Lectures at Zakopane Summer School “Aspects of Duality”, 13-22/06/08

From Classical to Quantum Integrability of Metsaev-Tseytlin Superstring (and back)

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Lecture I. Finite gap solution of classical string

- Gives a quick access to the full strong coupling spectrum of dimensions of planar N=4 SYM in terms of the moduli of the general algebraic curve of the model.

- Finite gap method was first done for $R_1 \times S^3$ subsector of superstring (KMMZ solutions). **V.K.,Marshakov,Minahan,Zarembo'04**

- Now: we do it to the full Metsaev-Tseytlin superstring model on $AdS_5 \times S^5$, including fermions, using classical integrability.
Beisert,V.K.,Sakai,Zarembo'05 **Bena,Roiban,Polchinski'02**

- Finite gap: Often a good starting point for quantization: simplctic structure identified, all one loop WKB is available…
N.Dorey,Vicedo'06 **Gromov,Vieira'06**

- Finite gap eqs. gave rise to the educated guess which led to all loop asymptotic Bethe ansatz eqs.(« discretization of alg.curve).
Arutynov,Frolov,Staudacher'04, **Beisert,Staudacher'05**

- What is missing? May be some hidden degrees of freedom, un upper level(s) of full loop Bethe ansatz.
- Lecture II: demonstration of "hidden" variables on $R_1 \times S^3$ subsector

Metsaev-Tseytlin superstring

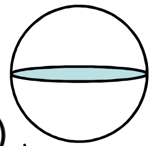
- It is a sigma model on the coset

$$AdS_5 \times S^5 \sim PSU(2,2|4)/\left(Sp(2,2)\times Sp(4)\right)$$

- Supergroup element g : $(4|4)\times(4|4)$ supermatrix of SU(2,2|4)

Isometry of sphere

$SU(4) \sim SO(6)$



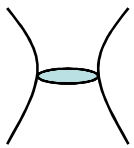
$B = C^\dagger \cdot \mathrm{diag}(1,1,-1,-1)$

$$g = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) \in PSU(2,2|4)$$

16 complex fermions

Isometry of Anti de Sitter (conformal group)

$SU(2,2) \sim SO(2,4)$



sphere

$$\frac{SU(4)}{Sp(4)} \sim \frac{SO(6)}{SO(5)} = S^5$$

Anti de Sitter

$$\frac{SU(2,2)}{Sp(2,2)} \sim \frac{SO(2,4)}{SO(1,5)} = AdS_5$$

- Decompose current:

$$J = -g^{-1}dg = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$$

$$J^{(0,2)} = \frac{1}{2}\begin{pmatrix} A \pm EA^TE & 0 \\ 0 & D \pm ED^TE \end{pmatrix}$$

$$E = \left(\begin{array}{c|c} 0 & 1 \\ \hline -1 & 0 \end{array}\right)_{4\times 4}$$

$$J^{(1,3)} = \frac{1}{2}\begin{pmatrix} 0 & B \mp iEC^TE \\ C \pm iEB^TE & 0 \end{pmatrix}$$

$$J = J^{(0)} + J^{(1)} + J^{(2)} + J^{(3)}$$

- Bosons: $J^{(2)} \in AdS_5 \times S^5$,
- To factor out: $J^{(0)} \in \mathrm{sp}(2,2) \times \mathrm{sp}(4)$,
- Fermions: $J^{(1,3)}$

Metsaev-Tseytlin String Action

$$S_{MT} = \frac{\sqrt{\lambda}}{4\pi} \mathrm{str} \int_{\mathcal{M}_2} \left[J^{(2)} \wedge *J^{(2)} - J^{(1)} \wedge J^{(3)} \right]$$

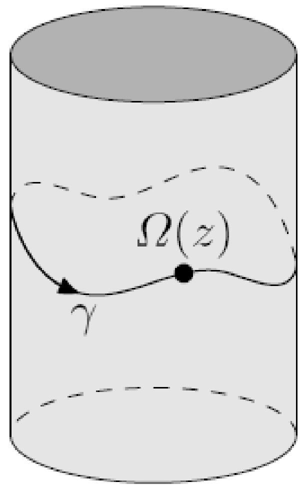
$$J = J^{(0)} + J^{(1)} + J^{(2)} + J^{(3)}$$

- Bosons: $J^{(2)} \in AdS_5 \times S^5,$
- To factor out: $J^{(0)} \in \mathrm{sp}(2,2) \times \mathrm{sp}(4),$
- Fermions: $J^{(1,3)}$

- Z_4-grading: $\mathcal{C} J^{(k)} \mathcal{C}^{-1} = i^k \ J^{(k)}$

$$\mathcal{C} = \left(\begin{array}{c|c} E & 0 \\ \hline 0 & -iE \end{array} \right), \quad E = \left(\begin{array}{c|c} 0 & 1 \\ \hline -1 & 0 \end{array} \right)_{4\times 4}$$

Monodromy matrix for MT superstring



- All Bianchi identities and eqs. of motion (current conserv.) are packed into a Lax eq.:

Bena,Roiban,Polchinski'02

$$(d + A(z)) \wedge (d + A(z)) = 0,$$

with connection depending on extra spectral parameter z

$$A(z) = J^{(0)} + \frac{1}{2}\left(z^{-2} + z^2\right) J^{(2)} + \frac{1}{2}\left(z^{-2} - z^2\right) * J^{(2)} + z^{-1} J^{(1)} + z\, J^{(3)}$$

- Monodromy matrix: $\Omega(z) = P \exp \oint A(z) d\sigma$

Conserved quantities: eigenvalues of $\Omega(z)$

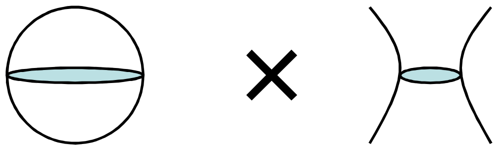
$$\{e^{i\tilde{p}_1(z)}, e^{i\tilde{p}_2(z)}, e^{i\tilde{p}_3(z)}, e^{i\tilde{p}_4(z)} || e^{i\hat{p}_1(z)}, e^{i\hat{p}_2(z)}, e^{i\hat{p}_3(z)}, e^{i\hat{p}_4(z)}\}$$

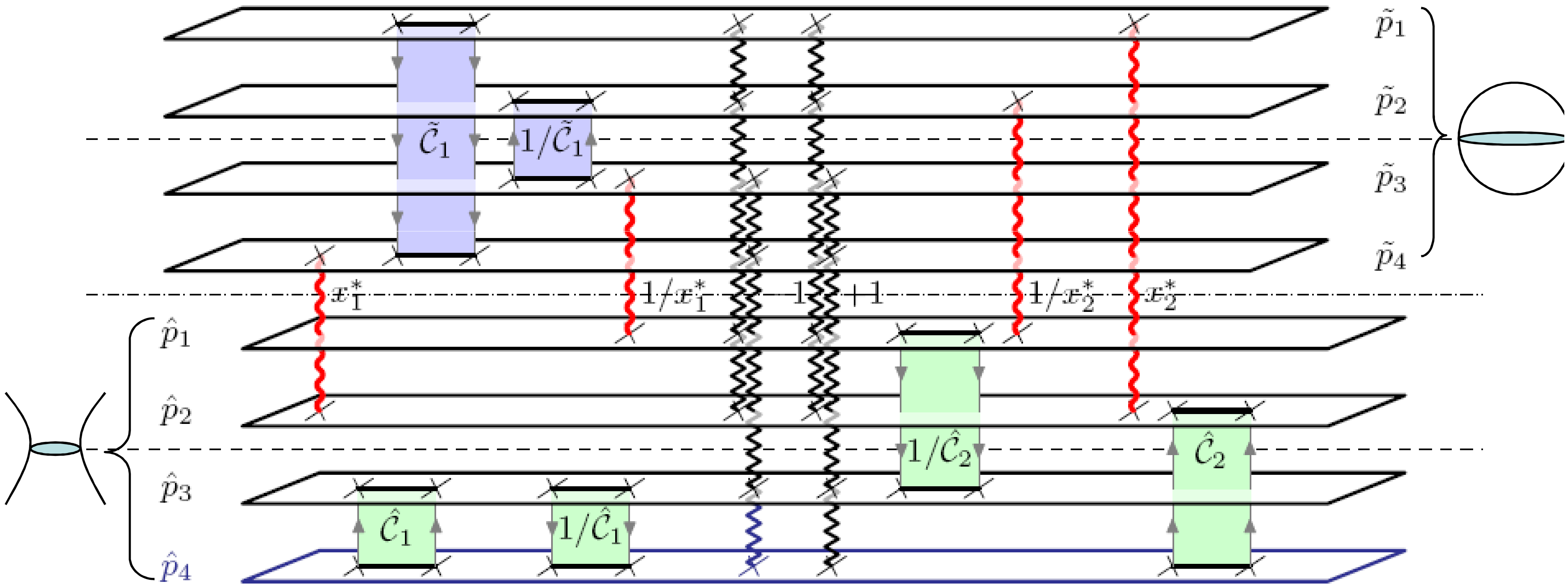
Finite gap method: Eigenvalues are found by solving a characteristic equation for Ω . They define Riemann Surface.

