Supergravitons from one loop perturbative $\mathcal{N} = 4$ SYM

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Plan

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2. BPS states at zero coupling/supergravitons
3. The dilatation operator at one loop
4. BPS states at one loop
5. Finite N, black holes
6. Summary

$\mathcal{N} = 4$ SYM

- the $\mathcal{N} = 4$ SYM on-shell fields
  - gauge field A^a_μ , scalar field $\phi^a_{[ij]}$, fermions $\psi^a_{\alpha,i}$, $\bar{\psi}^a_{\dot{\alpha},i}$
  $a = 1, \ldots, N^2 - 1 \leftrightarrow su(N)$,
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- global su(4) **R-symmetry**
- any state $\longrightarrow (\Delta, \underbrace{j_1, j_2}_{su(2)\times su(2)}, \underbrace{R_1, R_2, R_3}_{su(4)})$

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$$\{S_{\alpha i}, Q^{\beta j}\} = \delta_i^j (J_1)_\alpha^\beta + \delta_\alpha^\beta R_i^j + \frac{1}{2}\delta_i^j \delta_\alpha^\beta D$$

  $\longrightarrow$ BPS are protected

Oscillator picture

- operators are traces over $D_{\alpha\dot{\beta}}, F_{\alpha\beta}, \bar{F}_{\dot{\alpha}\dot{\beta}}, \phi_{ij}, \psi_{\alpha,i}, \bar{\psi}_{\alpha,i}$

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- dictionary (Beisert 03')

$$D^k F \longrightarrow (a^\dagger)^{k+2}(b^\dagger)^k|0\rangle,$$
$$D^k\psi \longrightarrow (a^\dagger)^{k+1}(b^\dagger)^k c^\dagger|0\rangle,$$
$$D^k\phi \longrightarrow (a^\dagger)^{k}(b^\dagger)^k c^\dagger c^\dagger|0\rangle,$$
$$D^k\bar\psi \longrightarrow (a^\dagger)^{k}(b^\dagger)^{k+1} c^\dagger c^\dagger c^\dagger|0\rangle$$
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$$D^k \bar{F} \longrightarrow (a^\dagger)^{k}(b^\dagger)^{k+2} c^\dagger c^\dagger c^\dagger c^\dagger|0\rangle$$

- an example: $D_{\alpha\dot{\beta}} D_{\gamma\dot{\delta}} \phi_{ij} \longrightarrow a^\dagger_\alpha a^\dagger_\gamma b^\dagger_{\dot{\beta}} b^\dagger_{\dot{\delta}} c^\dagger_i c^\dagger_j|0\rangle$

- in the planar limit $\longrightarrow$ single trace operators $Tr(\chi_1 \dots \chi_L)$

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- site index - $a_\alpha^\dagger,\ b_{\dot{\alpha}}^\dagger,\ c_i^\dagger \rightarrow a_{\alpha,t}^\dagger, b_{\dot{\alpha},t}^\dagger,\ c_{i,t}^\dagger$
- a generic state = combinations of $|s_1\rangle \otimes |s_2\rangle \dots |s_L\rangle$
  $|s_t\rangle = a_{\alpha,t}^\dagger, b_{\alpha,t}^\dagger, c_{i,t}^\dagger$ acting on $|0\rangle_t$
  $+$ constraint $\#a_t^\dagger - \#b_t^\dagger + \#c_t^\dagger = 2$ (central charge)

BPS states at zero coupling/supergravitons

- In $\{S_{\alpha i}, Q^{\beta j}\} = \delta_i^j (J_1)_\alpha^\beta + \delta_\alpha^\beta R_i^j + \frac{1}{2}\delta_i^j \delta_\alpha^\beta D$
  take $i, j, \alpha, \beta = 1$ $\;S = S_{11},\; Q = Q^{11},\; J_1 = (J_1)_1^1$

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- explicitly the index is

$$I = \sum_{\tilde{D}=0} (-1)^F t^{2D+2J_1} y^{2J_2} v^{R_2} w^{R_3}, \quad Z = \sum_{\tilde{D}=0} x^{2D} z^{2J_1} y^{2J_2} v^{R_2} w^{R_3}$$

- \- Z and I can be calculated exactly at $g = 0$
  \- for $g >> 1$ we can use the strong/weak coupling duality of AdS/CFT
  \- symm. of $\mathcal{N} = 4$ SYM $=$ symm. of $AdS_5 \times S_5$ superstring
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- result: I's match but Z's don't $\Rightarrow$ overcounting at g=0?

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  letters:

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$$Z_{s.t} = -\sum_{n=1}^{\infty} \frac{\phi(n)}{n} \ln(1 - z_B(t^n) - (-1)^{n+1} z_F(t^n))$$

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- single trace $\longrightarrow$ multiple trace

- $\lambda = 0$, finite N:

$$Z = \int DU exp \left\{ \sum_{n=1}^{\infty} [z_B(t^n) + (-1)^{n+1} z_F(t^n)] \frac{TrU^n TrU^{-n}}{n} \right\}$$

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  (deconfinement phase $\longrightarrow lnZ_{m.t} \sim N^2$)
- only qualitative agreement with BPS black holes solutions (given by Gutowski, Real 2x 04', Chong, Cvetic, Lu, Pope 05')

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$$H|s\rangle = \underbrace{|s_1\rangle|s_2\rangle}_{H_{12}}|s_3\rangle \dots |s_L\rangle + |s_1\rangle \underbrace{|s_2\rangle|s_3\rangle}_{H_{12}}|s_4\rangle \dots |s_L\rangle + \dots$$

