Topics in Cusped/Lightcone Wilson Loops

by

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- Lecture 1. Pedagogical Introduction

Wilson loops with cusps, their renormalization, relation to twist-two operators, the role in string/gauge correspondence, minimal surface in $AdS_5 \otimes S^5$ for cusped loops

- Lecture 2. Perturbation Theory: two loops and beyond

exact sum of ladders, explicit two loops and the anomaly terms, results in double logarithmic approximation, problems with planar QFT

- Lecture 3. Cusped Loop Equation

modern formulation of LE, SUSY extension, UV regularization, specifics of cusped loops, anomalous dimension from LE

Wilson Loops

Non-Abelian phase factor

$$U(C) = \boldsymbol{P}\,\mathrm{e}^{\mathrm{i}g\int_C A_\mu(x)\mathrm{d}x^\mu} \stackrel{\text{def}}{=} \prod_{x\in C}\left(1+\mathrm{i}gA_\mu(x)\mathrm{d}x^\mu\right)$$

parallel transporter in non-Abelian Yang–Mills field

$\mathrm{tr}\,U(C)$ is gauge-invariant for closed C

Wilson loop v.e.v. (average in Euclidean formulation)

$$W(C) = Z^{-1}\int \mathcal{D}A_\mu\,\mathcal{D}\bar{\psi}\mathcal{D}\psi\cdots\ \mathrm{e}^{\mathrm{i}S}\frac{1}{N}\mathrm{tr}\,U(C)$$

Importance of the Wilson loops (large N):

- observables are expressed via sum-over-path of $W(C)$
- dynamics is entirely reformulated via $W(C)$

$W(C)$ obeys the loop equation (closed equation on loop space)

Typical loops essential in the sum-over-path are cusped

Renormalization of smooth Wilson loops

For smooth loops

Gervais, Neveu (1980)

Polyakov (1980)

Vergeles, Dotsenko (1980)

$$W(g;C) = \mathrm{e}^{-\text{const.}\, L(C)/a}\, W_{\mathrm{R}}(g_{\mathrm{R}};C)$$

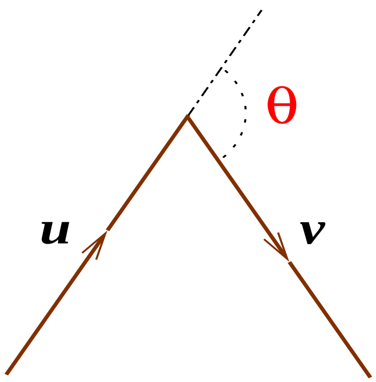
where W_{R} is finite after the charge renormalization $g \Longrightarrow g_{\mathrm{R}}$
and a is a certain (gauge-invariant) UV cutoff

The exponential comes from the renormalization of the mass of a heavy test particle propagating along the loop.
It does not emerge in dimensional regularization

Renormalization of cusped Wilson loops

An additional logarithmic divergency appear for cusped loops

Polyakov (1980)



Segment of a closed loop near the cusp.
θ is the cusp angle formed by the vectors u and v:
$\cosh\theta = \frac{u\cdot v}{\sqrt{u^2}\sqrt{v^2}}$

The cusped Wilson loop is multiplicatively renormalizable

Brandt, Neri, Sato (1981)

$$W(g;\Gamma) = Z(g,;\theta)W_{\mathrm{R}}(g_{\mathrm{R}};\Gamma)$$

where (the divergent factor of) $Z(g;\theta)$ depends on the cusp angle θ

This is true if Γ has no light-cone segments

Cusp anomalous dimension

The definition

$$\gamma_{\text{cusp}}(g;\theta) = -a\frac{\mathrm{d}}{\mathrm{d}a}\ln Z(g;\theta)$$

The limit of large θ

Korchemsky, Radyushkin (1987)

$$\gamma_{\text{cusp}}(g;\theta) \stackrel{\theta\to\infty}{\longrightarrow} \frac{\theta}{2}f(g)$$

The same function f appear in the anomalous dimensions of twist two conformal operators with large spin

Relation to twist-two operators

Anomalous dimensions of twist-two operators

$$O_J^{(F)} = \frac{1}{N} \operatorname{tr} F_{\mu\cdot} (\nabla_\cdot)^{J-2} F_{\mu\cdot}$$

$$O_J^{(\Psi)} = \bar{\Psi} \gamma_\cdot (\nabla_\cdot)^{J-1} \Psi$$

with Lorentz spin J (measurable in deep inelastic)

Also

$$O_J^{(\Phi)} = \frac{1}{N} \operatorname{tr} \Phi (\nabla_\cdot)^{J} \Phi$$

in $\mathcal{N} = 4$ SYM.

Notation: $\nabla_\cdot \equiv \nabla_\mu \xi_\mu \qquad \xi^2 = 0$
— symmetrization and subtraction of traces

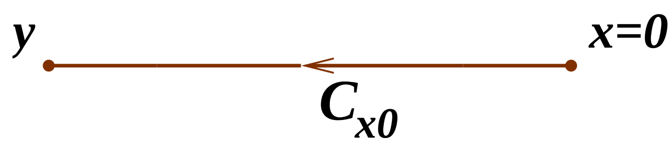
$(\nabla_\cdot)^J$ is in fact a (Gegenbauer) polynomial in $\overleftarrow{\nabla}_\cdot$ and $\overrightarrow{\nabla}_\cdot$
— conformal operators

Brodsky, Frishman, Lepage, Sachrajda (1980)
Y. M. (1981)
Ohrndorf (1982)