So does $y_k(z) = -izp'_k(z)$ (single valued on R.S.)

Riemann surface classical string

V.K., Marshakov, Minahan, Zarembo'04 (KMMZ)
Beisert, V.K., Sakai,Zarembo'05





$$x = \frac{1+z^2}{1-z^2}$$

Algebraic curve of quasi-momentum

$y_k(z) = -izp_k'(z)$ - good variables, having only:

- branch cuts at $\tilde{z}_i, \hat{z}_j$ where same grading e.v.'s cross;

Cut C_{ij} : $p_i^+ - p_j^- = 2\pi n_{ij}$

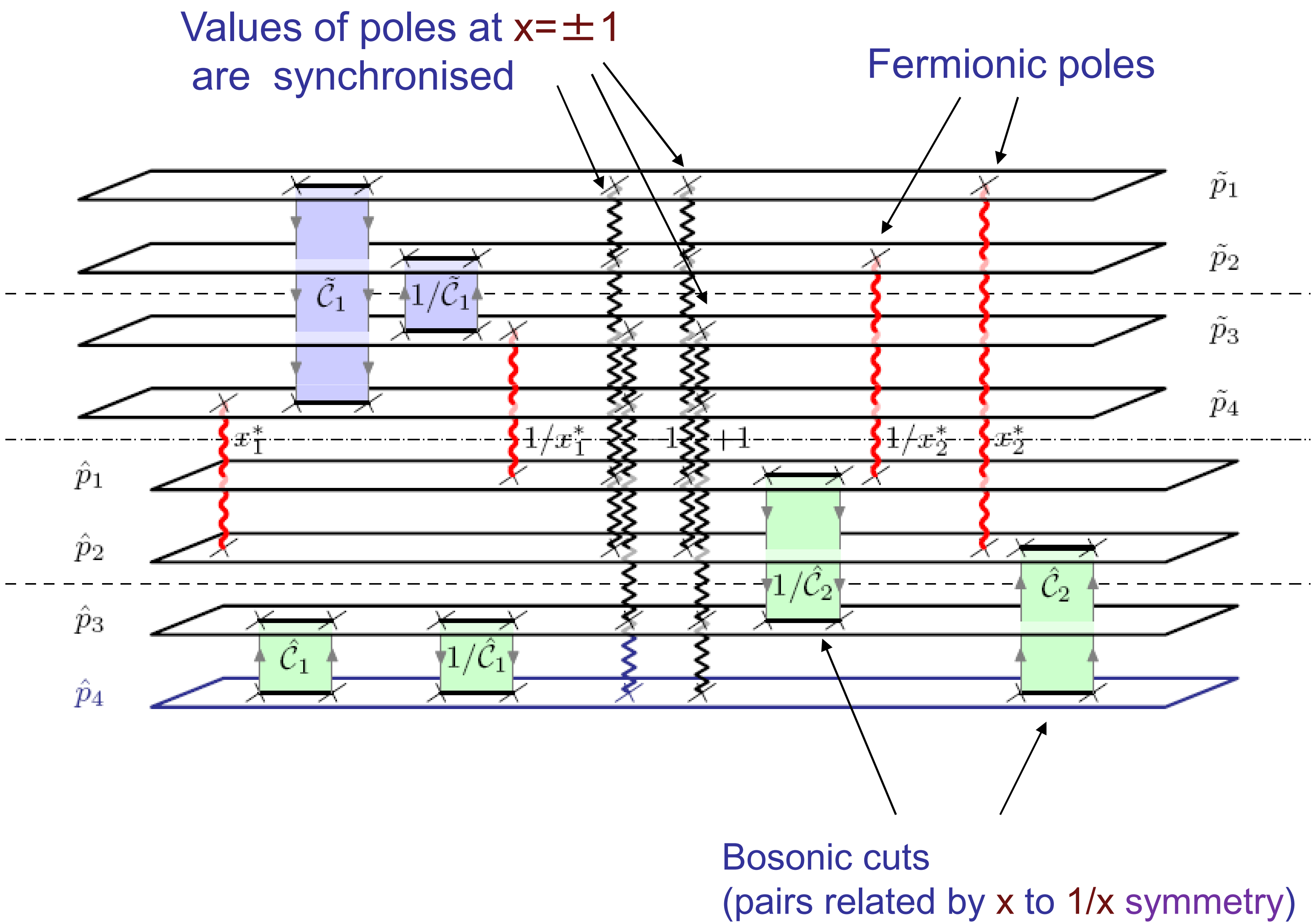
- poles at z_j^* where opposite grading e.v.'s cross.

Corresponding (1|1)x(1|1) sub-supermatrix of $\Omega(z)$

$$\left(\begin{array}{c|c} a & b \\ \hline c & d \end{array}\right) = u(z) \left(\begin{array}{c|c} \frac{bc}{a-d} + a & 0 \\ \hline 0 & \frac{bc}{a-d} + d \end{array}\right) u^{-1}(z)$$

$bc \to < bc > \sim \hbar$ - Fermionic condensate

8-Sheet Riemann surface



Inversion symmetry and Virasoro

- Important non-perturbative symmetry of string (and SYM !):

$$\Omega(1/x) = \mathcal{C}\Omega^{-ST}(x)\mathcal{C}^{-1}$$

induces a monodromy:

$$\tilde{y}_k(1/x) = -\tilde{y}_{k'}(x), \qquad (1,2,3,4) \leftrightarrow (2,1,4,3)$$

$$\hat{y}_k(1/x) = -\hat{y}_{k'}(x), \qquad (5,6,7,8) \leftrightarrow (6,5,8,7)$$

- Virasoro constraints:

$$\mathrm{str}\ (J^{(2)} \pm *J^{(2)})^2 = 0$$

They make the poles of all S^5 and AdS_5 quasimomenta at $x=\pm 1$ equal.

Global Charges

- Conserved harges: angular momenta, spins J_1, J_2, J_3, S_1, S_2

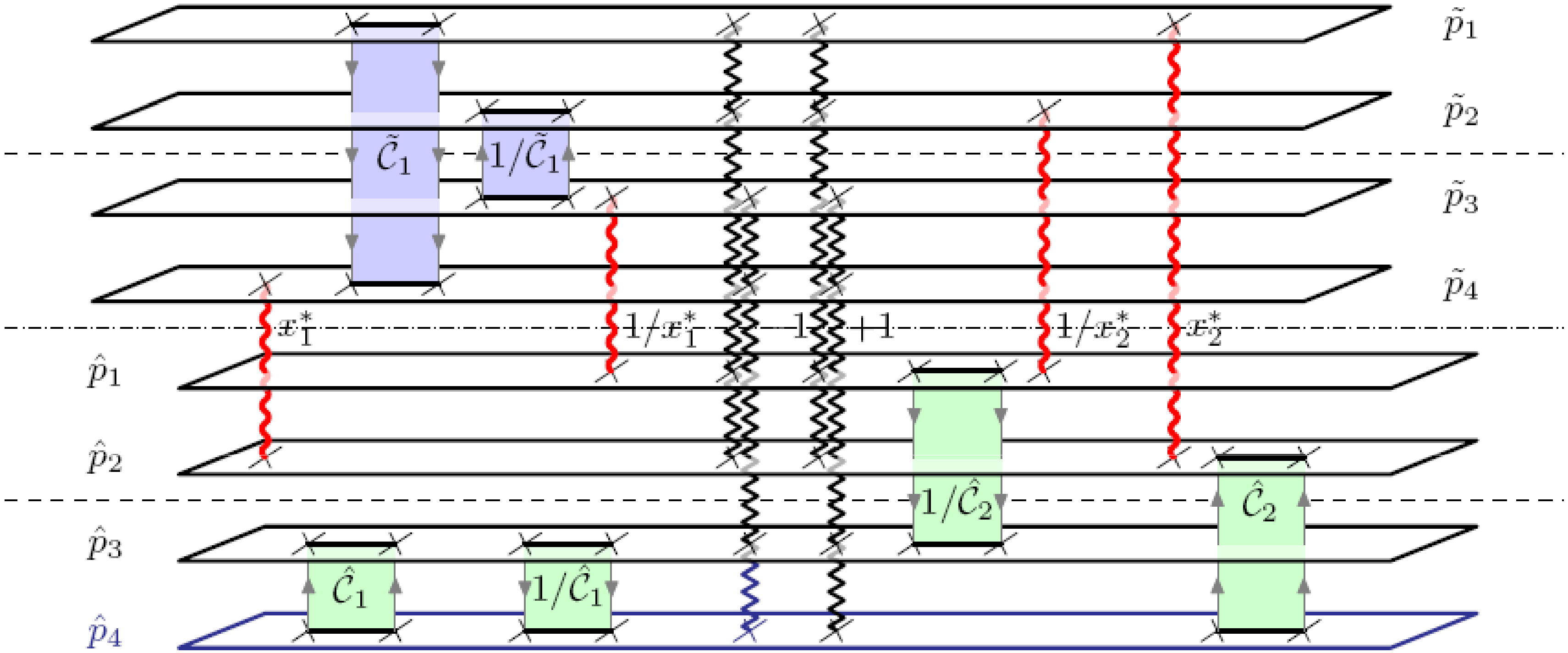
  and energy *E*: defined from the asymptotics at x=0,∞.

$$\mathbf{S^5}: \quad \tilde{p}_1(x) = -\frac{2\pi}{\sqrt{\lambda}}(J_1 + J_2 - J_3)\frac{1}{x} + \dots, \quad etc.$$

$$\mathbf{AdS^5}: \quad \hat{p}_1(x) = \frac{2\pi}{\sqrt{\lambda}}(E + S_1 - S_2)\frac{1}{x} + \dots, \quad \text{etc.}$$

- Energy E of state $\to$ dimension Δ of operator in SYM.
  One reads it off from the large *x* asymptotics.

