Cusp anomalous dimension
in maximally supersymmetric Yang-Mills theory

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Cusp anomalous dimension:

- logarithmic growth of the anomalous dimensions of high-spin Wilson operators,
- Sudakov asymptotics of elastic form factors,
- the gluon Regge trajectory,
- infrared singularities of on-shell scattering amplitudes,

B. Basso, G. P. Korchemsky, J. K., hep-th/0708.3933,
Phys. Rev. Lett. 100, 091601, (2008)

AdS/CFT correspondence

[Maldacena, Polyakov, Klebanov, Gubser, '98; Witten, '99]

String theory	$\Longleftrightarrow$	Super Y-M with $\mathcal{N} = 4$
Energy of folded strings		Anomalous dimension
rotating in AdS$_3$		$\gamma_S^{(L=2)}(g) = 2\Gamma_{\mathrm{cusp}}(g) \ln S + \ldots$
semiclasical expansion		for strong coupling,
		high spin S and twist $= 2$

$$\Gamma_{\mathrm{cusp}}(g) = 2g - \frac{3 \ln 2}{2\pi} + O(1/g)\,, \qquad g = \frac{\sqrt{\lambda}}{4\pi},$$

with $\lambda = g_{\mathrm{YM}}^2 N_c$ being t' Hooft coupling.

1-loop calculation [Gubser, Klebanov, Polykov, Frolov, Tseytlin '02]

string Bethe ansatz [Casteill, Kristjansen '07]

Beisert-Eden-Staudacher equation '06, '07

Fluctuation density (FT of Bethe roots distribution [Korchemsky '95)]

$$\hat{\sigma}_g(t) = \frac{t}{e^t - 1}\left[K_g(2gt, 0) - 4g^2 \int_0^\infty dt' K_g(2gt, 2gt')\hat{\sigma}_g(t')\right],$$

which predicts the cusp anomalous dimension

for arbitrary values of the coupling constant $\Gamma_{\text{cusp}}(g) = 8g^2\hat{\sigma}_g(0)$.

$$K_g(t,t') = \sum_{n,m=1}^{\infty} z_{nm}(g) \frac{J_n(t)J_m(t')}{tt'}$$

Weak coupling expansion gives

$$2\Gamma_{\text{cusp}}(g) = 8g^2 - \tfrac{8}{3}\pi^2 g^4 + \tfrac{88}{45}\pi^4 g^6 - 16\left(\tfrac{73}{630}\pi^6 + 4\zeta(3)^2\right) g^8 +$$

$$+32\left(\tfrac{887}{14175}\pi^8 + \tfrac{4}{3}\pi^2\zeta(3)^2 + 40\zeta(3)\zeta(5)\right) g^{10} +$$

$$-64\left(\tfrac{136883}{3742200}\pi^{10} + \tfrac{8}{15}\pi^4\ \zeta(3)^2 + \tfrac{40}{3}\pi^2\zeta(3)\zeta(5) + 210\zeta(3)\zeta(7) + 102\zeta(5)^2\right) g^{12} + \dots$$

also the four loop value calculated by [Bern, Czakon, Dixon, Kosower and Smirnov '07]

from QCD maximal tanscendentality [Lipatov et al. '06]

Strong coupling expansion in numerical approach

Expanding over Bessel functions and truncate the series

[Benna, Benvenuti,Klebanov, Scardicchio'06]

$\widehat{\sigma}_g(t) = \frac{t}{e^t-1}\sum_{n=1}^{M} s_n(g)\frac{J_n(2gt)}{2gt}$ and $s_{n\geq M+1} = 0$

