldots J(\sigma_n) \rangle_{[\mu_{d,\,g}]}$$

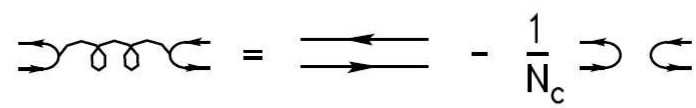
- First step towards simplicity: organization



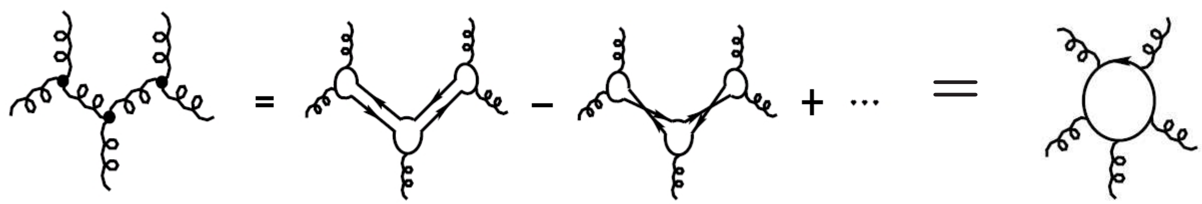
$$\mathrm{Tr}[T^a[T^b, T^c]]\Big[\eta_{\mu_1\mu_2}(p_1 - p_2)_{\mu_3} + \eta_{\mu_2\mu_3}(p_2 - p_3)_{\mu_1} + \eta_{\mu_3\mu_1}(p_3 - p_1)_{\mu_2}\Big]$$

- Vertex and propagator color flow





- Color decomposition of a Feynman diagram



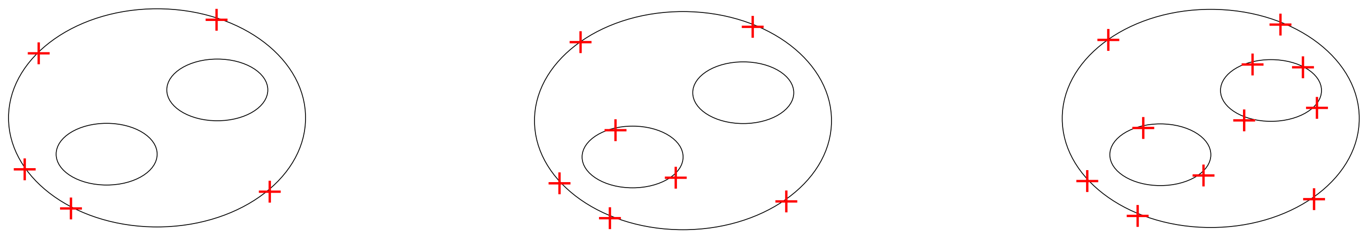
+ permutations

- End result: $\mathcal{A}_n = \sum_{\sigma \in S_n/\mathbb{Z}_n} \mathrm{Tr}[T^{a_{\sigma(1)}} \dots T^{a_{\sigma(n)}}] A_n(\sigma(1), \dots, \sigma(n))$

Berends, Giele; Mangano, Parke, Xu; Bern, Kosower

• First step towards simplicity: organization

More transparent origin: gauge theory amplitudes as a limit of open string scattering amplitudes



▪ End result: Berends, Giele; Mangano, Parke, Xu; Bern, Kosower

$$\mathcal{A}_n = \sum_{\sigma \in S_n/\mathbb{Z}_n} \mathrm{Tr}[T^{a_{\sigma(1)}} \ldots T^{a_{\sigma(n)}}] A_n(\sigma(1), \ldots, \sigma(n)) + \text{multitraces}$$

Second step toward simplicity: "right notation"

- Chinese magic: Xu, Chang, Zhang;
  Berends, Kleiss, De Causmaecker, Gastmans, Stirling, Troost, Wu; Gunion, Kunszt

$$p^\mu \;\to\; (p^\mu \sigma_\mu)^{\alpha\dot\alpha} = p^{\alpha\dot\alpha} \;\to\; p^\mu p_\mu = 0 \Leftrightarrow \det(p^\mu \sigma_\mu) = 0 \;\Rightarrow p^{\alpha\dot\alpha} = \lambda^\alpha \tilde\lambda^{\dot\alpha}$$

$$2p\cdot q = \langle pq\rangle[qp] \qquad \langle pq\rangle = \epsilon_{\alpha\beta}\lambda_p^\alpha \lambda_q^\beta \qquad [pq] = -\epsilon_{\dot\alpha\dot\beta}\lambda_p^{\dot\alpha}\lambda_q^{\dot\beta}$$

