Q-ball Solutions in Signum-Gordon Models

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Problem

Consider a scalar field theory given by Lagrange density function

$$\mathcal{L} = \partial_\mu \psi \partial^\mu \bar{\psi} - U\left(\psi \bar{\psi}\right),$$

where U is a potential term. Due to the phase invariance the charge Q

$$Q = i \int dV \left(\partial_0 \psi \bar{\psi} - \partial_0 \bar{\psi} \psi\right)$$

is constant during the field evolution.
Q-ball is a field configuration that minimizes energy functional for the given charge $Q > 0$.

Q-ball Ansatz

The energy functional for the given charge Q has the form:

$$\mathcal{E} = \int dV \left[\partial_0\psi\partial_0\bar{\psi} + \partial_i\psi\partial_i\bar{\psi} + U\left(\psi\bar{\psi}\right)\right] +$$

$$+\omega\left(Q - i\int dV\left[\partial_0\psi\bar{\psi} - \partial_0\bar{\psi}\psi\right]\right).$$

ω is a Lagrange multiplier. Some manipulations suggest the Q-ball *Ansatz*

$$\psi = e^{i\omega x_0}F(r).$$

F is supposed to be a spherical symmetric function, hence to depend on radial coordinate r only. The equation for the profile function F is following

$$\triangle F = -\omega^2 F + \frac{\partial U(F)}{\partial F}.$$

For some values of charge Q the solutions of the above equation, called also nontopological solitons, are absolutely stable.

Complex Signum Gordon - Models

Complex signum-Gordon model (CsGM) is a scalar field theory defined by the Lagrange function

$$\mathcal{L} = \partial_\mu \psi \partial^\mu \bar{\psi} - \lambda |\psi|, \quad \lambda > 0.$$

The potential term may be regarded as a limiting case of smooth potential, e.g.

$$U(\psi) \sim \lim_{\kappa \to 0^+} \sqrt{\kappa + |\psi|^2}$$

Such a regularization fixes the meaning of the term $\partial|\psi|/\partial\bar{\psi}$ which appears in equation derived from the $\mathcal{L}$.

$$\frac{\partial|\psi|}{\partial\bar{\psi}} = \begin{cases} 0 & \textit{if} \quad \psi = 0 \\ \frac{\psi}{2|\psi|} & \textit{otherwise}. \end{cases}$$

Q-ball in CsGM

The discussion of Q-ball *Ansatz* applies to the model.
Substitution $\psi = F(r)exp(i\omega x_0)$ results in the equation (in three spatial dimensions)

$$F''(r) + \frac{2}{r}F'(r) = \frac{\lambda}{2}sgn(F) - \omega^2 F$$

($sgn(0) = 0$). New variable and function

$$y = \omega r \quad f(y) = \frac{2\omega^2}{\lambda}F(r)$$

make the equation free of parameters

$$f'' + \frac{2}{y}f' = sgn(f) - f.$$

E(Q) relation

The charge Q and energy E may then be expressed as follows

$$Q = \pi \frac{\lambda^2}{\omega^6} \int_0^\infty f^2 \, y^2 dy,$$

$$E = \pi \frac{\lambda^2}{\omega^5} \int_0^\infty \left[f^2 + (\partial_y f)^2 + 2|f| \right] \, y^2 dy.$$

Note that the integrals in the above formulas merely give numerical coefficients.

The formulas make also evident the relation between energy and charge

$$E \sim Q^{5/6}.$$

This favours one (big) Q-ball configuration over any other (multi) Q-ball configuration.

The Solution

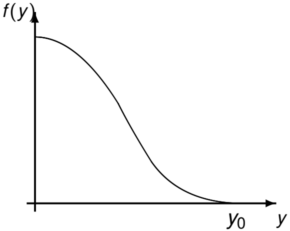
The conditions for physically relevant solution are:

- $f'(0) = 0$,
- finitness of the energy and the charge.

Such a solution has the form

$$f(y) = \begin{cases} 1 - \frac{y_0}{\sin(y_0)} \frac{\sin(y)}{y} & 0 < y < y_0, \\ 0 & y > y_0, \end{cases}$$

where $y_0 \approx 4.4934 \quad (\tan y_0 = y_0)$.



Remarks

1. The solution illustrates a distinctive feature of the signum Gordon models: the field reaches its vacuum value on a finite distance. The way solutions approach the vacuum value is determined by the mass term in the theory. Compactness of solutions in CsGM can be intuitively interpreted as a consequence of "infinite mass".
2. There is an infinite family of other ("excited") single Q-ball solutions.
3. Putting any number of Q-balls far apart from each other (so that they do not overlap) generates new solutions.

