Partons and jets at strong coupling (I)

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Outline

- Lecture I : Partons and jets in QCD at weak coupling

  (the benchmark for comparing with strong–coupling results from AdS/CFT)

  - Introduction & Motivations (why study finite–temperature and/or high–energy problems at strong coupling ?)
  - The situation at weak coupling (pQCD, phenomenology)

- Lecture II : A high–energy current in AdS/CFT
  - Methodology (black hole, wave equations)
  - The vacuum problem as a warm up

- Lecture III : $\mathcal{R}$–current in a strongly–coupled plasma
  - Results & Physical discussion
  - General consequences for high–energy scattering

- *Original part based on work in collaboration with Yoshitaka Hatta and Al Mueller (arXiv:0710.2148, 0710.5297, and 0803.2481)*

AdS/CFT : General introduction

- String theory methods for strongly–coupled gauge theories
- String theory in the posture of an 'epicycle' (J. Ambjorn), a tool, a non–perturbative representation for the gauge theory, particularly suitable for the strong–coupling problem
- Proving, or falsifying, string theory is here not an issue
- Rather, some real issues are
  - how efficiently can we make use of this tool ?
  - what is its widest field of application ?
  - and what are the limitations ?

AdS/CFT : General introduction

- A conjecture, most firmly established for very special, 'maximally supersymmetric and conformal', gauge theories
  - no confinement, no asymptotic freedom, no asymptotic states, no fundamental fermions ...
  - pretty far away from day-to-day QCD
- Essentially, all the calculations to date refer to the large–N_c, or supergravity, approximation
- How to go beyond these limitations ?
  - get closer to QCD
  - perform more accurate calculations (beyond large N_c)
- How to efficiently make use of what we know already ?

Introduction

- QCD at finite temperature (but not too high !) :

  $T \; = \; 2 \div 5 \, T_c$ with $T_c \sim 200$ MeV (deconfinement)

  ▷ particularly promising playground for AdS/CFT techniques

- ... and also a very important one

  ▷ possibly connected to real world

- Strong indications from different sources

  - experimental results for heavy ion collisions at RHIC
  - lattice QCD
  - problems with perturbation theory at finite temperature

  ... that the relevant coupling is quite strong

  ('strongly–coupled quark–gluon plasma', or sQGP)

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- Some potential drawbacks of AdS/CFT
  - conformal symmetry
  - lack of confinement

  ... are presumably less important in this particular regime (deconfined phase, small 'trace anomaly')

Introduction

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- Minimal formulation of AdS/CFT at finite temperature

  ◆ $\mathcal{N} = 4$ SYM at finite $T \longleftrightarrow AdS_5 \times S^5$ Black Hole

  ▷ unambiguous 'first-principle' calculations

  (no model–dependent 'deformations' : IR cutoff, D–Branes)

  ▷ strong–coupling limit $\lambda \to \infty$: 'supergravity'

  (relatively simple calculations)

Introduction

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- The simplest technical and conceptual context to study the problem of high–energy scattering at strong coupling
  - the $\mathcal{N} = 4$ SYM plasma : the 'simplest' target at strong coupling
  - similar results for 'hadronic' targets require more work and more modeling

Introduction

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  $T \ = 2 \div 5\, T_c$ with $T_c \sim 200$ MeV (deconfinement)

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- ... and also a very important one

  ▷ possibly connected to real world

- Some very robust results/physical scenarios
  - viscosity/entropy ratio
  - absence of ‘quasiparticles’ (resonances)
  - trailing string, limiting velocity, ...
  - quasi–democratic branching

  ▷ universality

  (similar results for all theories with known holographic dual)

Introduction

- Interesting conceptual questions
  - onset of hydrodynamic behaviour
  - approach to thermal equilibrium
  - degrees of freedom in a strongly–coupled plasma (thermodynamics, transport coefficients, hydro)
  - scattering off a strongly–coupled plasma
- ... some of which are easier to think of at strong coupling
  - a ‘perfect fluid’ is a strongly–coupled one !
- Only one intrinsic momentum scale (at equilibrium) : T

  ... as compared to a hierarchy of scales at weak coupling:

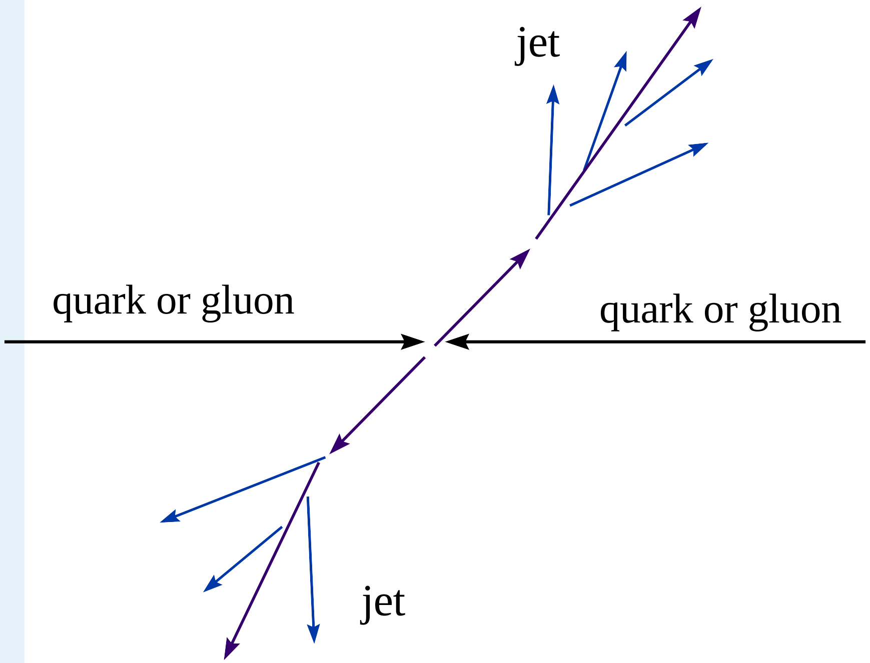
$$T \gg gT \gg g^2T$$

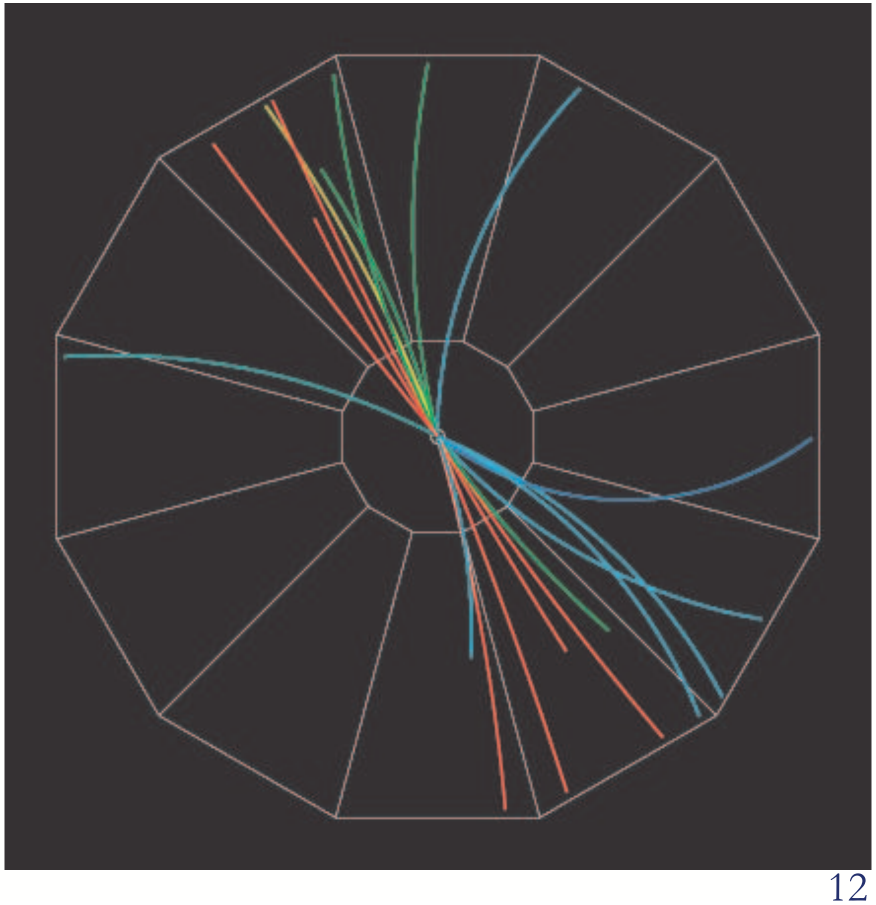
Introduction

- Typical space–time scales
  - hydrodynamics: large space–time separations $\gg 1/T$
  - thermodynamics, quasiparticles: $\sim 1/T$
  - high–energy scattering: $\ll 1/T$
- With reference to QCD, strong–coupling methods are expected to work better for large distances $\gtrsim 1/T$