Relation to twist-two operators (cont.1)

The relation can be understood from open Wilson loops

$$O(C_{y0}) = \bar{\psi}(y)\boldsymbol{P}\,\mathrm{e}^{\mathrm{i}g\int_0^y \mathrm{d}\xi^\mu A_\mu}\psi(0)$$



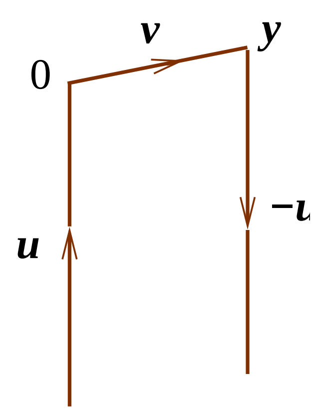
with matter fields attached at the ends

Standard triangular diagrams comes from

$$\left\langle \psi(\infty,\vec{y})O(C_{y0})\bar{\psi}(\infty,\vec{0})\right\rangle \propto W(\Pi)$$

as mass of matter fields $\to \infty$

Π-shaped
Wilson loop



Relation to twist-two operators (cont.2)

Remember that the propagator in an external field A_μ

$$\left\langle \psi_i(x)\bar{\psi}_j(y)\right\rangle_\psi \stackrel{\text{large } N}{=} \sum_{C_{yx}} \left[\mathrm{e}^{\mathrm{i}g\int_{C_{yx}} \mathrm{d}\xi^\mu A_\mu} \right]_{ij} \stackrel{\text{mass}\to\infty}{\propto} \left[\mathrm{e}^{\mathrm{i}g\int_{C_{yx}^{(\text{min})}} \mathrm{d}\xi^\mu A_\mu} \right]_{ij}$$

and thus straight vertical lines appear in $\sqcap$

The central segment of $\sqcap$ is near the light-cone (to kill twists higher than 2).
$\sqcap$ has two cusps with $\theta \to \infty$.

This is how the light-cone Wilson loop appear

Light-cone Wilson Loops

For Π-shaped loop (1 light cone)

Korchemsky, Marchesini (1993)

$$W(\Pi) = \mathrm{e}^{-\frac{1}{2}f(\lambda)\ln^2\frac{T}{a}+\mathrm{const.}(\lambda)\ln\frac{T}{a}+\mathrm{finite}(\lambda)}$$

with the same $f(\lambda)$ as before.
v^μ is along the light cone ($v^2 = 0$) and $y_\mu = v_\mu T$.

For Γ-shaped loop (2 light cones)

Alday, Maldacena (2007)

$$W(\Gamma) = \mathrm{e}^{-\frac{1}{2}f(\lambda)\ln\frac{T}{a}\ln\frac{S}{a}+g(\lambda)(\ln\frac{T}{a}+\ln\frac{S}{a})+\mathrm{finite1}(\lambda)}$$

both v^μ and u^μ are along the light cones ($v^2 = 0$, $u^2 = 0$) and $y_\mu = v_\mu T$, $x_\mu = u_\mu S$.

Most probably it gives the same $f(\lambda)$ but is not proved

SYM Wilson Loops

Extension to $\mathcal{N}=4$ SYM   Maldacena (1998)

$$W_{\mathsf{SYM}}(C) = \left\langle \frac{1}{N} \operatorname{tr} \boldsymbol{P}\, \mathrm{e}^{\mathrm{i} g \oint_C \mathrm{d}\sigma \left(\dot{\xi}^{\mu} A_{\mu} + |\dot{\xi}| n^i \Phi_i\right)} \right\rangle$$

with unit vector n^i ($n^2=1$) and 6 scalars Φ_i $(i=1,\cdots,6)$

No relative i in Minkowski space

Adjoint Wilson loop

$$\operatorname{tr}_A U = |\operatorname{tr} U|^2 - 1$$

Due to factorization at large N

$$\underbrace{\left\langle \frac{1}{N^2} \operatorname{tr}_A U(C) \right\rangle}_{\text{adjoint}} = \underbrace{\left\langle \frac{1}{N} \operatorname{tr} U(C) \right\rangle^2}_{\text{fundamental}}$$

Same results as in QCD hold and some more

BPS for a straight line inside the light-cone

$$W_{\mathsf{SYM}}(|) = 1$$

Motivation (since 2002)

AdS/CFT prediction for the anomalous dimension of twist-two operators with large (Lorentz) spin Gubser, Klebanov, Polyakov (2002)

$$\Delta - J - 2 = f(\lambda) \ln J \qquad \text{large } J \tag{1}$$

$$f(\lambda) = \frac{\sqrt{\lambda}}{\pi} \qquad \text{large } \lambda = g^2_{\mathrm{YM}} N \tag{2}$$

from spectrum of closed folded string which is rotating in AdS_5

Same result holds for the cusp anomalous dimension at large θ from minimal surface in supergravity approximation to AdS/CFT

Kruczenski (2002)

Y.M. (2002)

(2) has been remarkable reproduced recently from the spin chains (and much more results) Staudacher et al. (2006)

Same $f(\lambda)$ appears in MHV gluon amplitudes

Bern, Dixon, Smirnov (2005)

and is reproduced for large λ from AdS/CFT

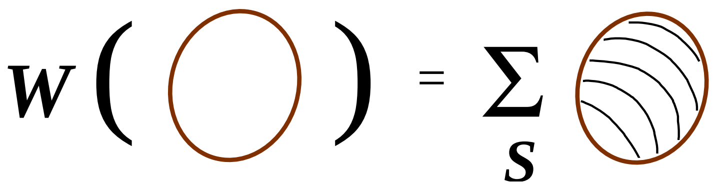
Alday, Maldacena (2007)