- More standard origin:

$$\sum_{s=\pm} u_s(p)\bar u_s(p) = -\not{p} + m$$

take $m \to 0$ and project onto chiral components

$$u_\pm(p) = \frac{1}{2}(1 \pm \gamma_{-1})u(p) = \begin{cases} |p^+\rangle = |p\rangle = u(p)_\alpha \\ |p^-\rangle = |p] = \bar u(p)_{\dot\alpha} \end{cases}$$

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$$2p\cdot q = \langle pq\rangle[qp] \qquad \langle pq\rangle = \epsilon_{\alpha\beta}\lambda_p^\alpha \lambda_q^\beta \qquad [pq] = -\epsilon_{\dot\alpha\dot\beta}\lambda_p^{\dot\alpha}\lambda_q^{\dot\beta}$$

- Interpretation in terms of standard quantities

$$u_\pm(p) = \frac{1}{2}(1\pm\gamma_{-1})u(p) = \begin{cases} |p^+\rangle = |p\rangle = u(p)_\alpha \\ |p^-\rangle = |p] = \bar u(p)_{\dot\alpha} \end{cases}$$

- Polarization vectors: $k^\mu \epsilon_\mu(k) = 0$ $\epsilon_\mu(k) \sim \epsilon_\mu(k) + x k_\mu$

$$\epsilon^+_\mu(k,\xi) = \frac{\langle\xi|\gamma_\mu|k]}{\sqrt{2}\langle\xi k\rangle} \qquad \epsilon^+_{\alpha\dot\alpha}(k,\xi) = \sqrt{2}\,\frac{\xi_\alpha\tilde\lambda_{\dot\alpha}}{\langle\xi k\rangle}$$

$$\epsilon^-_\mu(k,\xi) = -\frac{[\xi|\gamma_\mu|k\rangle}{\sqrt{2}[\xi k]} \qquad \epsilon^-_{\alpha\dot\alpha}(k,\xi) = -\sqrt{2}\,\frac{\lambda_\alpha\tilde\xi_{\dot\alpha}}{[\xi k]}$$

where ξ is an arbitrary null vector $\quad \xi_\mu \sigma^\mu_{\alpha\dot\alpha} = \xi_\alpha \tilde\xi_{\dot\alpha}$

- Some identities, etc.

$\epsilon \cdot k = 0$ $\qquad\qquad\qquad\qquad$ $\epsilon \cdot \xi = 0$

Gordon: $\quad 2k^\mu = \langle k|\sigma^\mu|k] = [k|\sigma^\mu|k\rangle$

Fierz: $\quad [i|\gamma_\mu|j\rangle[k|\gamma_\mu|l\rangle = 2[ik]\langle jl\rangle$

Schouten: $\langle ij\rangle\langle kl\rangle + \langle ik\rangle\langle lj\rangle + \langle il\rangle\langle jk\rangle = 0$

Properties of color-ordered amplitudes

- Cyclicity:

$$A(2,3,\ldots,n,1) = A(1,2,\ldots,n).$$

- Reflection:

$$A(n,n-1,\ldots,1) = (-1)^n A(1,2,\ldots,n).$$

- Dual Ward (or Sub-Cyclic) Identity:

$$\sum_{C(1,\ldots,n-1)} A(1,2,3,\ldots,n) = 0,$$

n – fixed; $C(1,\ldots,n-1)$ set of cyclic permutations of $\{1,\ldots,n-1\}$

- Generalized dual Ward identity:

$$\sum_{\text{Perm}(i,j)} A(i_1,\ldots,i_m,j_1,\ldots,j_k,n+1) = 0, \qquad 1 \le m \le n-1, \qquad m+k=n,$$

where the sum is taken over permutations of the set $(i_1,\ldots,i_m,j_1,\ldots,j_k)$ which preserve the order of the $(i_1,\ldots,i_m)$ and $(j_1,\ldots,j_k)$ separately.