  - hydrodynamics, thermodynamics, transport coefficients, ... quasiparticles
  - such topics are addressed by R. Peschanski and D. Son
- However, the experimental results for heavy–ion collisions mostly refer to 'hard probes'

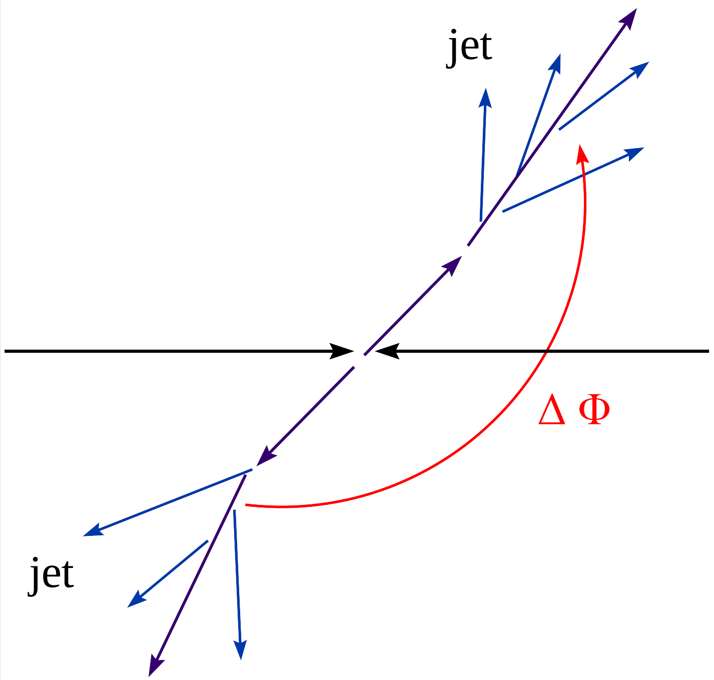
  'hard' = high energy, momentum, virtuality ... $\gg T$
- This will be the general topics of this set of lectures