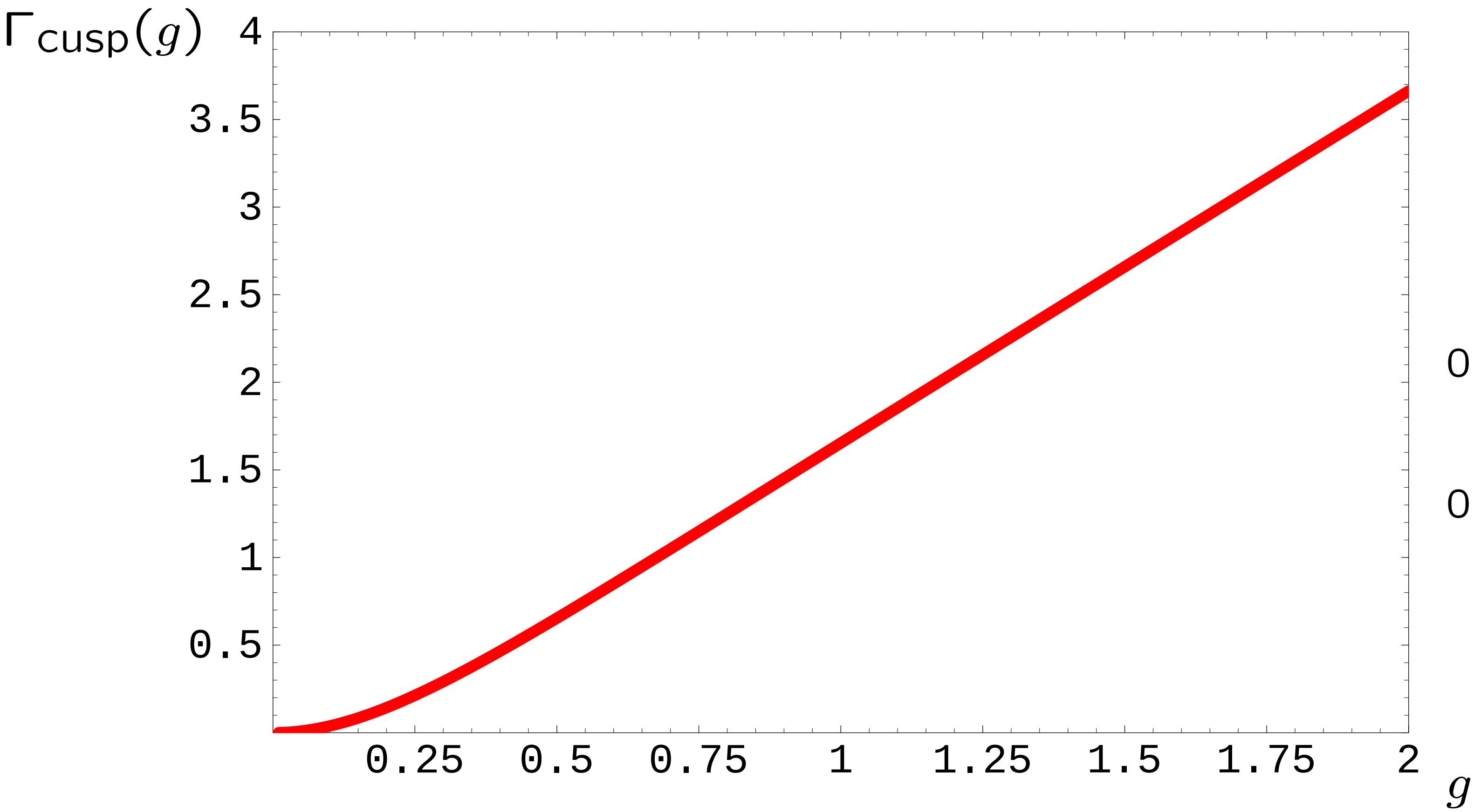
Integral equation becomes matrix equation

$s(g) = \frac{1}{1+K(g)} \cdot h$ and $\Gamma_{\text{cusp}}(g) = 4g^2 s_1(g)$

with $K(g)$ being complicated $g-$dependent matrix $(M \times M)$

and $h = (1, 0, 0, 0, 0, \ldots)$ is a boundary condition vector

Numerical results



$0.661907 = \frac{3\ln 2}{\pi}$

$0.0232 =?$

$$2\Gamma_{\text{cusp}}(g) = 4.000000g - 0.661907 - 0.0232g^{-1} + \dots$$

[Benna, Benvenuti, Klebanov and Scardicchio '07]

$$\Gamma_{\text{cusp}}(g) = 2g + \mathcal{O}(g^0)$$

Simple strong coupling expansion

Do not truncate the Bessel series (infinite-dimension matrices)

$[1 + K(g)] \cdot s(g) = h$ [Alday et al. '07],[Kostov,Serban,Volin '07]

[Beccaria, De Angelis, Forini'07], [Lipatov et al. '07]

Expand matrix $K(g)$ and solution $s(g)$ in powers of $1/g$

$$s_n(g) = \frac{1}{g} s_n^{(0)} + \frac{1}{g^2} s_n^{(1)} + \frac{1}{g^3} s_n^{(2)} + \ldots$$

$$K(g) = g K^{(0)} + K^{(1)} + \frac{1}{g} K^{(2)} + \ldots$$

Leading order solution $s^{(0)} = [K]^{-1} \cdot h +$ [zero modes]