The harmonic action H_{12}

- consider two (initial) neighboring sites (i and $i+1$)

$$|v\rangle = |n_{a_1}, \ldots, n_{c_4}\rangle \otimes |\, m_{a_1}, \ldots, m_{c_4}\rangle,$$

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- -to calculate $\langle v' \mid H \mid v\rangle$ consider all oscillator hopping so that $|v\rangle \to |v'\rangle$ (c.c.c., -1 factors included)
  -associate a probability,
  -sum over all possibilities

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- we are looking for

$$Z_L(a_2, b_1, b_2, c_2, c_3, c_4) =$$

$$\sum D_{n_{a_2} n_{b_1} n_{b_2} n_{c_2} n_{c_3} n_{c_4} L} a_2^{n_{a_2}} b_1^{n_{b_1}} b_2^{n_{b_2}} c_2^{n_{c_2}} c_3^{n_{c_3}} c_4^{n_{c_4}},$$

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- we can determine only finite number of $D_{n_{a_2},\ldots,n_{c_4},L}$. Can we guess Z_L?

- ...after some guess work

$$Z_L(a_2, b_1, b_2, c_2, c_3, c_4) = \frac{P}{(1 - a_2 b_1)(1 - a_2 b_2)},$$

$$P = \sigma_{L,L,0} + a_2 \sigma_{L,L-1,0} + a_2^2 \sigma_{L-1,L-1,0}$$

$$+(b_1 + b_2)\left(\sigma_{L,L,1} + a_2 \sigma_{L,L-1,1} + a_2^2 \sigma_{L-1,L-1,1}\right)$$

$$+b_1 b_2 \left(\sigma_{L,L,2} + a_2 \sigma_{L,L-1,2} + a_2^2 \sigma_{L-1,L-1,2}\right),$$

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- where

$$\sigma_{n_1,n_2,n_3} = \begin{vmatrix} c_2^{n_1+2} & c_3^{n_1+2} & c_4^{n_1+2} \\ c_2^{n_2+1} & c_3^{n_2+1} & c_4^{n_2+1} \\ c_2^{n_3} & c_3^{n_3} & c_4^{n_3} \end{vmatrix} \Big/ \begin{vmatrix} c_2^2 & c_3^2 & c_4^2 \\ c_2 & c_3 & c_4 \\ 1 & 1 & 1 \end{vmatrix}$$

- checked for
  $L = 2, 3, 4, 5$
  $\to 0 \leq n_{a_2}, n_{b_1}, n_{b_2} \leq 10$ and $0 \leq n_{c_2}, n_{c_3}, n_{c_4} \leq L$

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- $\rightarrow$ single trace $\rightarrow$ multi trace

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- another check - $Z_1 = Z_{lett.}$!

- old/new variables

$$Z = \sum_{\tilde{D}=0} x^{2D} z^{2J_1} y^{2J_2} v^{R_2} w^{R_3} =$$

$$\sum_{\tilde{D}=0, c.c.c} a_2^{n_{a_2}} b_1^{n_{b_1}} b_2^{n_{b_2}} c_2^{n_{c_2}} c_3^{n_{c_3}} c_4^{n_{c_4}},$$

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- hence the dictionary

$$a2 = x^2 z, \quad b1 = 1/y \quad b2 = y,$$

$$c2 = x/v \quad c3 = xv/w \quad c4 = xw$$

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- substituting to our result $\longrightarrow$ exact supergravity prediction!

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- 1)$Z_{m.t.}$ has no singularities for $a, b, c < 1$

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- 2)no (meaning of) letters when $\lambda > 0 \rightarrow$ effective letters

$$Z_{s.t.} = -\sum_{n=1}^{\infty} \frac{\phi(n)}{n} \ln[1 - z_B^{eff.}(t^n) - (-1)^{n+1} z_F^{eff.}(t^n)]$$

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- coeff. of $z_B^{eff.}(x), (a_n)$ and $z_F^{eff.}(x), (b_n)$: negative integers $\rightarrow$ commutative letters

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$$Z_{s.t.} = -\sum_{n=1}^{\infty} \frac{\phi(n)}{n} \ln[1 - z_B^{eff.}(t^n) - (-1)^{n+1} z_F^{eff.}(t^n)]$$

- coeff. of $z_B^{eff.}(x), (a_n)$ and $z_F^{eff.}(x), (b_n)$: negative integers $\rightarrow$ commutative letters
- choice: $a_2 = x^3, c_2 = c_3 = c_4 = x, b_1 = b_2 = 1$
  reason: # BH BPS parameters = # BPS parameters -1
  parameter constraint $\Rightarrow C = n_{a_2} + n_c/3$

Finite N, black holes

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- plethystic formalisms (Benvenuti, Feng, Hanany, He 06')

$$\prod_{n=1}^{\infty} \frac{(1+gx^n)^{a_n}}{(1-gx^n)^{b_n}} = \sum_{N=1}^{\infty} Z_N(x) g^N \quad \to \ln Z_N \sim N$$

Summary

- BPS protected $\Rightarrow$ Z is λ independent
  Z at $\lambda = 0$ different from Z at $\lambda >> 1$ (from AdS/CFT) - overcounting??
  Turn on $0 < \lambda << 1 \Rightarrow$ full agreement!
  BPS states vs.BPS black holes $\longrightarrow$ non planar analysis required