Riemann surface of the curve



- Algebraic curve encodes all "action" variables;
- "Angle" variables defined by incomplete holomorphic integrals.
- Possible to restore corresponding classical string motion (see for S3xR1 sector [Dorey,Vicedo'06]).
- Good start for quantization (symmetry $x \to 1/x$ important!)

Fixing the Curve

- Zeroes and poles of spectral super-determinant give the curve:

$$\mathcal{D}(y,z) = \mathrm{sdet}\left(y - \left[u(z)\left(-izp'(z)\right)u^{-1}(z)\right]\right) = \{\frac{0}{0}\}$$

$$\mathcal{D}(y,z) \sim \frac{\tilde{F}_4(x)y^4 + \tilde{F}_2(x)y^2 + \tilde{F}_1(x)y + \tilde{F}_0(x)}{\hat{F}_4(x)y^4 + \hat{F}_2(x)y^2 + \hat{F}_1(x)y + \hat{F}_0(x)} = \{\frac{0}{0}\}$$

with $x = (1+z^2)/(1-z^2)$.

$$\tilde{F}_4(x) = x^4 \prod_{a=1}^{2\tilde{A}}(x-\tilde{x}_a^+)\prod_{a=1}^{2\tilde{A}}(x-\tilde{x}_a^-)\prod_{a=1}^{2A^*}(x-x_a^*)^2$$

$$\hat{F}_4(x) = x^4 \prod_{a=1}^{2\hat{A}}(x-\hat{x}_a^+)\prod_{a=1}^{2\hat{A}}(x-\hat{x}_a^-)\prod_{a=1}^{2A^*}(x-x_a^*)^2$$

- These polynomials define the moduli of the algebraic curve.

- Read off the energy of solution Δ(L,J,S,s,…) from asymptotics at x=∞ .

Fixing the moduli...

The polynomials, along with the moduli of the curve, are fixed from:

- Asymptotics at $x=\infty$ (charges);
- Equal poles at at $x=\pm 1$ (Virasoro conditions);
- Zero A-cycles

$$\oint_{A_k} d\tilde{p}(x) = 0$$

- singlevaluedness on R.S.

- Integer B-cycles (mode numbers):

$$\oint_{B_k} d\tilde{p}(x) = 2\pi n_k$$

- unimodularity of Ω,

n_k - mode numbers of string oscillators

- Bohr-Sommerfeld conditions, WKB quantization:

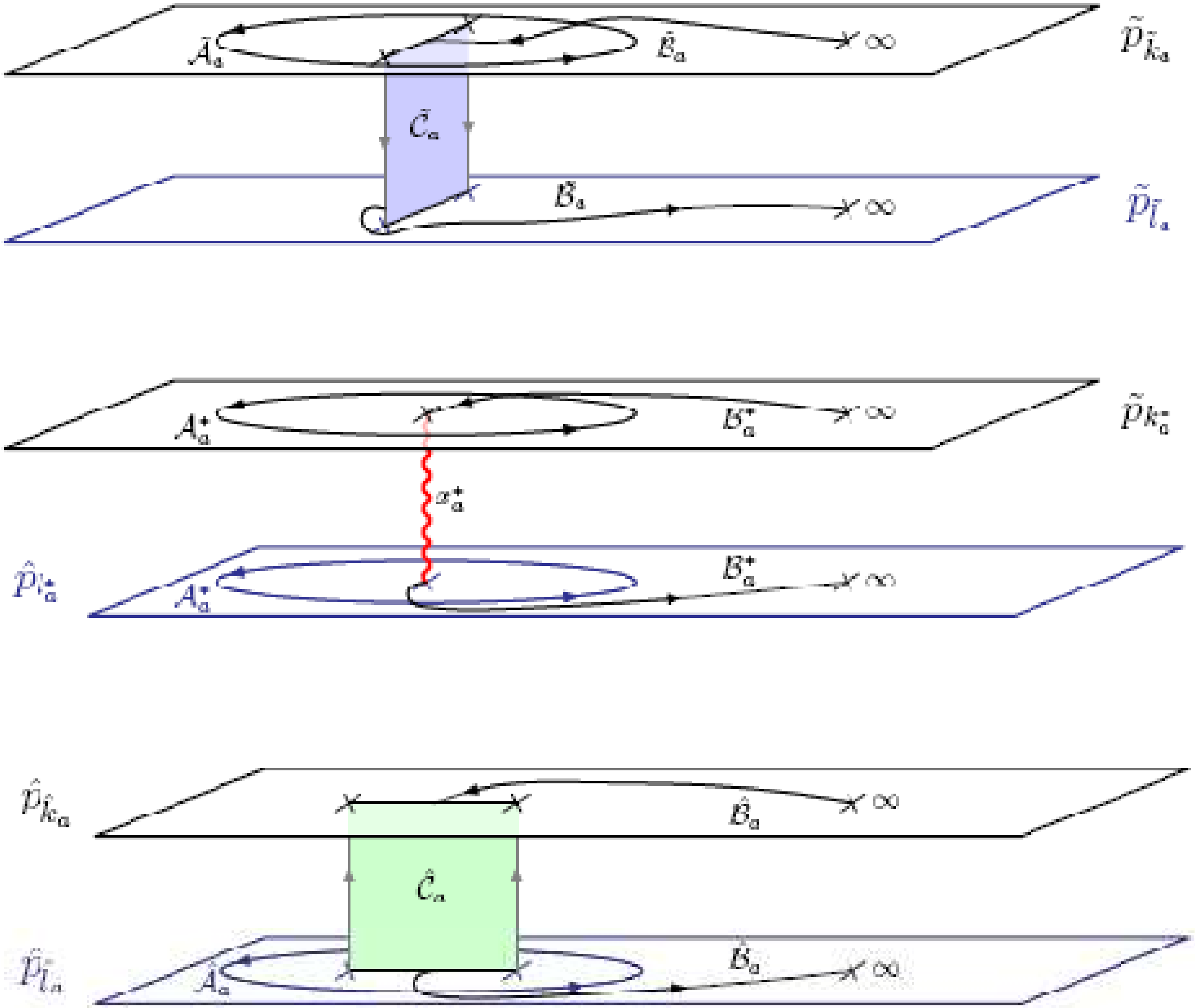
$$\oint_{A_k} dx \left(1 - \frac{1}{x^2}\right) \, p(x) \;=\; \oint_{A_k} du \, p(x(u)) \;=\; s_k + \frac{1}{2},$$

$$u = x + 1/x$$

- "filling fractions" (higher int's of motion)