AdS/CFT for Wilson Loops

Maldacena (1998)
Rey, Yee (1998)

The correspondence

$$W_{\mathsf{SYM}}(C) = \sum_{S:\partial S = C} \mathrm{e}^{\mathrm{i}A_{IIB \text{ on } AdS_5 \otimes S^5}}$$



$$C = \left(x^\mu(\sigma), \int^\sigma \mathrm{d}\sigma \, |\dot{x}| n^i\right)$$

— loop in the boundary of $AdS_5 \otimes S^5$
e.g. $n^i = (1,0,0,0,0,0) \Rightarrow$ 4D contour $x^\mu(\sigma)$

Circular loop:

AdS — Berenstein, Corrado, Fischler, Maldacena (1998)
Drukker, Gross, Ooguri (1999)

CFT — Erickson, Semenoff, Zarembo (2000)
Drukker, Gross (2001)

(perfect agreement)

AdS/CFT for Wilson Loops (cont.)

Rectangular loop (or antiparallel lines):

AdS

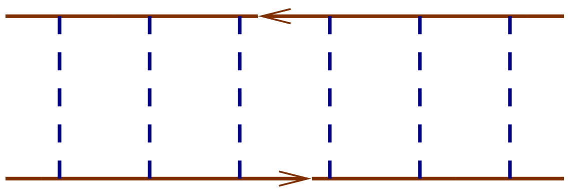
Maldacena (1998)
Rey, Yee (1998)

(minimal surface in $AdS_5 \otimes S^5$)

CFT

Erickson, Semenoff, Szabo, Zarembo (1999)
Erickson, Semenoff, Zarembo (2000)

(summation of ladder diagrams)



AdS:

$$V(R) = -\frac{4\pi^2\sqrt{2\lambda}}{\Gamma^4(1/4)R}$$

SYM:

$$V(R) = -\frac{\sqrt{\lambda}}{\pi R}$$

The discrepancy is attributed to interaction diagrams

But the SYM coefficient is what is needed for the cusp anomalous dimension

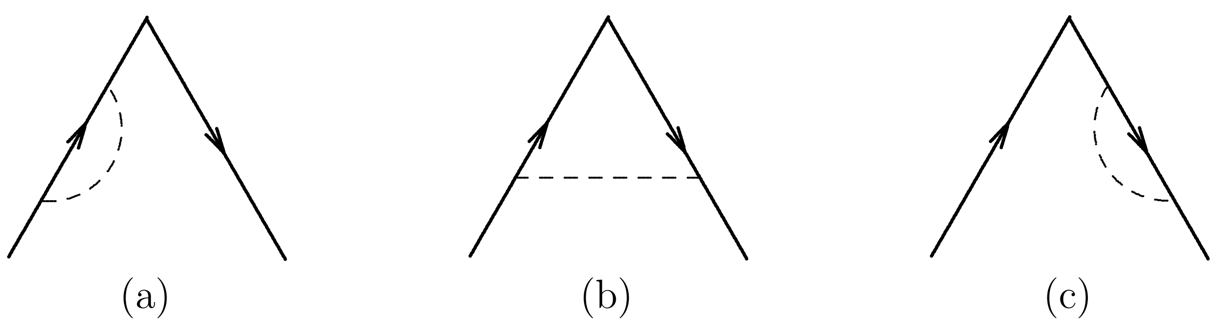
Perturbation Theory

Order λ (one loop)

$$W(\Gamma) = 1 - \frac{\lambda}{2} \int_{-\infty}^{+\infty} d\sigma_1 \int_{-\infty}^{+\infty} d\sigma_2 \left[\dot{x}^\mu(\sigma_1)\dot{x}_\mu(\sigma_2) - |\dot{x}(\sigma_1)||\dot{x}(\sigma_2)|\right] \times D\left(x(\sigma_1) - x(\sigma_2)\right)$$

with (scalar) propagator in d-dimensions

$$D(x) = -\frac{\Gamma\left(d/2 - 1\right)}{4\pi^{d/2}}[-x^2]^{1-d/2}$$

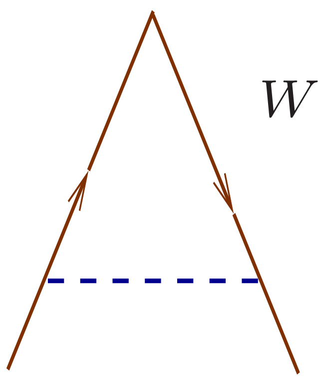


diagrams of order λ

Diagrams (a) and (c) vanish (gluons are cancelled by scalars)

One-loop perturbation theory

Only one diagram is nonvanishing



$$W(\Gamma) = 1 - \frac{\lambda}{4\pi^2}(\cosh\theta - 1)\int \mathrm{d}s \int \mathrm{d}t\, \frac{1}{s^2 + 2st\cosh\theta + t^2}$$
$$= 1 - \frac{\lambda}{4\pi^2}\frac{\cosh\theta - 1}{\sinh\theta}\theta \ln\frac{L}{a}$$
$$\xrightarrow{\text{large } \theta} 1 - \frac{\lambda}{4\pi^2}\theta \ln\frac{L}{a}$$

which yields $\Longrightarrow$ $f(\lambda) = \dfrac{\lambda}{2\pi^2}$

No mass-renormalization term $-\lambda/4\pi a$ as is in QCD

One-loop perturbation theory (cont.)