Ambiguity is fixed by the constraint from numerics $s_{2k-1}^{(0)} = s_{2k}^{(0)}$

$$s_{2k-1}^{(0)} = s_{2k}^{(0)} = (-1)^{k+1} \frac{\Gamma(k+\frac{1}{2})}{\Gamma(k)\Gamma(\frac{1}{2})}$$

For the next-to-leading order constraints which fixing zero modes

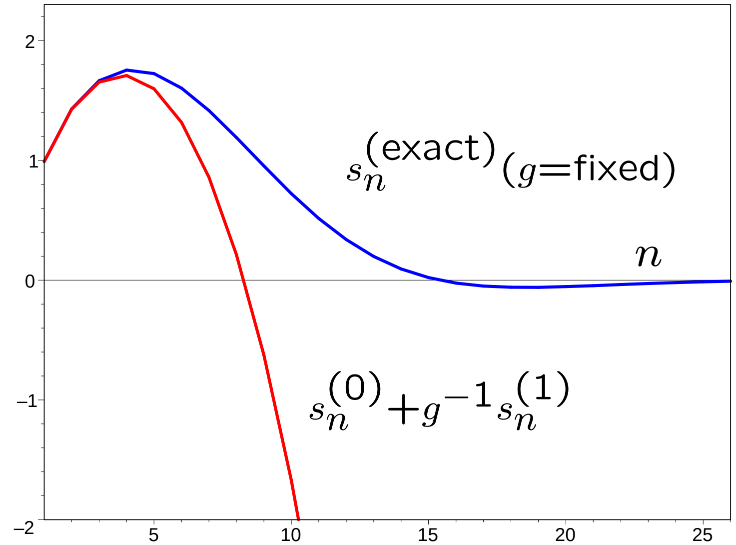
are more complicated

Main Idea

The $n-$dependence of the coefficients

$$s_n^{\pm}(g) = \frac{1}{2}(s_{2n-1}(g) \pm s_{2n}(g))$$

$1/g$ expantion fails for large n



Our approach:

(i) Construct solution for $s_n(g)^{\pm}$ in the region $n \sim 1$ and parametrize the comtribution of (zero modes) by set of yet unknown coefficients $c_p^{\pm}(g)$

(ii) Construct asymptotic solution for $s_n^{\pm}(g)$ in the region $n \gg 1$

(iii) Sew two asymtotic expressions for $s_n^{\pm}(g)$ in intermediate region $n \sim g^{1/2}$ and determine the infinite set of zero mode coefficients $c_p^{\pm}(g)$

One can do it without knowing the exact solution performing the scaling limit

$n, g \to \infty$ where $x = (n - \frac{1}{4})^2/g$ =fixed

Fixing zero mode coefficients

Changing variables one can solve BES equation

$$\Gamma_{\text{cusp}}(g) = 2g + \sum_{p=1}^{\infty} \frac{1}{g^{p-1}} \left[\frac{2c_p^-(g)}{\sqrt{\pi}} \Gamma(2p - \tfrac{3}{2}) + \frac{2c_p^+(g)}{\sqrt{\pi}} \Gamma(2p - \tfrac{1}{2}) \right]$$

$$\text{where } c_p^{\pm}(g) = \sum_{r \geq 0} g^{-r} c_{p,r}^{\pm}.$$

The expansion coeffcients $s_m^{\pm}(g)$ should have correct

scaling behaviour in the scaling limit

$m, g \to \infty$ for $x = (m - \frac{1}{2})^2/g$ =fixed

$$s_m(g)^{\pm} = \frac{(gx)^{-1/4}}{g\sqrt{\pi}} \left[\gamma_{\pm}^{(0)}(x) + \frac{\gamma_{\pm}^{(1)}(x)}{gx} + \mathcal{O}(1/g^2) \right]$$

where expansion of $\gamma_{\pm}^{(r)}(x)$ runs in integer positive powers of x

$\gamma_{\pm}^{(r)}(x)$ should have faster-than-power decrease at large x

Quantisation conditions

In the leading order one gets

$$\left(\sum_{p\geq 0} s^p\, c^{+}_{p,0}\Gamma(p-\tfrac{1}{4})\right) = 2[\Gamma(\tfrac{3}{4})]^2 \frac{\Gamma\left(1-\frac{s}{2\pi}\right)}{\Gamma\left(\frac{3}{4}-\frac{s}{2\pi}\right)} + O(1/g),$$