Dorey,Vicedo'05
Gromov,Vieira'07

- No singularities but bosonic cuts, fermionic poles and poles at $x=\pm 1$

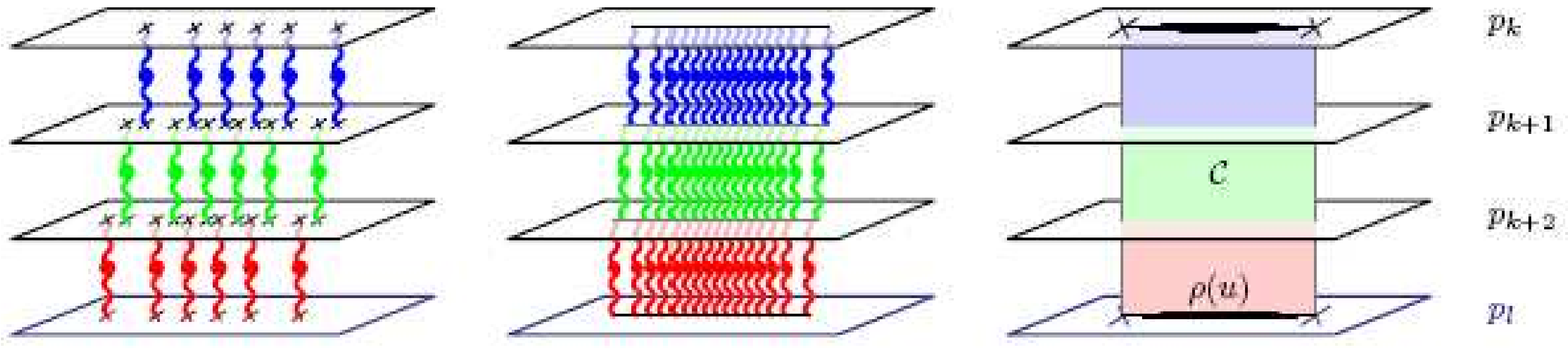


$\tilde{A}_a$
$\tilde{B}_a$
∞
$\tilde{p}_{\tilde{k}_a}$
$\tilde{C}_a$
$\tilde{B}_a$
∞
$\tilde{p}_{\tilde{l}_a}$
$A_a^\star$
$B_a^\star$
∞
$\tilde{p}_{k_a^\star}$
$x_a^\star$
$\hat{p}_{l_a^\star}$
$A_a^\star$
$B_a^\star$
∞
$\hat{p}_{\hat{k}_a}$
$\hat{B}_a$
∞
$\hat{C}_a$
$\hat{p}_{\hat{l}_a}$
$\hat{A}_a$
$\hat{B}_a$
∞

How to quantize this superstring?

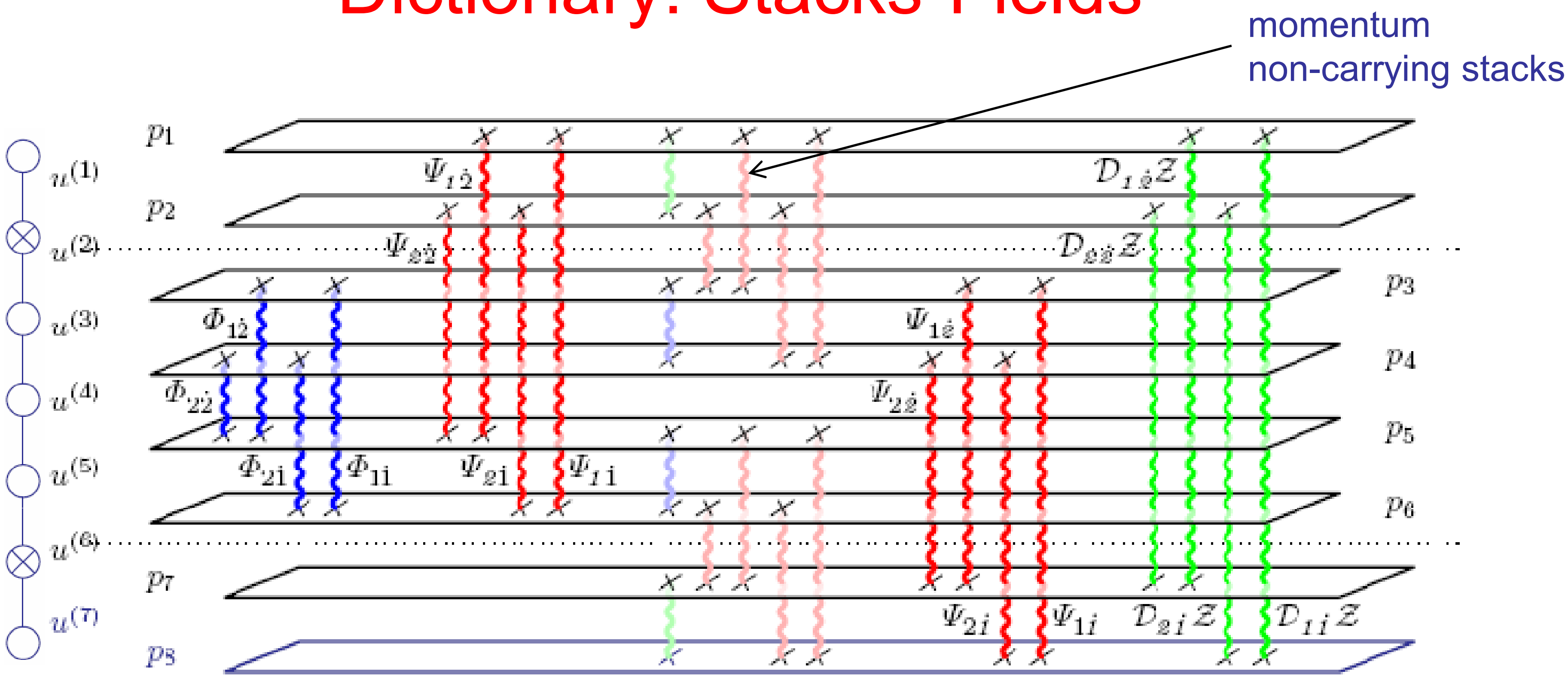
• Condensation of SYM Bethe roots matches string cuts

• Formation of cuts from strings of stacks:



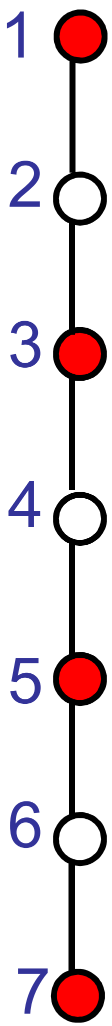
Beisert,V.K.,Sakai,Zarembo'05(b)

Dictionary: Stacks-Fields



- To each field of SYM corresponds a Bethe root or stack of roots.
- Bosonic roots with the same mode number n_k condense into cuts in the scaling limit of long operators.
- Fermionic roots stay apart.

- Algebraic curves of string and SYM coincide by the appropriate identification of parameters: AdS/CFT correspondence!



psu(2,2|4)

Asymptotic Bethe Ansatz

(discretization of algebraic curve)

Beiserst,Staudacher'05

$$1 = \prod_k \qquad e_{-2}\left(u_j^{(2)} - u_k^{(2)}\right) e_{+1}\left(u_j^{(2)} - u_k^{(3)}\right)$$

$$1 = \prod_k e_{-1}\left(u_j^{(3)} - u_k^{(2)}\right) \qquad r_{+}\left(u_j^{(3)}, u_k^{(4)}\right)$$

dressing phase

$$\left(\frac{x_j^{(4)+}}{x_j^{(4)-}}\right)^L = \prod_k \sigma^2(x_j^{(4)}|x_k^{(4)}) r_{-}\left(u_j^{(5)}, u_k^{(4)}\right) e_{+2}\left(u_j^{(4)} - u_k^{(4)}\right) r_{-}\left(u_j^{(3)}, u_k^{(4)}\right)$$

$$1 = \prod_k r_{+}\left(u_j^{(5)}, u_k^{(4)}\right) \qquad e_{-1}\left(u_j^{(5)} - u_k^{(6)}\right)$$

$$1 = \prod_k e_{+1}\left(u_j^{(6)} - u_k^{(5)}\right) e_{-2}\left(u_j^{(6)} - u_k^{(6)}\right)$$

$$e_k(u) = \frac{u + ik/2}{u - ik/2},$$

$$r_{\pm}(u, \tilde{u}) = \frac{x - \tilde{x}^{\pm}}{x - \tilde{x}^{\mp}}$$

$$u = x + 1/x$$

$$x^{\pm} = x(u \pm i/2)$$

- Completely fixes dimensions of long operators of N=4 SYM!

  (by rapidities of the middle node)

$$\mathcal{E} - L = \sqrt{\lambda} \sum_j \left(\frac{i}{x_j^{+}} - \frac{i}{x_j^{-}}\right)$$

- **Two coupled "Hubbards", can be derived by algebraic BA for Hubbard chain**

Lecture II. Quantization of superstring

Gubser,Polyakov,Klebanov'00
Frolov,Tseytlin'02
....
Bena, Roiban,Polchinski'02
V.K.,Marshakov,Minahan,Zarembo'04
Beisert,V.K.,Sakai,Zarembo'05
....

Integrability of N=4 SYM in 1,2,3,… loops

Lipatov'00
Minahan,Zarembo'02
Beisert,Kristjansen,Staudacher'02
Beisert,Staudacher'03
....

Algebraic curve of classical string on AdS5xS5 (finite gap)

Asymptotic Bethe ansatz (ABA)

Arutyunov,Frolov,Staudacher'05
Beisert,Staudacher'05

Eden,Beisert,Staudacher,07

S-matrix
+
Dressing factor

Staudacher'04,
Beisert'05
Janik'05
Beisert,Hernandez,Lopez'07
....

Extra (hydden) level of ABA ?
TBA ?
??????????

Full quantum solution for short operators (finite size closed string)

Gromov,V.K.'06
Janik'07
....