- Conjugation/CPT: invariance under $+ \leftrightarrow -$ with simultaneous $\lambda \leftrightarrow \bar{\lambda}$.

$$A(\lambda_i, \bar{\lambda}_i, \eta_{iA}) = \int d^{4n}\psi \ \ \text{exp}\left[i \sum_{i=1}^{n} \eta_{iA} \psi_i^A \right] A(\bar{\lambda}_i, \lambda_i, \psi_i^A).$$

- Soft-Gluon Limit: in the limit $p_1 \to 0$ any amplitude behaves as

$$A^{\text{tree}}(1^+, 2, \ldots, n) \longrightarrow \frac{\langle n\, 2\rangle}{\langle n\, 1\rangle \langle 1\, 2\rangle} A^{\text{tree}}(2, \ldots, n).$$

- Collinear Limits: $(p_1 \to zp$ and $p_2 \to (1-z)p$ with $p^2 = 0$; or $z = \xi \cdot p_1 / \xi \cdot p)$

$$A_n^{(L)}(1, 2, 3, \ldots) \longrightarrow \sum_{l=0}^{L} \sum_{\chi=\pm} \ \ \text{Split}_{-\chi}^{(l)}(1, 2, z) A^{(L-l)}(p^\chi, 3, \ldots)$$

$$A^{\text{tree}}(1^+, 2^+, 3, \ldots) \longrightarrow \frac{1}{\sqrt{z(1-z)}} \frac{1}{\langle 1\, 2\rangle} A(p^+, 3, \ldots)$$

$$A^{\text{tree}}(1^+, 2^-, 3, \ldots) \longrightarrow \frac{z^2}{\sqrt{z(1-z)}} \frac{1}{[1\, 2]} A(p^+, 3, \ldots) + \frac{(1-z)^2}{\sqrt{z(1-z)}} \frac{1}{\langle 1\, 2\rangle} A(p^-, 3, \ldots)$$

- Multi-particle Poles: Color-ordered amplitudes can only have poles in channels corresponding to a sum of cyclically adjacent momenta going on-shell.

  - Tree level:

$$A_n(1,\ldots,n) \longrightarrow \sum_{\chi=\pm} A_{m+1}(1,\ldots,m,p^{\chi})\frac{i}{p_{1,m}^2}A_{n-m+1}(m+1,\ldots,n,p^{-\chi})$$

where $p_{1,m} = p_1 + p_2 + \cdots + p_m$ with $p_{1,m}^2 \to 0$

  - 1 loop:

$$A_n^{1\,\text{loop}}(1,\ldots,n) \longrightarrow \sum_{\chi=\pm} \Big[\; A_{m+1}^{\text{tree}}(1,\ldots,m,p^{\chi})\frac{i}{p_{1,m}^2}A_{n-m+1}^{1\,\text{loop}}(m+1,\ldots,n,p^{-\chi})$$

$$+ \; A_{m+1}^{1\,\text{loop}}(1,\ldots,m,p^{\chi})\frac{i}{p_{1,m}^2}A_{n-m+1}^{\text{tree}}(m+1,\ldots,n,p^{-\chi})$$

$$+ \; A_{m+1}^{\text{tree}}(1,\ldots,m,p^{\chi})\frac{i\mathcal{F}(1\ldots n)}{p_{1,m}^2}A_{n-m+1}^{\text{tree}}(m+1,\ldots,n,p^{-\chi}$$

Important amplitudes (for the next 3 days):

- $A(p_1^{\pm}, p_2^{+} \dots p_n^{+}) = 0$

V1. Use spinor helicity; choose the reference vectors of likewise-helicity gluons to be the same and equal to the momentum of an opposite helicity gluon

V2. SUSY Ward identities: act with Q on $\Lambda^+ g^+ \dots g^+$ and $\Lambda^+ g^- \dots g^+$. In susy theories they vanish to all orders in perturbation theory

- MHV and $\overline{MHV}$ amplitudes

$$A(1^+ ... i^- ... j^- ... n^+) = \frac{\langle ij \rangle^4}{\prod_{s=1}^n \langle s, s+1 \rangle} \quad \text{and} \quad \begin{cases} + \leftrightarrow - \\ \langle\rangle \leftrightarrow [] \end{cases}$$

- Hidden structure of tree-level *on shell* (gluon) amplitudes:

- MHV: $A_n(i^-, j^-) = \frac{\langle ij\rangle^4}{\prod_{i=1}^n \langle i(i+1)\rangle} \delta^4(\sum_i \lambda_i^\alpha \tilde{\lambda}_i^{\dot{\alpha}})$

Quadratic constraint

Fourier-transform $\tilde{\lambda}$: $\tilde{A}_n(i^-, j^-) = \int \prod_k d^2\tilde{\lambda}_k e^{i\sum_l [\mu_l, \tilde{\lambda}_l]} A_n(i^-, j^-)$