Exact formula

$$W(S,T;a,b) = 1 - \frac{\lambda}{4\pi^2}(\cosh\theta - 1)\int_a^S \mathrm{d}s \int_b^T \mathrm{d}t\, \frac{1}{s^2 + 2st\cosh\theta + t^2}$$

$$= \ 1 - \frac{\lambda}{8\pi^2}\frac{\cosh\theta - 1}{\sinh\theta}\Big(\mathrm{Li}_2(-\frac{T}{S}\,\mathrm{e}^{\theta}) - \mathrm{Li}_2(-\frac{T}{S}\,\mathrm{e}^{-\theta}) - \mathrm{Li}_2(-\frac{T}{a}e^{\theta})$$

$$+\mathrm{Li}_2(-\frac{T}{a}e^{-\theta}) - \mathrm{Li}_2(-\frac{b}{S}\,\mathrm{e}^{\theta}) + \mathrm{Li}_2(-\frac{b}{S}\,\mathrm{e}^{-\theta}) + \mathrm{Li}_2(-\frac{b}{a}\,\mathrm{e}^{\theta}) - \mathrm{Li}_2(-\frac{b}{a}\,\mathrm{e}^{-\theta})\Big)$$

where Li_2 is Euler's dilogarithm

$$\mathrm{Li}_2(z) = \sum_{n=1}^{\infty}\frac{z^n}{n^2} = -\int_0^z \frac{dx}{x}\ln(1-x)$$

which obeys the relation

$$\mathrm{Li}_2\left(-e^{\Omega}\right) + \mathrm{Li}_2\left(-e^{-\Omega}\right) = -\frac{1}{2}\ln^2\Omega - \frac{\pi^2}{6}$$

It is used to extract the double logarithms

Double-Logarithmic Approximation

Again at one loop

$$W(S,T;a,b) = 1 - \frac{\lambda}{4\pi^2}(\cosh\theta - 1)\int_a^S \mathrm{d}s \int_b^T \mathrm{d}t \, \frac{1}{s^2 + 2st\cosh\theta + t^2}$$

The double-logarithmic region of integration, is

$$t\,\mathrm{e}^{-\theta} \lesssim s \lesssim t\,\mathrm{e}^{\theta} \qquad \text{or} \qquad s\,\mathrm{e}^{-\theta} \lesssim t \lesssim s\,\mathrm{e}^{\theta}$$

so write it in DLA

$$W(S,T;a,b) = 1 - \beta \int_b^T \frac{\mathrm{d}t}{t} \int_{\max\{a,t\,\mathrm{e}^{-\theta}\}}^{\min\{S,t\,\mathrm{e}^{\theta}\}} \frac{\mathrm{d}s}{s}$$

$$\Longrightarrow = 1 - 2\beta\theta \ln\frac{T}{b} \qquad \text{very large } S \ , \quad \text{very small } a$$

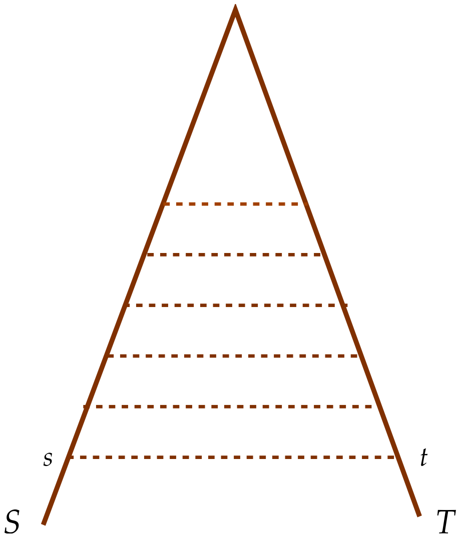
reproducing the above result

$$\Longrightarrow = 1 - \beta \ln\frac{T}{b} \ln\frac{S}{a} \qquad \text{very large } \theta$$

reproducing the 2 light-cone result

Sum of Ladder Diagrams

Bethe–Salpeter equation



$$\mathcal{G}(S,T) = 1 - \frac{\lambda(\cosh\theta - 1)}{4\pi^2}\int_a^S \mathrm{d}s \int_b^T \mathrm{d}t \frac{\mathcal{G}(s,t)}{s^2 + 2st\cosh\theta + t^2}$$

1 light-cone limit: $\mathrm{Re}\,\theta \to \infty$ with fixed $T_{\mathrm{l.c.}} = 2T\,\mathrm{e}^{\theta}$

$$\mathcal{G}(S,T;a,b) = 1 - \beta\int_a^S ds\int_b^T dt\,\frac{\mathcal{G}(s,t;a,b)}{\alpha s^2 + st} \qquad (3)$$

where

$$\beta = \frac{\lambda}{8\pi^2} \qquad \alpha = \frac{u^2}{2u\cdot v} = \pm 1$$

(remember that $v^2 = 0$ for the light-cone direction)

$\alpha = 0$ for 2 light cones when additionally $u^2 = 0$

The Ladder Equation

Differentiating Eq. (3) we obtain

$$S\frac{\partial}{\partial S}T\frac{\partial}{\partial T}\mathcal{G}\left(S,T;a,b\right) = -\frac{\beta}{1+\alpha S/T}\,\mathcal{G}\left(S,T;a,b\right) \tag{4}$$

and analogously

$$a\frac{\partial}{\partial a}b\frac{\partial}{\partial b}\mathcal{G}\left(S,T;a,b\right) = -\frac{\beta}{1+\alpha a/b}\,\mathcal{G}\left(S,T;a,b\right)$$

with the boundary conditions

$$\mathcal{G}(a,T;a,b) = \mathcal{G}(S,b;a,b) = 1\,. \tag{5}$$

New variables

$$X = \ln\frac{S}{a} - \ln\frac{T}{b} \qquad Y = \ln\frac{S}{a} + \ln\frac{T}{b}$$