"Toy" model for hydden level: σ-model on $S^3 x R_1$

Gromov,V.K.'06

- Classical motion can be limited by a subset $S^3 \times R_1 \subset S^5 x AdS_5$
- Polyakov string action

world sheet metric

AdS time direction

$$S = \frac{\sqrt{\lambda}}{4\pi} \int d\sigma d\tau \sqrt{\det h} \left[(\nabla X_a)^2 - (\nabla t)^2 \right], \qquad X_1^2 + \ldots + X_4^2 = 1$$

- Conformal gauge: $h_{\alpha\beta}(\sigma,\tau) = e^{\phi(\sigma,\tau)} \eta_{\alpha,\beta}$

- Virasoro conditions: $T_{++} = T_{--} = 0$

$$Y = (X_1, X_2, X_3, X_4, t)$$

$$T_{\alpha\beta} = \frac{\delta S}{\delta h_{\alpha\beta}(\sigma,\tau)} = \partial_\alpha Y \cdot \partial_\beta Y - \tfrac{1}{2}\eta_{\alpha\beta}(\partial Y)^2$$

- Gauge for AdS "time": $t(\sigma,\tau) = \tfrac{1}{2}\kappa_+(\tau+\sigma) + \tfrac{1}{2}\kappa_-(\tau-\sigma)$

Equivalent SU(2)xSU(2) principal chiral field

- S^3 coset $\longrightarrow$ SU(2) group manifold:

$$j_a = g^{-1}\partial_a g, \qquad g = \begin{pmatrix} X_1 + iX_2 & X_3 + iX_4 \\ -X_3 + iX_4 & X_1 - iX_2 \end{pmatrix} \in SU(2)$$

- Action:

$$S = -\frac{\sqrt{\lambda}}{8\pi}\int d\sigma d\tau \ \mathrm{Tr}\ j_a^2$$

- Global charges (rotational momenta):

$$J_L = \frac{\sqrt{\lambda}}{4\pi}\int d\sigma\ \mathrm{Tr}\ \left(i\partial_0 g\, g^\dagger \tau^3\right), \qquad J_R = \frac{\sqrt{\lambda}}{4\pi}\int d\sigma\ \mathrm{Tr}\ \left(i g^\dagger \partial_0 g\, \tau^3\right)$$

- Virasoro conditions: $\mathrm{tr} j_\pm^2(\sigma,\tau) = 2\kappa_\pm^2$

- Energy and momentum of sigma model:

$$E^{\mathrm{cl}} \pm P^{\mathrm{cl}} = -\frac{\sqrt{\lambda}}{8\pi}\int \mathrm{tr}\ [j_0 \pm j_1]^2 d\sigma = \frac{\sqrt{\lambda}}{2}\kappa^2$$

- For string: level matching (no time windings): $P = 0$: $\kappa_+ = \kappa_- = \kappa$

- AdS "Energy" = dim. of a SYM operator:

time translation generator

$$\Delta = \int_0^{2\pi} d\sigma\ \frac{\delta S}{\delta[\partial_\tau t(\sigma,\tau)]} = \sqrt{\lambda}\,\kappa$$

S-matrix for $SU_L(2)xSU_R(2)$ chiral field

- S-matrix:

$$\hat{S}(\theta) = \hat{S}_L(\theta) \times \hat{S}_R(\theta) \quad \Rightarrow$$

Satisfies the Yang-Baxter eqs., unitarity, crossing and analyticity:

permutation

where

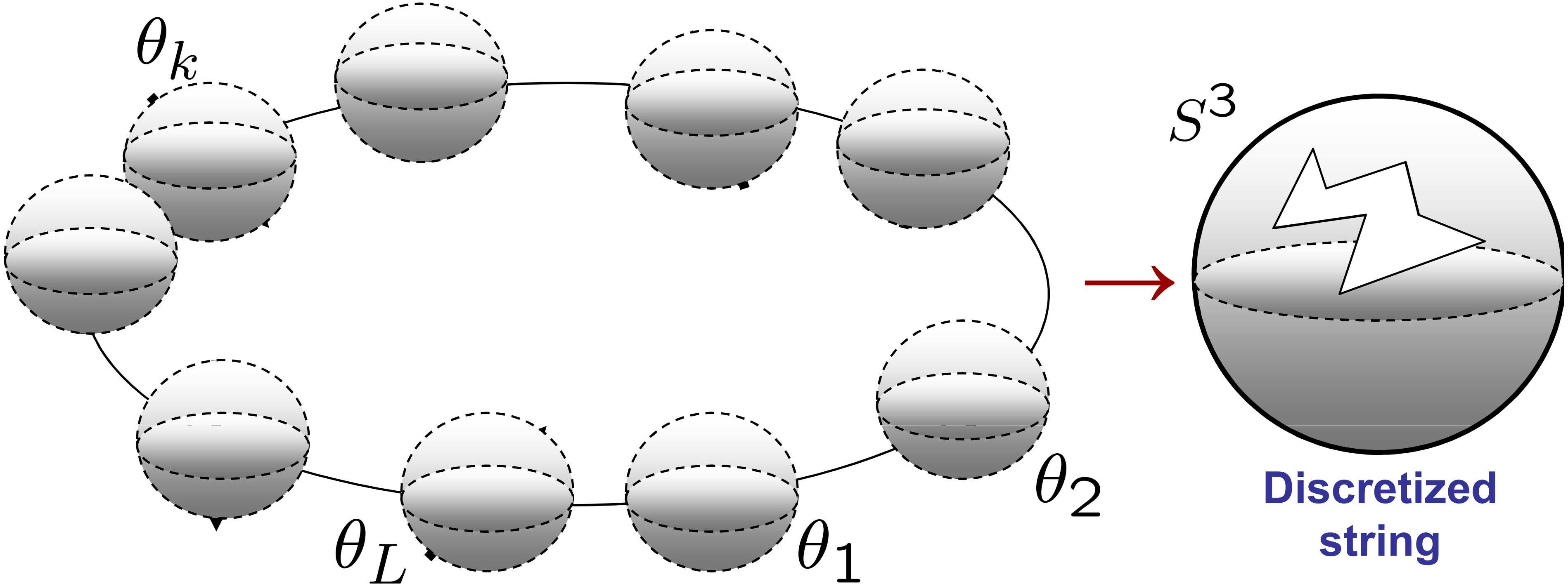
$$\hat{S}_{L,R}(\theta) = S_0(\theta) \frac{\theta - i\hat{\mathcal{P}}}{\theta - i}$$

$$S_0(\theta) = \frac{\Gamma\left(-\frac{\theta}{2i}\right)\Gamma\left(\frac{1}{2}+\frac{\theta}{2i}\right)}{\Gamma\left(\frac{\theta}{2i}\right)\Gamma\left(\frac{1}{2}-\frac{\theta}{2i}\right)} \quad \to \exp\left(-\frac{i}{\theta}\right), \qquad \theta \to \pm\infty$$

Al.&A.Zamolodchikov'79

«Coulomb» asymptotics

Particles on a ring as dynamical spin chain



$E = m_0 \ \cosh\theta$

$p = m_0 \ \sinh\theta$

• isotopic degrees of freedom: SO(4)~SU(2)xSU(2)

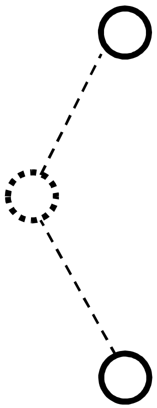
S-matrix $\quad \widehat{S}(\theta) = \widehat{S}_L(\theta) \times \widehat{S}_R(\theta)$

• Equivalent to σ-model on S^3, a subsector of superstring

- Periodicity condition defining the states:

$$e^{-im_0\mathcal{L}\sinh\pi\theta_\alpha}\ |\psi> = \prod_{\beta=\alpha+1}^{L}\ \hat{S}\left(\theta_\alpha-\theta_\beta\right)\prod_{\gamma=1}^{\alpha-1}\ \hat{S}\left(\theta_\alpha-\theta_\gamma\right)\ |\psi>$$

- Bethe equations (diagonalization of periodicity condition):



$$1 = \prod_{\beta}^{J_u}\frac{u_j-\theta_\beta-i/2}{u_j-\theta_\beta+i/2}\prod_{i\neq j}^{J_u}\frac{u_j-u_i+i}{u_j-u_i-i},$$

$$e^{-im_0\mathcal{L}\sinh\pi\theta_\alpha} = \prod_{\beta\neq\alpha}^{L} S_0^2\left(\theta_\alpha-\theta_\beta\right)\prod_{j}^{J_u}\frac{\theta_\alpha-u_j+i/2}{\theta_\alpha-u_j-i/2}\prod_{k}^{J_v}\frac{\theta_\alpha-v_k+i/2}{\theta_\alpha-v_k-i/2},$$

$$1 = \prod_{\beta}^{J_v}\frac{v_k-\theta_\beta-i/2}{v_k-\theta_\beta+i/2}\prod_{l\neq k}^{J_v}\frac{v_k-v_l+i}{v_k-v_l-i},$$