(use integral representation of δ-function)

$$\tilde{A}_n(i^-, j^-) = \int d^4x \frac{\langle ij\rangle^4}{\prod_{i=1}^n \langle i(i+1)\rangle} \prod_k \delta^2(\lambda_i^\alpha + x_{\alpha\dot{\alpha}}\mu_i^{\dot{\alpha}})$$

Linear constraints

$\diamond$ in (λ, μ) space (twistor space) MHV amplitudes are localized on complex lines

$\longrightarrow$ all but two quadruples (λ_i, μ_i) are linearly dependent

$\diamond$ To expose this structure complex momenta are useful

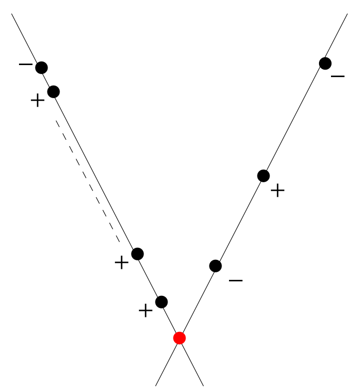
- Probe available amplitudes for such ‘collinearity’
  (case by case study of each helicity configuration)

$$0 = F_{ijk} = \lambda^{\alpha}_{[i}\lambda^{\beta}_{j}\mu^{\dot\gamma}_{k]} \longrightarrow \langle ij\rangle \frac{\partial}{\partial \tilde\lambda^{k}_{\dot\gamma}} + \langle jk\rangle \frac{\partial}{\partial \tilde\lambda^{i}_{\dot\gamma}} + \langle ki\rangle \frac{\partial}{\partial \tilde\lambda^{k}_{\dot\gamma}} = 0$$

◇ determine the twistor space representation of known amplitudes

- intersecting complex lines; as many as $n_- - 1$
- each of them contains at most two (λ_i, μ_i) quadruples corresponding to negative helicity gluons
- consistent with color ordering

e.g.



in the $(- + - + \cdots + -)$ amplitude

MHV diagrams

Cachazo, Svrcek, Witten

- interpret each line as an MHV amplitude

$\longrightarrow$ 'invent' a way to glue together MHV amplit's into N^{k}MHV ones

- Rules and issues
  - number of "vertices" is $k-2$ (nr. of intersecting lines)
  - assignment of legs consistent color ordering
  - use Feynman propagators to connect MHV vertices
  - off-shellness of internal legs
    - need holomorphic spinor P_α for an off-shell momentum $P^2 \neq 0$
    - introduce a constant antiholomorphic spinor $\tilde{\eta}_{\dot{\alpha}}$
    - define $P_\alpha = P|\tilde{\eta}]$

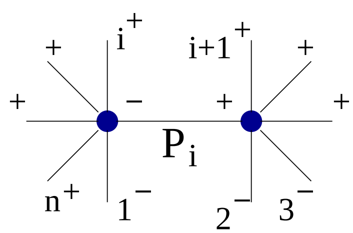
MHV diagrams

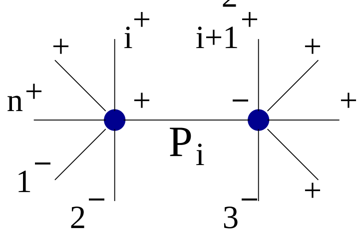
Cachazo, Svrcek, Witten

$$A_{n_-,n_+} = \sum_{\mathcal{D}|\mathrm{card}\mathcal{D}=n_- -1} A_k^{\mathrm{MHV}} \prod_{\{ij\}\in\mathrm{Links}} \frac{1}{P_{ij}^2}$$

- assignment of external legs – consistent with color ordering
- off-shell leg: constant spinors – similar to light-cone gauge

Example: $A_n(- - - + \cdots +)$





$$\sum_{i=3}^{n-1} \left[\frac{\langle 1P_i\rangle^3}{\langle P_i, i+1\rangle\langle i+1, i+2\rangle \ldots \langle n1\rangle}\right] \frac{1}{P_i^2} \left[\frac{\langle 23\rangle^3}{\langle P_i 2\rangle \ldots \langle iP_i\rangle}\right.$$