Variables separated

$$\left(\frac{\partial^2}{\partial X^2} - \frac{\partial^2}{\partial Y^2}\right)\mathcal{G} = \frac{\beta}{1+\alpha\frac{a}{b}\,\mathrm{e}^X}\,\mathcal{G} \quad \overset{\alpha S \ll T}{=} \quad \beta\,\mathcal{G}$$

— similar to equation of Erickson, Semenoff, Szabo, Zarembo (1999)

but with different boundary conditions

Exact Solution for Ladders ($\alpha = 0$)

The solution for $\alpha = 0$ is a Bessel function

$$\mathcal{G}_{\alpha=0}(S,T;a,b) = J_0\left(2\sqrt{\beta \ln\frac{S}{a}\ln\frac{T}{b}}\right)$$

which obviously obeys the boundary condition.

This can be easily shown by iterative solution of

$$\mathcal{G}_{\alpha=0}(S,T;a,b) = 1 - \beta\int_a^S \frac{\mathrm{d}s}{s}\int_b^T \frac{\mathrm{d}t}{t}\,\mathcal{G}_{\alpha=0}(s,t;a,b)$$

where the integrals over s and t decouple and both are logarithmic

$$\mathcal{G}_{\alpha=0}(S,T;a,b) = \sum_{n=0}^{\infty}(-\beta)^n\frac{\left(\ln\frac{S}{a}\right)^n}{n!}\frac{\left(\ln\frac{T}{b}\right)^n}{n!} = J_0\left(2\sqrt{\beta \ln\frac{S}{a}\ln\frac{T}{b}}\right)$$

Asymptotically

$$J_k(z) \sim \cos z \qquad\qquad \text{large} \quad z$$

which is not of the type expected for renormalization

Exact Solution for Ladders ($\alpha \neq 0$)

Olesen, Semenoff, Y.M. (2006)

The ansatz

$$\mathcal{G}\left(S,T;a,b\right) = \oint\limits_{C} \frac{\mathrm{d}\omega}{2\pi i\omega} \left(\frac{S}{a}\right)^{\sqrt{\beta}\omega} \left(\frac{T}{b}\right)^{-\sqrt{\beta}\omega^{-1}} F\left(-\omega, \alpha\frac{a}{b}\right) F\left(\omega, \alpha\frac{S}{T}\right),$$

where C is a contour in the complex ω-plane.
Motivated by the integral representation of the Bessel function J_0 at $\alpha = 0$ ($\Longrightarrow F = 1$).

The substitution into Eq. (4) reduces it to the hypergeometric equation ($\xi = \alpha S/T$)

$$\xi(1+\xi)F''_{\xi\xi} + [1+\sqrt{\beta}(\omega+\omega^{-1})](1+\xi)F'_{\xi} + \beta F = 0$$

whose solution is given by hypergeometric functions.

The main difficulty (solved) is how to draw the contour C to satisfy the boundary conditions (5).

Great Simplification at $S = T$

and $a = b$, $\alpha = -1$:

$$\mathcal{G}_{\alpha=-1}(T,T;a,a) = \frac{1}{\sqrt{\beta\tau(\tau - 2\pi i)}} J_1\left(2\sqrt{\beta\tau(\tau - 2\pi i)}\right)$$

with

$$\ln\frac{T}{a} = \tau \qquad \ln\left(-\frac{T}{a}\right) = \tau - i\pi$$

The Bessel function is similar to Erickson, Semenoff, Zarembo (2000) for a circular Wilson loop that has a random matrix model origin.
This is J_1 rather than I_1 because of Minkowski space.

Nothing good for the contribution of ladders to the cusp anomalous dimension. It is *not* of the form prescribed by renormalizability

$$W\left(\Gamma_{\text{l.c.}}\right) \propto \mathrm{e}^{-\frac{1}{4}f(\beta)\ln^2\frac{T}{\varepsilon}}$$

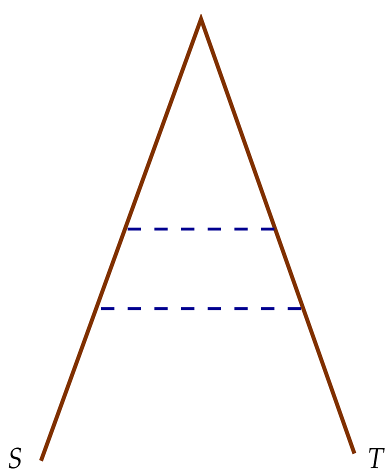
Miniconclusion: diagrams with interaction have to contribute

Two-Loop Ladder Diagram

Korchemsky, Radyushkin (1987)

Contribution to cusp anomalous dimension

$$\gamma_{\text{cusp}}^{(\text{lad})} = \frac{\lambda^2}{128\pi^4}\frac{(\cosh\theta - 1)^2}{\sinh^2\theta}\int_0^\infty \frac{d\sigma}{\sigma}\ln\left(\frac{1+\sigma\,\mathrm{e}^{\theta}}{1+\sigma\,\mathrm{e}^{-\theta}}\right)\ln\left(\frac{\sigma+\mathrm{e}^{\theta}}{\sigma+\mathrm{e}^{-\theta}}\right)$$
$$\to \frac{\lambda^2}{96\pi^4}\left(\theta^3 + \frac{\pi^2}{2}\theta + \mathcal{O}(1)\right)$$



θ^3 should be cancelled by interaction !!!