- θ-variables describe longitudinal motions of string, u,v “magnon” variables – the transverse.
- Nested and algebraic BA [Yang ’60’s,Sutherland’70’s,Leningrad school’80’s]

Conformal (classical) limit

In the classical (conformal) limit the dynamically generated mass is small

$$\mu \sim e^{-\frac{\sqrt{\lambda}}{2}} \to 0$$

$$\mu = m_0 \mathcal{L}$$

Rapidities and number of particles scale with the coupling

$$\theta_j \sim L \sim \sqrt{\lambda} \to \infty$$

Conformal (classical) limit for U(1) sector (no magnons)

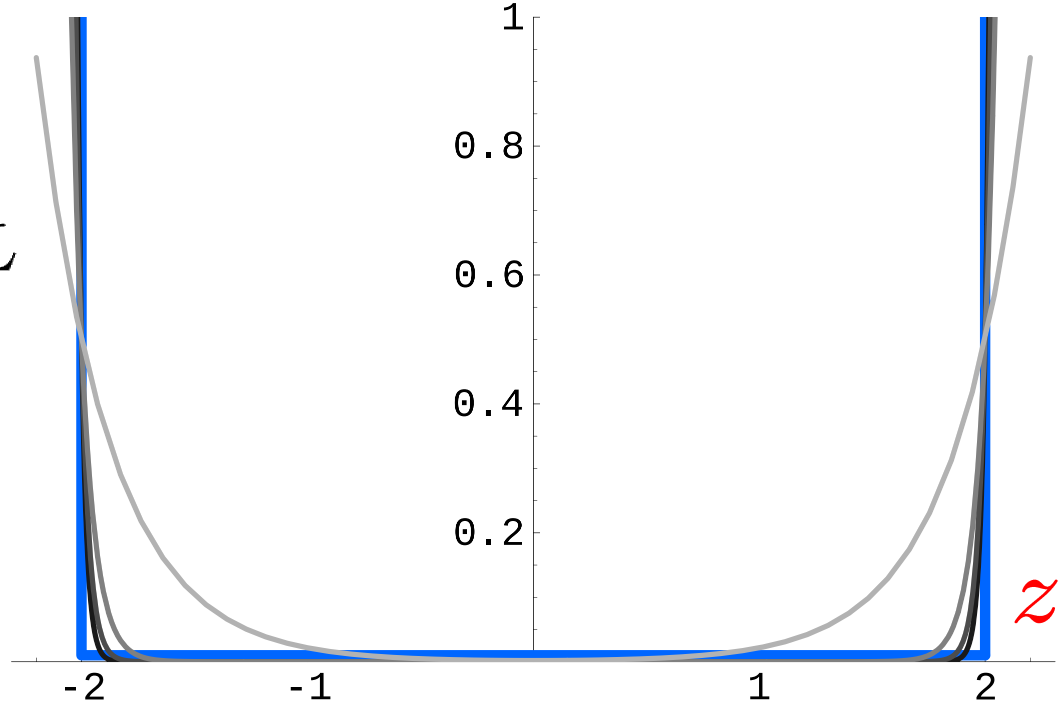
- For rescaled variable $z = \frac{4\pi}{\sqrt{\lambda}}\,\theta$ Bethe eqs. become

$$\mu \sinh\left(\frac{\sqrt{\lambda}}{4} z_\alpha\right) - 2\pi m_\alpha = -\frac{4\pi}{\sqrt{\lambda}} \sum_{\beta\neq\alpha}^{L} \frac{1}{z_\alpha - z_\beta}$$

2D Coulomb charges in potential $\mu \cosh \frac{\sqrt{\lambda}}{4\pi} z$

$$\mu \sim e^{-\frac{\sqrt{\lambda}}{2}} \to 0 \qquad \sqrt{\lambda} \sim L$$

Potential becomes a box



- No longitudinal motion:

$$m_1 = \cdots = m_L = m$$

Conformal (classical) limit for U(1) sector (no magnons)

- Bethe eqs. (similar to large N matrix model):

$$\mu \sinh\left(\frac{\sqrt{\lambda}}{4} z_\alpha\right) - 2\pi m_\alpha = -\frac{4\pi}{\sqrt{\lambda}} \sum_{\beta \neq \alpha}^{L} \frac{1}{z_\alpha - z_\beta}$$

for: $m_1 = \cdots = m_L = m$

becomes

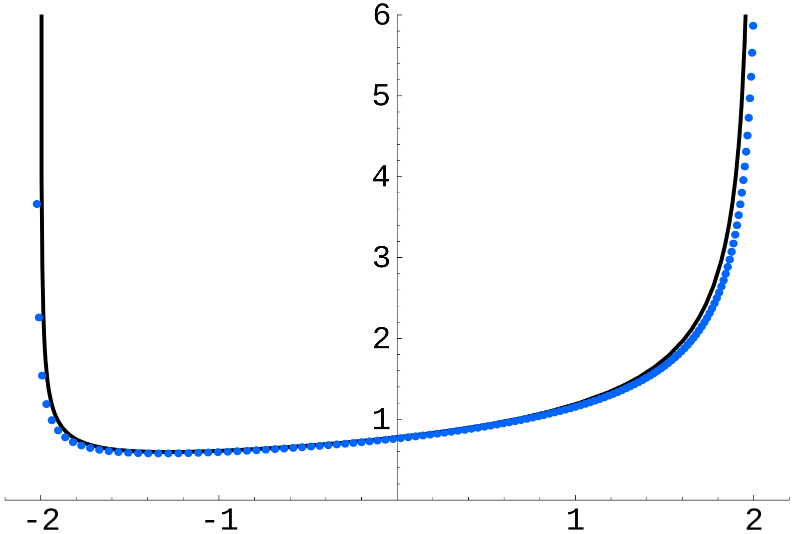
$$\not{G}(z) \equiv \frac{1}{2}\left(G(z+i0) + G(z-i0)\right) = -2\pi m, \qquad z \in \mathcal{C}_\theta = (-2, 2)$$

$$G(z) \equiv \sum_{\alpha=1}^{L} \frac{1}{\frac{\sqrt{\lambda}}{4\pi} z - \theta_\alpha} = \int_{-2}^{2} \frac{dz' \rho(z')}{z - z'} \to \not{G}(z) \pm i\pi\rho(z), \qquad z \in (-2, 2).$$

Classical limit of U(1) highest weight sector

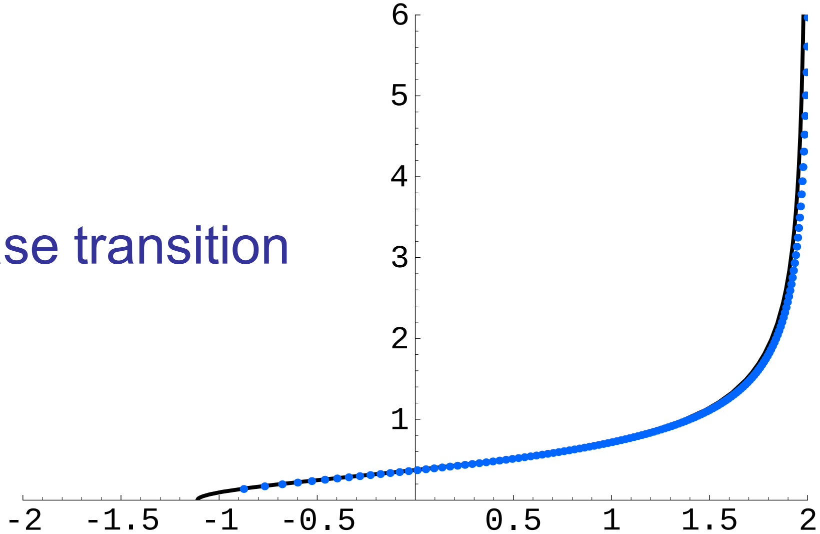
$$G = \frac{2\pi m \ z + \frac{4\pi L}{\sqrt{\lambda}}}{\sqrt{z^2 - 4}} - 2\pi m$$

$$G = 2\pi m \left(\frac{\sqrt{z - 2 + \frac{4L}{m\sqrt{\lambda}}}}{\sqrt{z-2}} - 1 \right)$$