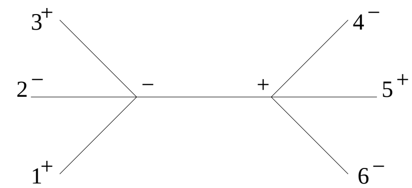
$$+ \sum_{i=3}^{n-1} \left[\frac{\langle 12\rangle^3}{\langle 2P_i\rangle\langle P_i, i+1\rangle \ldots \langle n1\rangle}\right] \frac{1}{P_i^2} \left[\frac{\langle 34\rangle^3}{\langle P_i 2\rangle \ldots \langle iP_i\rangle}\right]$$

MHV diagrams

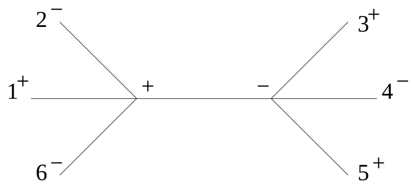
Cachazo, Svrcek, Witten

Gauge theory amplitudes can be computed by sewing together with Feynman propagators off-shell-continued MHV amplitudes: $\langle kP\rangle = \epsilon_{ab}\lambda_k^a P^{ab}\tilde{\eta}_b$ – arbitrary η

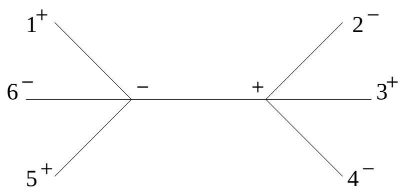
Another example: $A_6(+-+-+-)$ $\langle kP\rangle = \epsilon_{ab}\lambda_k^a P^{a\dot b}\tilde\eta_{\dot b}$ – arbitrary $\tilde\eta$



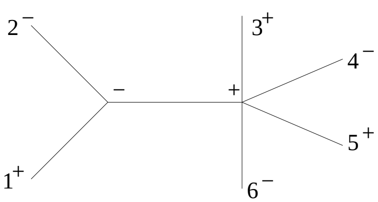
$$\frac{\langle 2p_{123}\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 3p_{123}\rangle\langle p_{123}1\rangle}\frac{1}{p_{123}^2}\frac{\langle 46\rangle^4}{\langle 45\rangle\langle 56\rangle\langle 6p_{123}\rangle\langle p_{123}4\rangle}$$



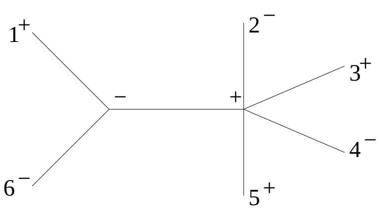
$$\frac{\langle 62\rangle^4}{\langle 61\rangle\langle 12\rangle\langle 2p_{612}\rangle\langle p_{612}6\rangle}\frac{1}{p_{612}^2}\frac{\langle 4p_{612}\rangle^4}{\langle 34\rangle\langle 45\rangle\langle 5p_{612}\rangle\langle p_{612}3\rangle}$$



$$\frac{\langle 6p_{561}\rangle^4}{\langle 56\rangle\langle 61\rangle\langle 1p_{561}\rangle\langle p_{561}5\rangle}\frac{1}{p_{561}^2}\frac{\langle 42\rangle^4}{\langle 23\rangle\langle 34\rangle\langle 4p_{561}\rangle\langle p_{561}2\rangle}$$



$$\frac{\langle 2p_{12}\rangle^3}{\langle p_{12}1\rangle\langle 12\rangle}\frac{1}{p_{12}^2}\frac{\langle 46\rangle^4}{\langle 34\rangle\langle 45\rangle\langle 56\rangle\langle 6p_{12}\rangle\langle p_{12}3\rangle} + 2\times(i\to i+2)$$



$$\frac{\langle p_{61}6\rangle^3}{\langle 61\rangle\langle 1p_{61}\rangle}\frac{1}{p_{61}^2}\frac{\langle 24\rangle^4}{\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 5p_{61}\rangle\langle p_{61}2\rangle} + 2\times(i\to i+2)$$

- the arbitrary spinor $\tilde{\eta}$ used to define the off-shell continuation drops out of the final result $\longrightarrow$ Lorentz inv. is restored
- the unphysical poles $1/\langle iP \rangle$ are spurious
- correct multi-particle singularities
- $k \geq 3$-particle poles are manifest
- collinear poles without change in n_- also manifest in all trees
- $+- \to +$ and $-- \to -$ only from 3-point MHV vertices

◇ Amplitudes computed from MHV vertices have the same poles and residues as the amplitudes computed from regular Feynman diagrams $\longrightarrow$ at tree level this guarantees that the two are the same

- possible reorganization in terms of non-MHV diagrams

◇ Vastly more efficient: e.g.