$\Longrightarrow$ not only ladder diagrams are essential

Similar results for the light-cone Wilson loop:

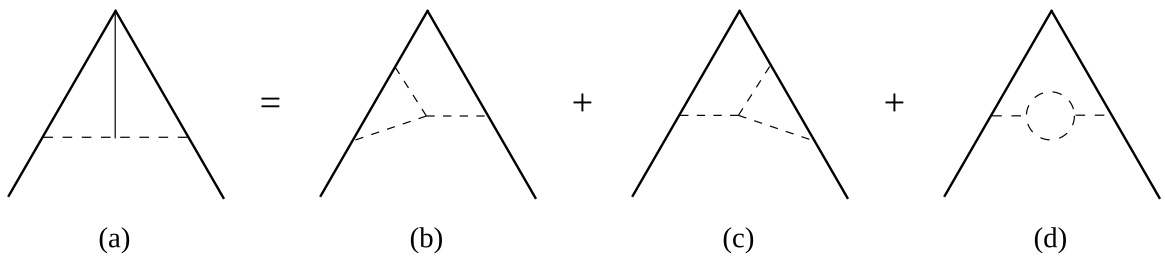
$$\mathcal{G}_{\text{l.c.}}^{\text{ladd.}} = 1 - \frac{\beta}{2}\ln^2\frac{T}{\varepsilon} + \frac{\beta^2}{12}\ln^4\frac{T}{\varepsilon} - \frac{\beta^2\pi^2}{12}\ln^2\frac{T}{\varepsilon}$$

$\ln^4\frac{T}{\varepsilon}$ is to be cancelled by diagrams with interaction

Surface Term

Olesen, Semenoff, Y.M. (2006)

Cancellation between three-gluon vertex and propagators is *not* complete



Surface term comes from integration by parts

$$\gamma_{\text{cusp}}^{\text{anom}} = -\frac{\lambda^2}{16\pi^4}\frac{\cosh\theta - 1}{\cosh\theta}\left(\int_0^{\theta} + \int_0^{\pi/2}\right)\frac{d\psi\,\psi}{1-\cosh^2\psi/\cosh^2\theta}\ln\frac{\cosh^2\theta}{\cosh^2\psi}$$

$$\to -\frac{\lambda^2}{96\pi^4}\left(\theta^3 + \pi^2\theta + \mathcal{O}(1)\right)$$

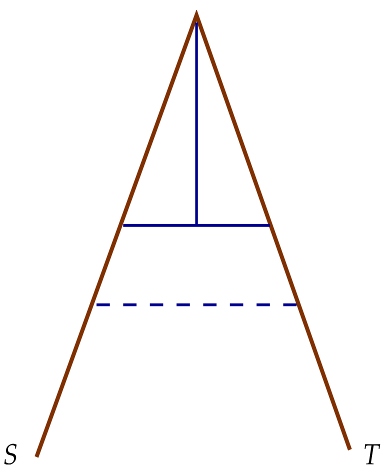
Two-loop cusp anomalous dimension

$$\gamma_{\text{cusp}} = \frac{\theta}{2}\left(\frac{\lambda}{2\pi^2} - \frac{\lambda^2}{96\pi^2}\right) + \mathcal{O}(\theta^0)$$

reproduces the known results

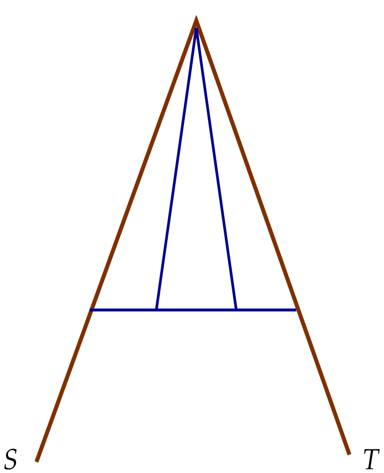
Higher-Order Surface Terms

A question arises whether the surface term of order β^2 is the only one (like an anomaly in QFT) or next order surface terms also appear. It can be answered in DLA.



order β^2 surface term dressed by a ladder + the ladder with 3 rungs do not provide exponentiation required for

$$W_{\mathrm{l.c.}}(\Gamma) = \mathrm{e}^{-\frac{\beta}{2}\mathcal{T}^2} \qquad \alpha S \gg T$$



most probably the surface term like this is required for the exponentiation in DLA

What is the equation which sums this kind of the surface terms and provides the exponentiation in DLA?

Loop equation in QCD

Schwinger–Dyson equation for Wilson loops

$$\nabla^{ab}_{\mu} F^{b}_{\mu\nu}(x) \stackrel{\text{w.s.}}{=} \hbar \frac{\delta}{\delta A^{a}_{\nu}(x)}$$

can be translated as $N \to \infty$ to the loop equation

Migdal, Yu.M. (1979)

$$\partial^{x}_{\mu} \frac{\delta}{\delta \sigma_{\mu\nu}(x)} W(C) = \lambda \oint_{C} \mathrm{d} y_{\nu}\, \delta^{(d)}(x-y)\, W(C_{yx})\, W(C_{xy})$$

which includes path and area derivatives

Vocabulary for translation into loop space

Ordinary space		Loop space	
$\Phi[A]$	Phase factor	$\Phi(C)$	Loop functional
$F_{\mu\nu}(x)$	Field strength	$\dfrac{\delta}{\delta\sigma_{\mu\nu}(x)}$	Area derivative
∇_μ^x	Covariant derivative	∂_μ^x	Path derivative
$\nabla \wedge F = 0$	Bianchi identity		Stokes functionals
$-\nabla_\mu F_{\mu\nu} = \delta/\delta A_\nu$	Schwinger–Dyson equations		Loop equations