$L > |m|\sqrt{\lambda}$

phase transition



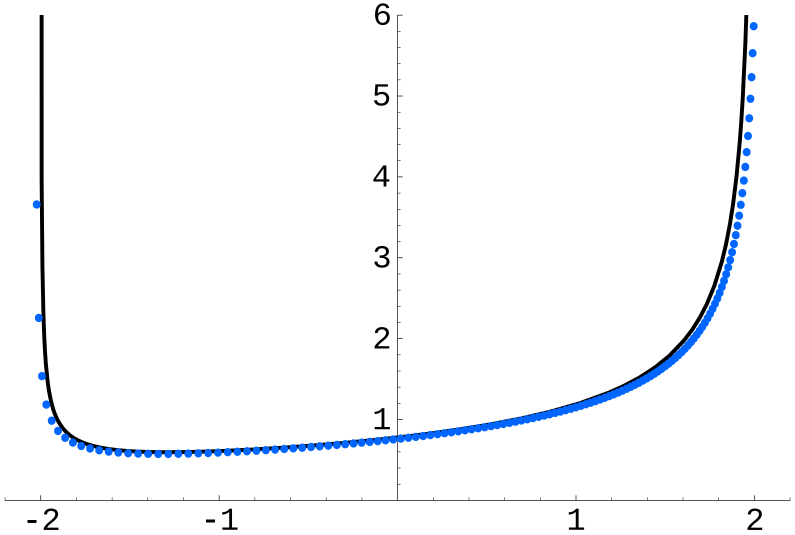
$L < |m|\sqrt{\lambda}$

- In terms of Zhukovsky variable $\quad x + \frac{1}{x} = z$

$$G\left(z(x)\right) = \frac{Ax + B}{x^2 - 1}$$

- BMN vacuum

Energy and Momentum



In general, we have

$$\rho(z) \equiv \frac{4\pi}{\sqrt{\lambda}} \sum \delta(z - z_\alpha) \simeq \frac{2\kappa_\pm}{\sqrt{z \mp 2}}$$

Energy and momenta can be read from these singularities,

$$E = \frac{\mu}{2\pi} \sum \cosh \pi\theta_\alpha$$
$$P = \frac{\mu}{2\pi} \sum \sinh \pi\theta_\alpha$$

$$E \pm P = \frac{\sqrt{\lambda}}{2\pi} \kappa_\pm^2$$

Derivation of asymptotic string Bethe ansatz

[Gromov,V.K.'06] [Arutynov,Frolov,Staudacher'04]

$$e^{-ip(\theta\alpha)} = \prod_{\beta(\neq\alpha)}^{L} S_0^2\left(\theta_\alpha - \theta_\beta\right) \prod_{j}^{J} \frac{\theta_\alpha - u_j + i/2}{\theta_\alpha - u_j - i/2},$$

$$1 = \prod_{\beta}^{L} \frac{u_j - \theta_\beta - i/2}{u_j - \theta_\beta + i/2} \prod_{k(\neq j)}^{J} \frac{u_j - u_k + i}{u_j - u_k - i}$$

Strategy: We solve the first eq. for θ's , exclude them from second eq. and obtain effective Bethe equation for the magnons

For Hubbard: [Rey,Serban,Staudacher'05]

- Long spin chain limit: $L \to \infty$
- Arbitrary number of magnons K
- θ's continuously distributed in a square box of size $\frac{\sqrt{\lambda}}{\pi}$.
- Impose a one cut distribution: analogue of Virasoro conditions.

Density of rapidities θ_α from our BAE's

Take log of first BAE and get a Riemann-Hilbert problem

$$\not{G}_\theta(z|\{u_j\}) + 2\pi m = i \sum_{j=1}^{K} \log \frac{z\frac{\sqrt{\lambda}}{4\pi} - u_j + i/2}{z\frac{\sqrt{\lambda}}{4\pi} - u_j - i/2}, \qquad |z| \leq 2$$

- Impose a one cut distribution: analogue of Virasoro conditions.

- Solution, in terms of Zhukovsky variables

$$z = x + 1/x$$
$$x \equiv \frac{1}{2}\left(z + \sqrt{z^2 - 4}\right)$$
$$y_j^{\pm} = x(u \pm i/2)$$

$$G\left(z(x)\right) = \frac{Ax + B}{x^2 - 1} + 2i \sum_{j=1}^{K} \log \frac{x - 1/y_j^+}{x - 1/y_j^-}$$

where:

$$A = \sum_{j=1}^{K} \left(\frac{2i}{y_j^+} - \frac{2i}{y_j^-}\right) + \frac{4\pi L}{\sqrt{\lambda}}, \qquad B = 4\pi m - 2i \sum_{j=1}^{K} \log \frac{y_j^+}{y_j^-}$$

- Exclude θ's from the 2-nd BAE $$\sum_{\beta=1}^{L} \log \frac{u_j - \theta_\beta - i/2}{u_j - \theta_\beta + i/2} = \dots$$

- We get

$$e^{i\mathrm{p}_k L} \equiv \left(\frac{y^+(u_k)}{y^-(u_k)}\right)^L = \prod_{j=1}^{J} \frac{u_k - u_j + i}{u_k - u_j - i}\, \sigma^2(u_k, u_j)$$

where

$$\sigma(u_k, u_j) = \frac{1 - 1/(y_j^+ y_k^-)}{1 - 1/(y_j^- y_k^+)} \left(\frac{(y_j^- y_k^- - 1)}{(y_j^- y_k^+ - 1)} \frac{(y_j^+ y_k^+ - 1)}{(y_j^+ y_k^- - 1)}\right)^{i(u_j - u_k)}$$

and

$$z = x + 1/x, \quad x(2\pi\sqrt{\lambda}\; z) \equiv \frac{1}{2}\left(z + \sqrt{z^2 - 4}\right), \quad y_j^\pm = x(u \pm i/2)$$

- We reproduced the conjectured AFS formula for string within SU(2) sector (operators with only Z,X scalar fields in SYM).

Dispersion relation

[Beisert,Dippel,Staudacher'04]

- From Bethe equation $$p = -i \log \frac{y^+}{y^-}$$

- Zero momentum condition imposed: $$\prod_j \frac{y_j^+}{y_j^-} = e^{-i\sum_j p_j} = 1$$

- Then the dispersion relation

$$\Delta \; = \; L + i\frac{\sqrt{\lambda}}{2\pi}\sum_{j=1}^{J}\left(\frac{1}{y_j^+} - \frac{1}{y_j^-}\right)$$

can be recast as

$$\Delta \; = \; L + \sum_{j=1}^{J}\left(\sqrt{1 + \frac{\lambda}{\pi^2}\sin^2\left(p_j/2\right)} - 1\right)$$

Energy (Dimension) is expressed trough the magnon momentum.
For small p the formula looks as a relativistic dispersion relation.
The periodicity w.r.t. the momentum is natural for the spin chains

*

Classical limit

- Taking $$L, J_u, J_v \sim \sqrt{\lambda}$$ (we restored J_v)

we get, to leading order in λ

$$\pi n_k = \frac{\frac{L}{2M}y_k + 2\pi m}{1 - y_k^2} + \frac{1}{y_k^2 - 1}\frac{1}{M}\sum_{l=1}^{J_v}\frac{1}{1/y_k - \tilde{y}_l} + \frac{y_k^2}{y_k^2 - 1}\frac{1}{M}\sum_{j\neq k}^{J_u}\frac{1}{y_k - y_j}$$

Then, defining

$$p(x) = \frac{\frac{L}{2M}x + 2\pi m}{1 - x^2} + \frac{1}{x^2 - 1}\frac{1}{M}\sum_{j=1}^{J_v}\frac{1}{1/x - \tilde{y}_j} + \frac{x^2}{x^2 - 1}\frac{1}{M}\sum_{j=1}^{J_u}\frac{1}{x - y_j}$$

we can see that p(x) defines a hyperelliptic algebraic curve of the classical KMMZ equations!

[V.K.Marshakov,Minahan,Zarembo'04]