with $c^{+}_{0,0} = -\frac{1}{2}$

$$\left(\sum_{p\geq 0} s^p \left[c^{-}_{p,0}\Gamma(p-\tfrac{3}{4}) + 2c^{+}_{p,0}(p-\tfrac{1}{4})\Gamma(p+\tfrac{1}{4})\right]\right) = \tfrac{1}{4}[\Gamma(\tfrac{1}{4})]^2 \frac{\Gamma\left(1-\frac{s}{2\pi}\right)}{\Gamma\left(\frac{1}{4}-\frac{s}{2\pi}\right)} + O(1/g),$$

with $c^{-}_{0,0} = 0$

which gives

$$c^{+}_{1,0} = -\frac{3\ln 2}{\pi} + \frac{1}{2} + O(1/g)\,, \qquad c^{-}_{1,0} = \frac{3\ln 2}{4\pi} - \frac{1}{4} + O(1/g).$$

$$\Longrightarrow \boxed{\Gamma_{\rm cusp}(g) = 2g - \frac{3\ln 2}{2\pi} + O(g^{-1})}$$

Similarly for next orders but much more complicated formulae

Strong coupling expansion $\mathcal{O}(g^{-11})$

$$\Gamma_{\rm cusp}(g+c_1) = 2g\Big[1 - c_2 g^{-2} - c_3 g^{-3} - \left(c_4 + 2\,c_2^2\right)g^{-4} +$$
$$-(c_5 + 23\,c_2c_3)g^{-5} - \left(c_6 + \tfrac{166}{7}\,c_2c_4 + 54\,c_3^2 + 25\,c_2^3\right)g^{-6} +$$
$$-\left(c_7 + \tfrac{1721}{29}\,c_2c_5 + \tfrac{1431}{7}\,c_3c_4 + 457\,c_2^2c_3\right)g^{-7} +$$
$$-\left(c_8 + \tfrac{6352}{107}\,c_2c_6 + \tfrac{12606}{29}\,c_3c_5 + \tfrac{7916}{49}\,c_4^2 + \tfrac{6864}{7}\,c_2^2c_4\right.$$
$$\left.+4563\,c_2c_3^2 + 374\,c_2^4\right)g^{-8} + O\left(g^{-9}\right)\Big],$$

where the expansion coefficients are given by

$$c_1 = \frac{3\ln 2}{4\pi},\quad c_2 = \frac{1}{16\pi^2}\mathsf{K},\quad c_3 = \frac{27}{2^{11}\pi^3}\zeta(3),\quad c_4 = \frac{21}{2^{10}\pi^4}\beta(4),$$
$$c_5 = \frac{43065}{2^{21}\pi^5}\zeta(5),\ c_6 = \frac{1605}{2^{15}\pi^6}\beta(6),\ c_7 = \frac{101303055}{2^{30}\pi^7}\zeta(7),\ c_8 = \frac{1317645}{2^{22}\pi^8}\beta(8),$$

with $\zeta(x)$ the Riemann zeta function, $\beta(x) = \sum_{n\geq 0}(-1)^n(2n+1)^{-x}$

the Dirichlet beta function and $\mathsf{K} = \beta(2)$ the Catalan's constant

Asymptotic expansion $\mathcal{O}(g^{-40})$

The asymptotic expansion is not Borel summable

$$\Gamma_{\text{cusp}}(g) \sim -g \sum_k \frac{\Gamma(k-\frac{1}{2})}{(2\pi g)^k} = g \int_0^\infty \frac{du\, u^{-1/2}\, \mathrm{e}^{-u}}{u - 2\pi g},$$

with the Borel transform having a pole at $u = 2\pi g$.

Ambiguity due to different prescriptions

to integrate over the pole is for large g

$$\delta\Gamma_{\text{cusp}}(g) \sim g^{1/2} \exp(-2\pi g)$$

FRS equation for $\gamma_S^{(L)}(g) = 2(\Gamma_{\text{cusp}}(g) + \epsilon(g, L)) \ln S + \ldots$

exponential corrections [Freyhult, Rej, Staudacher '08]

Solutions: [Basso, Korchemsky '08]

agree with $O(6)$ sigma model [Alday, Maldacena '07]

Summary

Analytical calculation of strong coupling expansion of $\Gamma_{\text{cusp}}(g)$

- scaling limit, quantization conditions, the right choice of variables

Strings

After our work Roiban and Tseytlin corrected their two loop
string results which now agree with our results