6g adjacent -: 220 Feynman diag. vs. 6 MHV diag.

6g alternating: large # of Feynman diag. vs. 9 MHV diag.

MHV rules have Lagrangian origin

Mansfield

earlier attempts by Gorsky, Rosly

Start with Yang-Mills light-cone Lagrangian:

$(A_0 = A_0 - A_3, A_{\bar{0}}, A_z = A_1 + iA_2, A_{\bar{z}})$

$$L = \mathrm{Tr}\,[A_z \Box A_{\bar{z}}] - \mathrm{Tr}\left[[A_{\bar{z}}, \partial_{\bar{0}} A_z]\,\frac{1}{\partial_{\bar{0}}}\,[A_z, \partial_{\bar{0}} A_{\bar{z}}]\right] - \mathrm{Tr}\left[\frac{\partial_{\bar{z}}}{\partial_{\bar{0}}} A_z\,[A_z, \partial_{\bar{0}} A_{\bar{z}}]\right] - \mathrm{Tr}\left[[A_{\bar{z}}, \partial_{\bar{0}} A_z]\,\frac{\partial_z}{\partial_{\bar{0}}} A_{\bar{z}}\right]$$

- Construct unitary/canonical transformation such that

$$\mathrm{Tr}\,[A_z \Box A_{\bar{z}}] - \mathrm{Tr}\left[\frac{\partial_{\bar{z}}}{\partial_{\bar{0}}} A_z\,[A_z, \partial_{\bar{0}} A_{\bar{z}}]\right] \mapsto \mathrm{Tr}\,[B_z \Box B_{\bar{z}}]$$

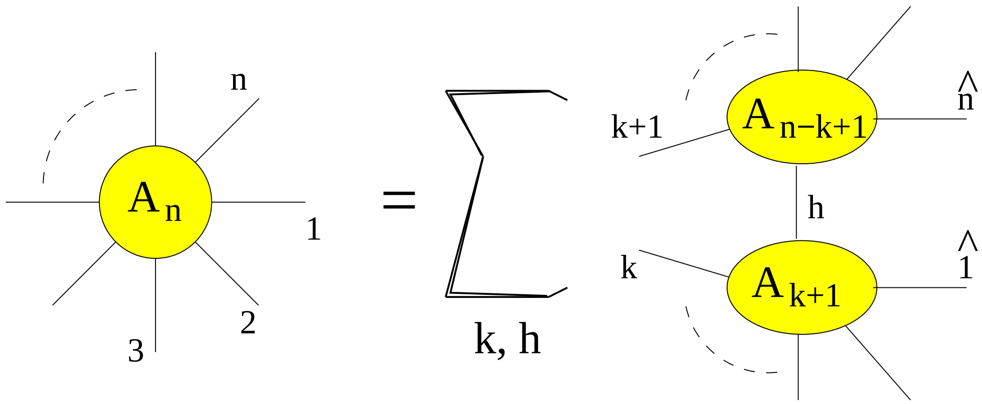
$\Pi_z = \partial_{\bar{0}} A_{\bar{z}} = \int d^3x \dfrac{\delta B_z}{\delta A_z} \partial_{\bar{0}} B_z$ (no change in path integral measure)

- Vertices: MHV amplitudes with same $\vec{p}$ as the off-shell fields
  4-dimensional definition introduces η!

- Still... for given N and n_-, number of MHV diagrams grows as

$$N = \frac{1}{n_-}\binom{n-3}{n_--2}\binom{n+n_--3}{n_--2} \sim n^{2n_--4}$$

We can do better with on-shell recursion relations



$$\widehat{p}_n = p_n + \frac{P^2_{1\ldots k}}{2[n|P_{1\ldots k}|1\rangle}\lambda_1\bar{\lambda}_n$$

$$\widehat{p}_1 = p_1 - \frac{P^2_{1\ldots k}}{2[n|P_{1\ldots k}|1\rangle}\lambda_1\bar{\lambda}_n$$

↑
no effect on
momentum conservation

Convoluted history:

- examples based on constraints following from 1-loop amplitudes
  Bern, DelDuca, Dixon, Kosower; RR, Spradlin, Volovich
- derived from 1-loop IR — Britto, Cachazo, Feng
- proved using basic field theory+cpx — Britto, Cachazo, Feng, Witten
- proved using largest time equation — Vaman, Yao

- Massless fields:

Key observation: momenta may be complex

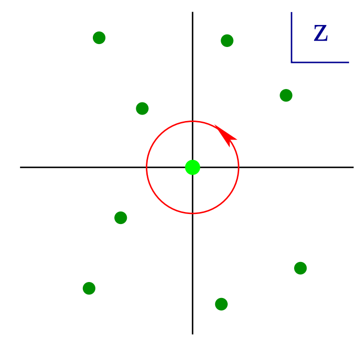
$$p_i \to p_i(z) = p_i + z\eta \qquad \text{such that} \qquad p_i + p_j = p_i(z) + p_j(z)$$
$$p_j \to p_j(z) = p_j - z\eta \qquad\qquad\qquad\qquad p_i(z)^2 = 0 = p_j(z)^2$$

$\eta = \lambda_i \tilde{\lambda}_j$

- Amplitude and propagators:
$$A_{1\ldots n} \mapsto A_{1\ldots n}(z)$$
$$P_{i,\ldots,i+k} \mapsto P_{i,\ldots,i+k}(z)$$

  – $A(0)$– original amplitude

$$\longrightarrow A_{1\ldots n} = \oint_{C_0} \frac{dz}{z} A_{1\ldots n}(z)$$

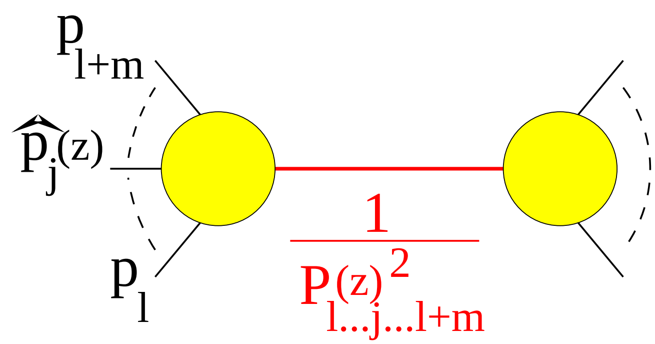


Properties:

- $A(z)$ is a rational function of z
- $A(z)$ has only simple poles in z

  – at $z = z_{lm}$ for which

$$P_{l,\dots,j,\dots,l+m}(z_{lm})^2 = 0$$



- $\lim_{z\to\infty} A(z) = 0$ nontrivial step: use MHV vertices and count z

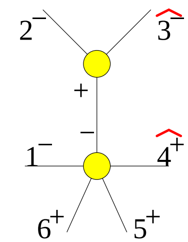
$\Rightarrow$ rotate contour: $A(z) = \sum_{lm} \frac{c_{lm}}{z - z_{lm}}$

- c_{lm} are products of amplitudes evaluated at $z = z_{lm}$

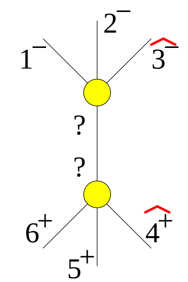
$$A = \sum_{l,m;h} A_L^h(z_{lm}) \frac{i}{P^2_{l\dots j\dots m}} A_R^{-h}(z_{lm}) \qquad z_{lm} = \frac{P^2_{l\dots j\dots m}}{2[j|P_{l\dots j\dots m}|i\rangle}$$

Example: split-helicity $(- - - + + +)$ amplitude

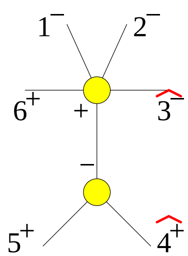
- determine z from on-shell condition of internal leg



$$\frac{\langle 2\hat{3}\rangle^3}{\langle\hat{3}\widehat{p_{23}}\rangle\langle\widehat{p_{23}}2\rangle}\,\frac{1}{p_{23}^2}\,\frac{\langle 1\widehat{p_{23}}\rangle^3}{\langle\widehat{p_{23}}\hat{4}\rangle\langle\hat{4}5\rangle\langle 56\rangle\langle 61\rangle}\qquad z=\frac{p_{23}^2}{\langle 4|P_{23}|3]}$$



$$0$$



$$\frac{[\widehat{p_{45}}6]^3}{[\widehat{p_{23}}6][61][12][2\hat{3}][\hat{3}\widehat{p_{23}}]}\,\frac{1}{p_{45}^2}\,\frac{[\hat{4}5]^3}{[5\widehat{p_{45}}][\widehat{p_{45}}\hat{4}]}\qquad z=\frac{p_{45}^2}{\langle 4|p_{45}|3]}$$