Loop-space Laplace equation

One more contour integration over y

$$\Delta W(C) = \lambda \oint_C \mathrm{d}x_\mu \oint_C \mathrm{d}y_\mu \, \delta^{(d)}(x-y)\, W(C_{yx})\, W(C_{xy})$$

Loop-space Laplacian

$$\Delta \equiv \oint_C \mathrm{d}x_\nu \, \partial_\mu^x \frac{\delta}{\delta\sigma_{\mu\nu}(x)} = \int_{\sigma_i}^{\sigma_f} \mathrm{d}\sigma \int_{\sigma-0}^{\sigma+0} \mathrm{d}\sigma' \frac{\delta}{\delta x_\mu(\sigma')} \frac{\delta}{\delta x_\mu(\sigma)}$$

is defined for much wider class of functionals than Stokes
This is important for SUSY extension

It is associated with the second-order Schwinger–Dyson equation

$$\int \mathrm{d}^d x \, \nabla_\mu F^a_{\mu\nu}(x) \frac{\delta}{\delta A^a_\nu(x)} \overset{\text{w.s.}}{=} \hbar \int \mathrm{d}^d x \, \mathrm{d}^d y \, \delta^{(d)}(x-y) \frac{\delta}{\delta A^a_\nu(y)} \frac{\delta}{\delta A^a_\nu(x)}$$

A non-perturbative gauge-invariant regularization Halpern, Yu.M. (1989)

$$\delta^{ab}\delta^{(d)}(x-y) \overset{\text{reg.}}{\Longrightarrow} \left\langle y \left| \left(\mathrm{e}^{a^2 \nabla^2/2} \right)^{ab} \right| x \right\rangle$$

Smearing of loop-space Laplacian

Smearing of loop-space Laplacian is needed to invert it, i.e. to produce the Green function

Smearing procedure (gets second-order operator from the first order)

$$\Delta^{(G)} = \int_0^1 \mathrm{d}\sigma \int_0^1 \mathrm{d}\sigma' \, G(\sigma,\sigma') \frac{\delta}{\delta x_\mu(\sigma')} \frac{\delta}{\delta x_\mu(\sigma)}$$

$$= \int_0^1 \mathrm{d}\sigma \, \fint_0^1 \mathrm{d}\sigma' \, G(\sigma,\sigma') \frac{\delta}{\delta x_\mu(\sigma')} \frac{\delta}{\delta x_\mu(\sigma)} + \Delta$$

with parametric-invariant

$$G(\sigma_1,\sigma_2) = \mathrm{e}^{-\left|\int_{\sigma_1}^{\sigma_2} \mathrm{d}\sigma \sqrt{\dot{x}^2(\sigma)}\right|/\varepsilon} \qquad (\varepsilon \ll L)$$

ε has the meaning of stiffness

Green function of functional Laplacian

Yu.M. (1988)

Loop-space Laplacian can be inverted to produce the Green function (useful for iterative solution)
The functional Laplace equation (with given $J[x]$)

$$\Delta^{(G)} W[x] = J[x]$$

with the proper choice of boundary conditions can be solved to give

$$W[x] = 1 - \frac{1}{2}\int_0^\infty \mathrm{d}A \left\{ \left\langle J[x+\sqrt{A}\xi] \right\rangle_\xi^{(G)} - \left\langle J[\sqrt{A}\xi] \right\rangle_\xi^{(G)} \right\}$$

The average over the loops $\xi(\sigma)$ is given by the path integral

$$\langle F[\xi] \rangle_\xi^{(G)} = \frac{\int_{\xi(0)=\xi(1)} D\xi \, \mathrm{e}^{-S} F[\xi]}{\int_{\xi(0)=\xi(1)} D\xi \, \mathrm{e}^{-S}}$$

with the local action

$$S = \frac{1}{4}\int_0^1 \mathrm{d}\sigma \{ \frac{\varepsilon}{\sqrt{\dot{x}^2(\sigma)}} \dot{\xi}^2(\sigma) + \frac{\sqrt{\dot{x}^2(\sigma)}}{\varepsilon} \xi^2(\sigma) \}$$

It extends the results of Gateux (early 1900's) for functional Laplacian

Iterative solution

In large-N Yang–Mills the regularized $J[x]$ is as above bilinear in W:

$$
\begin{aligned}
J^{(G)}[x] \;=\; & \lambda \int_0^1 \int_0^1 d\sigma_1 d\sigma_2 (1 - G(\sigma_1 - \sigma_2)) \dot{x}^\mu(\sigma_1) \dot{x}^\mu(\sigma_2) \\
& \times \int_{r(0)=x(\sigma_1)}^{r(a^2)=x(\sigma_2)} \mathcal{D}r \, \mathrm{e}^{-\frac{1}{2}\int_0^{a^2} d\tau \dot{r}^2(\tau)} \\
& \quad \times W(C_{x(\sigma_1)x(\sigma_2)} r_{x(\sigma_2)x(\sigma_1)}) W(C_{x(\sigma_2)x(\sigma_1)} r_{x(\sigma_1)x(\sigma_2)})
\end{aligned}
$$