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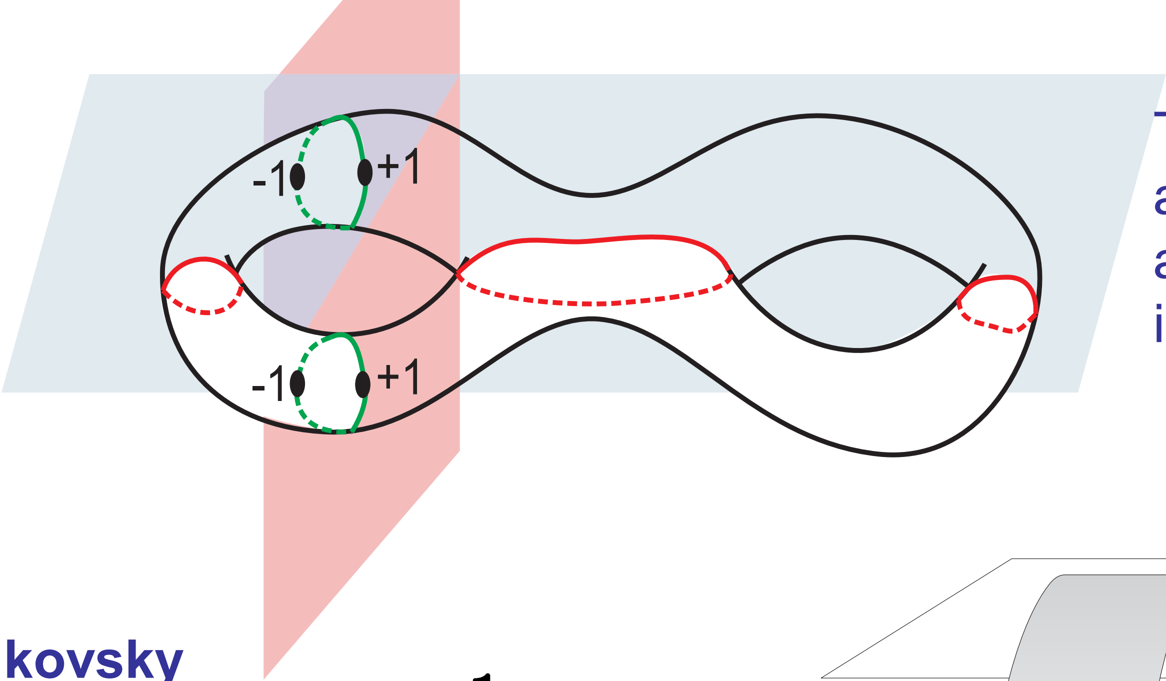
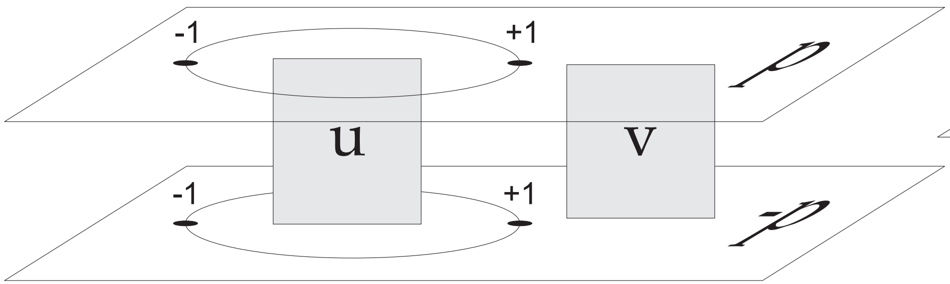
The geometric picture



The θ cuts we saw today are mapped to the poles at x=±1 we saw before in the finite gap method!

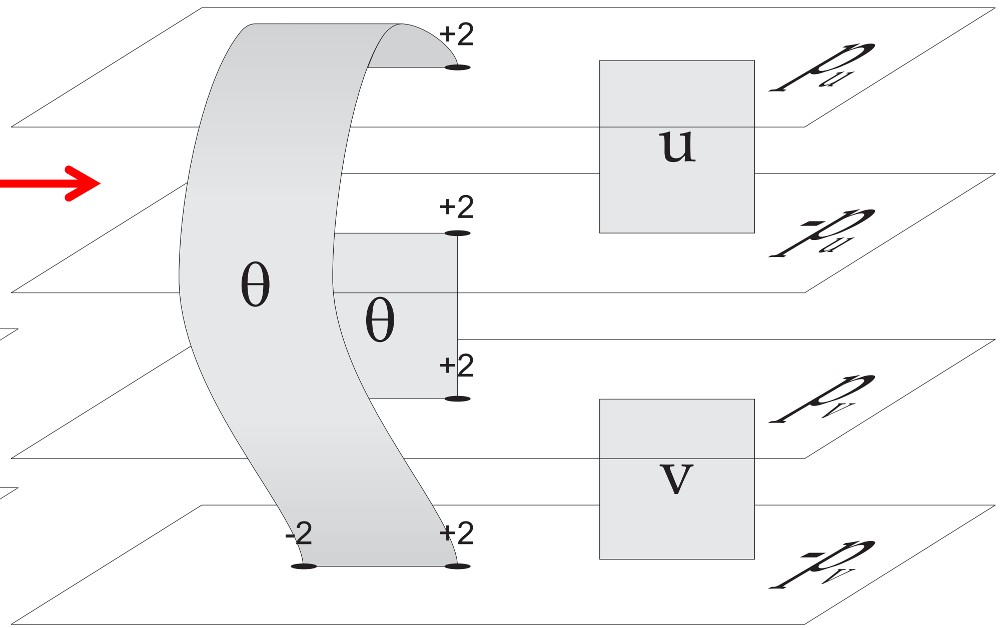
Zhukovsky map:

$$x + \frac{1}{x} = z$$



From classical finite gap

[V.K.Marshakov,Minahan,Zarembo'04; Minahan'05]



From Bethe ansatz

Conclusions and Problems

- Dimensions of SYM operators (= Energy of string) encoded into finite gap algebraic curve (same approximation as for amplitudes of F.A.)

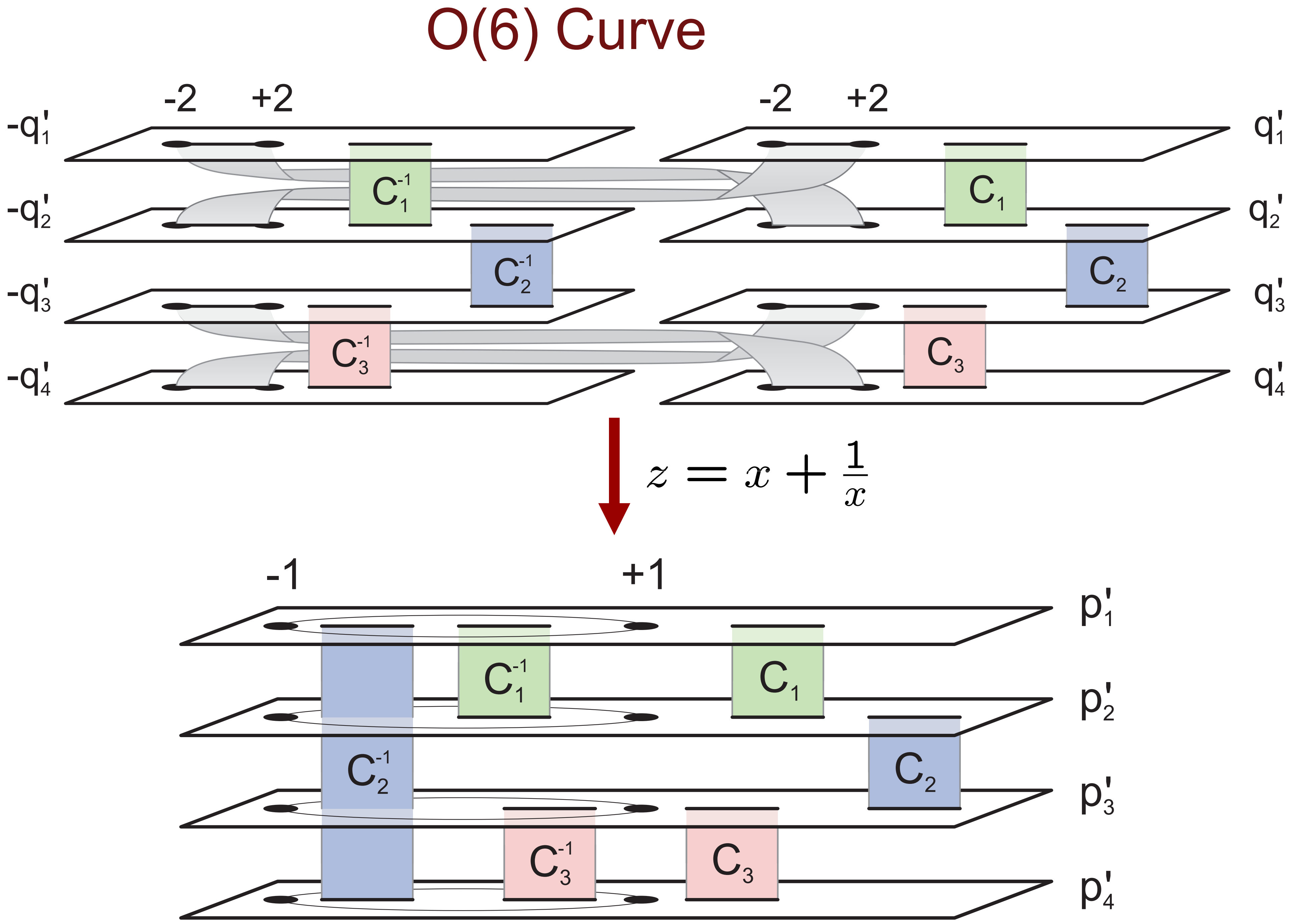
- Bohr-Sommerfeld quantisation (one loop) is relatively straightforward. Fermoins included at one loop. The curve describes their condensates.

- Classical limit: Roots of Bethe equations condense into cuts and this renders again the finite gap solution. Bethe eqs. – discretized algebraic curve.

- How to quantize the algebraic curve in a "scientific" way? At least regular loop expansion?

- Find classical hidden variables simplifying the curve (lifting the $x \to 1/x$ symmetry).

• END


O(6) Curve
-2
+2
-2
+2
-q'1
-q'2
-q'3
-q'4
q'1
q'2
q'3
q'4
C1-1
C2-1
C3-1
C1
C2
C3
z = x + 1/x
-1
+1
p'1
p'2
p'3
p'4
C1-1
C1
C2-1
C2
C3-1
C3