Just shake...

$$A_{1^-2^-3^-4^+5^+6^+}=\frac{1}{\langle 5|p_{34}|2]}\left(\frac{\langle 1|p_{23}|4]^3}{[23][34]\langle 56\rangle\langle 61\rangle p_{234}^2}+\frac{\langle 3|p_{45}|6]^3}{[61][12]\langle 34\rangle\langle 45\rangle p_{345}^2}\right)$$

Main feature: generality

- works the same way with fields of different spins
  - spinors
  - scalars
  - massive
- simplicity related to appearence of spurious singularities
- issues with numerical implementation due to shifts, but simple enough for many-point analytic calculations
  - e.g. $A_{(-)^m(+)^n}$, $A_{(-)^m(+)^n(-)(+)^p}$, $A_{(+-)^6}$

Homework:

Find $A_{(+-)^n}$ for generall n

Completely determines the tree-level S-matrix!

- Massive fields:

– No difference of principle

$$\frac{1}{P^2_{l\ldots j\ldots l+m} + M^2_{l\ldots m}} \mapsto \frac{1}{P^2_{l\ldots j\ldots l+m}(z) + M^2_{l\ldots m}}$$

$$A \mapsto A(z) = \sum_{l,m,h} A^h_L(z) \frac{1}{P^2_{l\ldots j\ldots l+m}(z) + M^2_{l\ldots m}} A^{-h}_R(z)$$

More complicated building blocks and residues but same principle

Badger, Glover, Khoze, Svrcek

- Shifts: $\begin{array}{l} p_i \to p_i(z) = p_i + z\eta \\ p_j \to p_j(z) = p_j - z\eta \end{array}$ $\mapsto$ $\eta \cdot p_i = \eta \cdot p_j = \eta^2 = 0$

- simple solution if $m_i = 0$ or $m_j = 0$; otherwise complicated

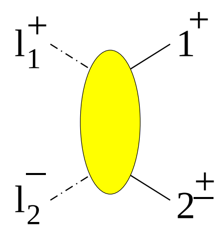
$A = \oint \frac{dz}{z} A(z)$ use instead poles at $z_{l\ldots j\ldots l+m} = -\frac{P^2_{l\ldots j\ldots l+m} + M^2_{l\ldots m}}{2\eta \cdot P_{l\ldots j\ldots l+m}}$

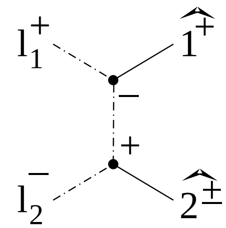
Example: $gg\phi\bar{\phi}$ with massive scalars $m_\phi \neq 0$ (say Higgs-glue e.t.)

Badger, Dixon, Glover, Khoze

Minimal coupling $\mapsto$ off-shell vertex: $\quad V_3(l_1^+, k^\mu, l_2^-) = \frac{1}{\sqrt{2}}(l_2^\mu - l_1^\nu)$

$$\longrightarrow \quad A_3(l_1^+, k^+, l_2^-) = A_3(l_1^-, k^+, l_2^+) = \frac{\langle q_1|k|l_1]}{\langle q_1 k\rangle} \quad \text{with} \quad q_1 = \widehat{k_2}$$

$A_4(l_1^+, 1^+, 2^+, l_2^-) \; = $ 

$=$ 

$$= A_4(l_1^+, \hat{1}^+, -\hat{P}^-)\frac{1}{P^2 + m_\phi^2}A_4(\hat{P}^+, \hat{2}^+, l_2^-)$$

$$= \frac{\langle 2|l_1|1]\langle \hat{1}|l_2|\hat{2}]}{\langle 12\rangle^2((l_1+k_1)^2 + m_\phi^2)} = \frac{\mathrm{Tr}\ [\not{k}_1 \not{l}_1 \not{k}_2 \not{l}_1]}{\langle 12\rangle^2((l_1+k_1)^2 + m}$$

$$\overset{\hat{k}_1\cdot l_1 = 0}{=} - \frac{m_\phi^2\,[12]}{\langle 12\rangle((l_1+k_1)^2 + m_\phi^2)}$$

Summary

- Spinor helicity; relation to twistors and curves
- MHV rules
  - efficient; expose factorization properties
  - yield relatively compact expressions
  - Lagrangian
- On-shell recursion relations
  - efficient for analytic calculation; compact expressions
  - simplicity comes with potential spurious poles
  - generality

Next step: loops from trees