CsGM with gauge symmetry

Natural extension of the presented model is a generalization of the global symmetry to the local one. In that case the Lagrangian has the form

$$\mathcal{L} = (\partial^\mu - ieA^\mu)\,\bar{\psi}\,(\partial_\mu + ieA_\mu)\,\psi - U\left(\bar{\psi}\psi\right) - \frac{1}{4}F^{\mu\nu}F_{\mu\nu},$$

where A_μ is gauge field.
The *Ansatz* is now as follows

$$\psi = F(r)\exp(i\omega x_0),$$
$$A_0 = A_0(r),$$
$$A_j = 0.$$

The equation for A_0 expresses (nonlinear) Gauss law.
The condition for energy minimum sets the equation for F.

Field equations

The equations are

$$\triangle A_0(r) = 2e\,(\omega + eA_0(r))\,F^2(r),$$

$$\triangle F(r) = -\,(\omega + eA_0(r))^2\,F(r) + \frac{dU}{dF}.$$

In our case $U(\psi) = 2\lambda|\psi|$. Then it is convenient to introduce new functions and to rescale the r variable:

$$G(y) = \sqrt[3]{\frac{2e^2}{\lambda}}F(r), \quad B(y) = \frac{\omega + eA(r)}{\sqrt[3]{\sqrt{2}\lambda e}}, \quad y = r\sqrt[3]{\sqrt{2}\lambda e}.$$

In this notation the equations have the form

$$\frac{d^2G}{dr^2} + \frac{2}{r}\frac{dG}{dr} = sgn(G) - GB^2,$$

$$\frac{d^2B}{dr^2} + \frac{2}{r}\frac{dB}{dr} = BG^2.$$

Solutions

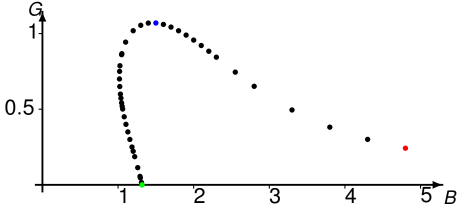
For the same reasons as in the previous model compact solutions are expected, i.e. there exists a point y_0 such that $G(y) = 0$ for $y > y_0$. In that region the electric potential is known

$$B(y) = \omega - \frac{Q}{y}, \quad y > y_0.$$

The matching condition is again continuity of the solutions and their first derivatives.
No analytical solution of the equations is known so far. To get some insight in the theory we have investigated them numerically.

Solutions for small Q

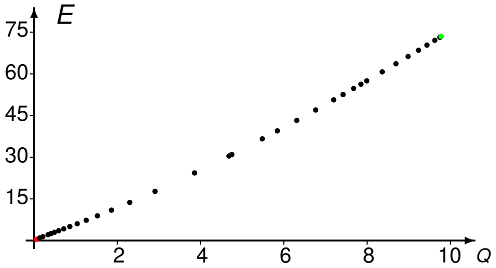
The figure shows the plane $(B(0), G(0))$. The plotted points correspond to the "initial" values of the functions $B(y)$, $G(y)$ for which solutions exist.



One can find solutions for arbitrarily large $B(0)$. They have small charges and energies. They weakly interact with the gauge field, thus may be described in terms of the model with global symmetry.

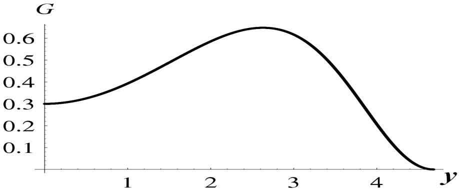
Solutions for medium values of Q

The solutions starting from small $B(0)$ are much harder to explore. If we knew how to solve the system, we could draw a curve interpolating between plotted points (previous slide). The curve would have a striking feature: a "dead end" at $G(0) = 0$, $B(0) \cong 1.317$. The corresponding solution has roughly the largest energy E and charge Q (see below) of the solutions from the dotted curve.

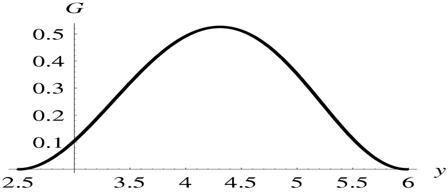


Solutions for large Q

The profile function G close to the "dead end" solution does not resemble much a ball configuration with maximal matter density in the middle.



This motivated conjecture about existence of Q-shell solutions. It has turned out to be true.



Remarks

- The relations E(Q) and R(Q) (with R being the external radius of a solution) make clear that the Q-shell solutions are a natural continuation of the Q-ball solutions.
- The shell solutions admit arbitrarily large charges and energies.
- The reasoning based on E(Q) relation indicates instability for larger Q-balls. The dynamics of an energetically possible collapse is complex and the solutions may turn out to be stable against small disturbances.
- There is a huge variety of "excited" Q-balls and Q-shells.

Literature:

1. H. Arodź, J.Lis, Phys. Rev. D **77**, 107702 (2008).
2. Kimeyeone Lee *et al.*, Phys. Rev. D **39**, 1665 (1989).
3. H. Arodź, J. Lis, in prepatation