Iterative solution in λ recovers perturbation theory

All that can be deduced from the general formula

$$
\left\langle \mathrm{e}^{\mathrm{i}\sqrt{A}\int d\sigma \dot{p}(\sigma)\xi(\sigma)} \right\rangle_\xi^{(G)} = \mathrm{e}^{-A\int d\sigma \int d\sigma' \dot{p}(\sigma) G(\sigma-\sigma') \dot{p}(\sigma')/2}
$$

where $p^\mu(\sigma)$ ($p^\mu(0) = p^\mu(1)$) represents a momentum-space loop

The triple gluon vertex appears from the uncertainty $\varepsilon \times 1/\varepsilon$

Cusped Loop Equation

Cusped loop equation for $\mathcal{N}=4$ SYM Drukker, Gross, Ooguri (1999)
for supersymmetric loops $\boldsymbol{C}=\{x_\mu(\sigma),Y_i(\sigma);\zeta(\sigma)\}$ ($\zeta(\sigma)$ denotes the Grassmann odd component)

$$\boldsymbol{\Delta}\ln W(\boldsymbol{C})\big|_{\boldsymbol{C}=\Gamma} = \lambda\int \mathrm{d}\sigma_1\int \mathrm{d}\sigma_2\,\left(\dot{x}_\mu(\sigma_1)\dot{x}_\mu(\sigma_2)-|\dot{x}_\mu(\sigma_1)||\dot{x}_\mu(\sigma_2)|\right) \times\delta^{(4)}(x_1-x_2)\frac{W(\Gamma_{x_1x_2})W(\Gamma_{x_2x_1})}{W(\Gamma)}$$

where

$$\boldsymbol{\Delta}=\lim_{\eta\to 0}\int \mathrm{d}s\int_{s-\eta}^{s+\eta}\mathrm{d}s'\left(\frac{\delta^2}{\delta x^\mu(s')\delta x_\mu(s)}+\frac{\delta^2}{\delta Y^i(s')\delta Y_i(s)}+\frac{\delta^2}{\delta\zeta(s')\delta\bar{\zeta}(s)}\right)$$

is the supersymmetric extension of the loop-space Laplacian and $\dot{Y}^2=\dot{x}^2$, $\zeta=0$ after acting by $\boldsymbol{\Delta}$.

The RHS $\sim(La)^{-1}$ for smooth loops but $\sim a^{-2}$ for cusped loops (was L/a^3 in QCD)

Cusped Loop Equation (cont.1)

It can be shown for cusped Wilson loops

$$\Delta \ln W(C)|_{C=\Gamma} = -\frac{\mathrm{d}}{\mathrm{d}a^2} \ln W(\Gamma)$$

$$\Longrightarrow \frac{2}{a^2}\gamma_{\text{cusp}}(\theta,\lambda) = \lambda \int \mathrm{d}\sigma_1 \int \mathrm{d}\sigma_2 \, \left(\dot{x}_\mu(\sigma_1)\dot{x}_\mu(\sigma_2) - |\dot{x}_\mu(\sigma_1)||\dot{x}_\mu(\sigma_2)|\right) \times \delta_a^{(4)}(x_1 - x_2) \frac{W(\Gamma_{x_1 x_2}) W(\Gamma_{x_2 x_1})}{W(\Gamma)}$$

- is observed to order λ by Drukker, Gross, Ooguri (1999)
- is verified to order λ^2 for arbitrary θ Olesen, Semenoff, Yu.M. (2006):

The ladder diagram of order λ^2 comes iteratively from the ladder diagram of order λ

Cusped Loop Equation (cont.2)

The anomaly diagram is reproduced when gluon is attached to the regularizing path $r_{x_1 x_2}$ by the formula Migdal, Yu.M. (1981)

$$\int\limits_{\substack{z(0)=x\\ z(\tau)=y}} \mathcal{D}z(t)\, \mathrm{e}^{-\int_0^\tau \mathrm{d}t\, \dot{z}^2(t)/2} \int_x^y \mathrm{d}z^\mu \delta^{(d)}(z-u)$$

$$= \frac{1}{2}\int_0^\infty \mathrm{d}\tau_1 \int_0^\infty \mathrm{d}\tau_2\, \delta\left(\tau-\tau_1-\tau_2\right)$$

$$\times \frac{1}{(2\pi\tau_1)^{d/2}}\, \mathrm{e}^{-(x-u)^2/2\tau_1} \frac{\overleftrightarrow{\partial}}{\partial u_\mu} \frac{1}{(2\pi\tau_2)^{d/2}}\, \mathrm{e}^{-(y-u)^2/2\tau_2}$$

The loop equation may be useful for next orders in λ

Some comments about large-N QCD

$|\dot{x}|$ can be neglected near the light-cone $\Longrightarrow$ same cusped loop equation as in QCD

This may indicate that γ_{cusp} coincide while the difference is absorbed by charge renormalization

This may be because SUSY is broken by construction
(the presence of a cusp)

Conclusions

- Cusped Wilson loops are convenient for study anomalous dimensions

- Minimal surface of open string reproduces GKP closed string calculation

- Ladder diagrams themselves do not give a reasonable result (the need of diagrams with interaction)

- Cancellation of interaction diagram to order λ^2 is not complete for $\mathcal{N}=4$ SYM (a surface term remains)

- Results in DLA indicate that higher-order interaction diagrams are also essential

- Loop equation has specific features for cusped loops

- There are indications that cusp anomalous dimension could be the same as for QCD

- Challenging problem to obtain $\sqrt{\lambda}$ for large λ by perturbation theory