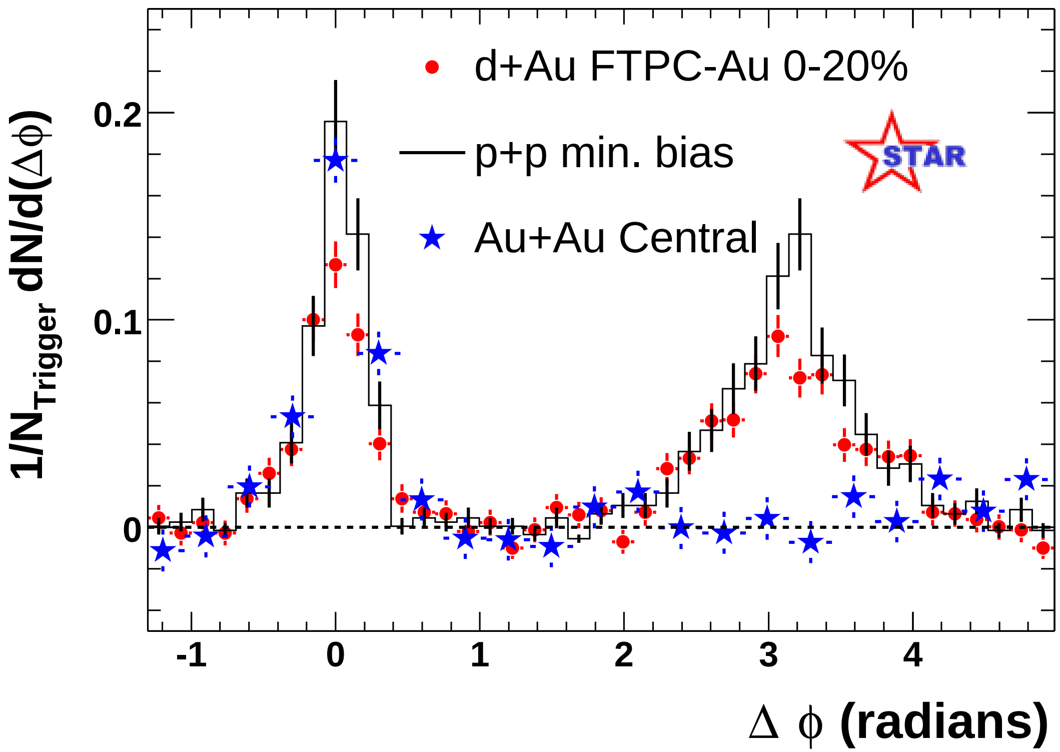
Jets in proton–proton collisions





Jets in proton–proton collisions

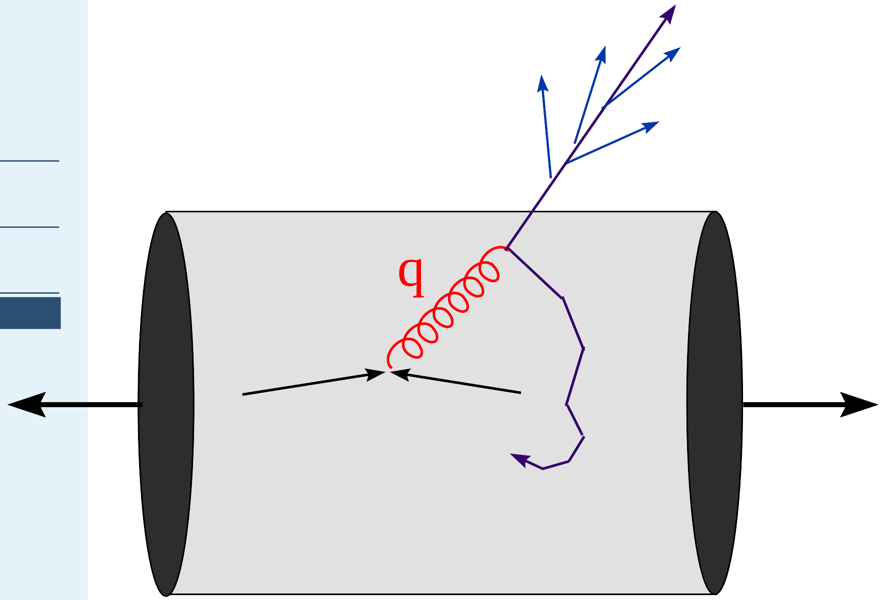


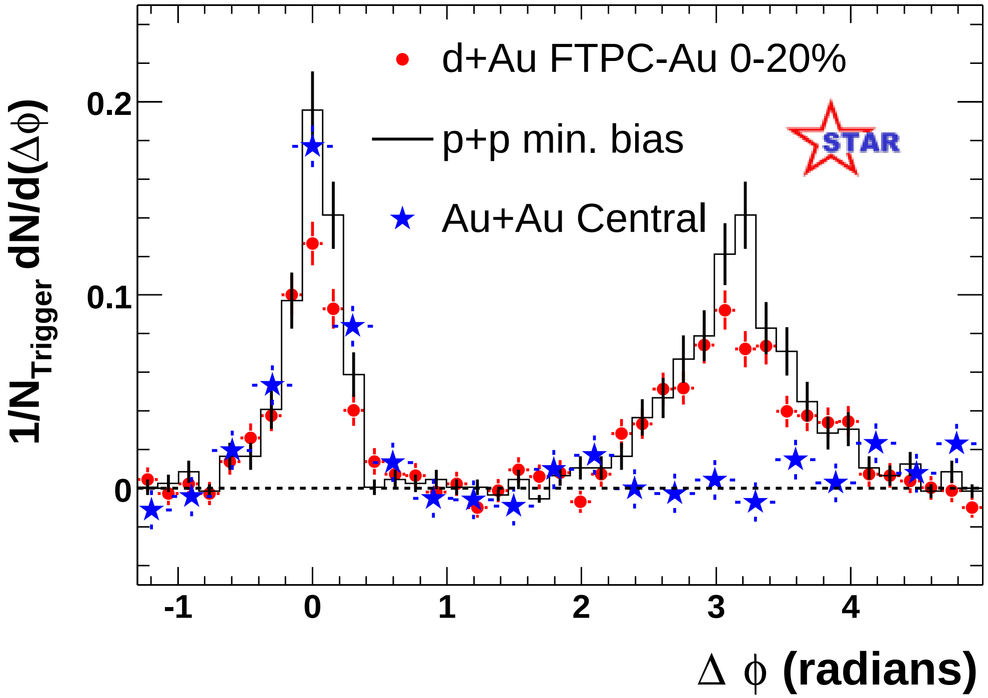


- Azimuthal correlations between the produced jets:

  a peak at $\Delta\Phi = 180°$

Nucleus–nucleus collision: Jet quenching



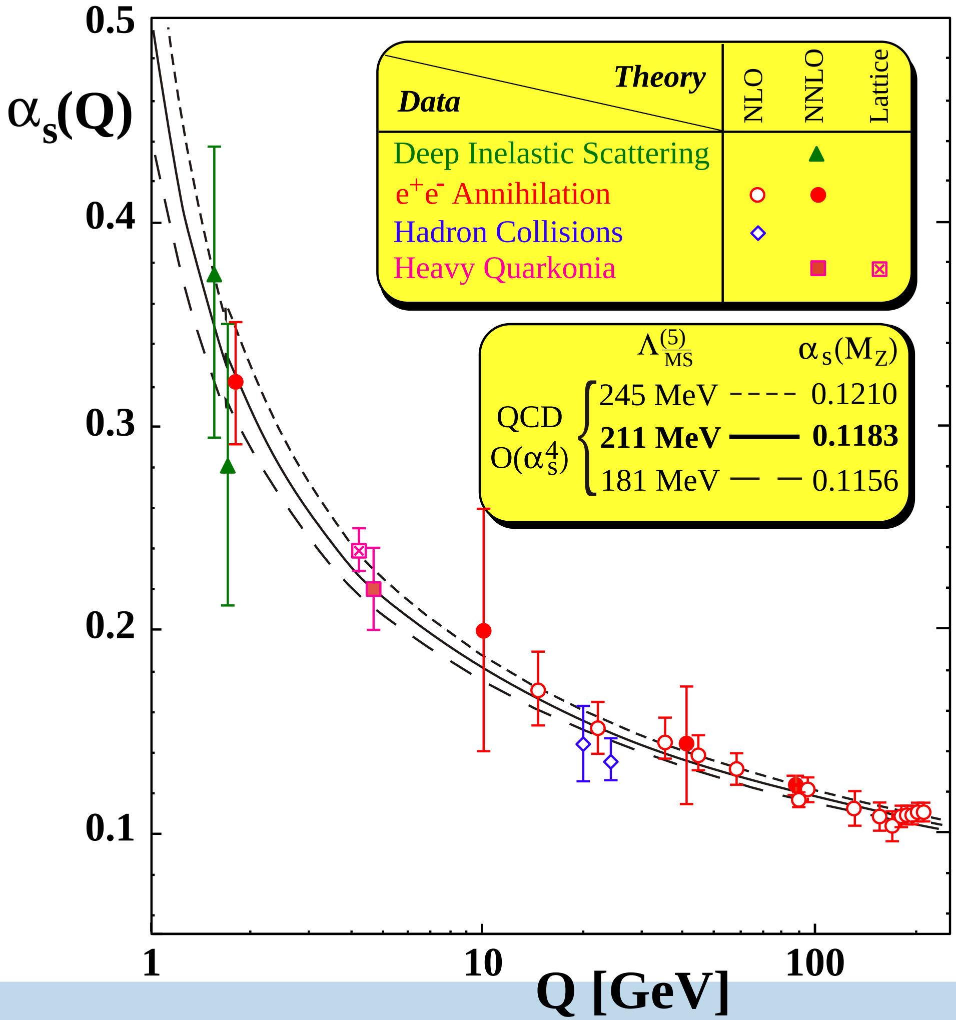


- The "away–side" jet has disappeared !
  absorbtion (or energy loss, or "jet quenching") in the medium
- The matter produced in a heavy ion collision is opaque
  high density, strong interactions, ... or both

The QCD running coupling

- What is the relevant value of the QCD running coupling ?

$$\alpha_s(Q) \propto \frac{1}{\ln\left(Q^2/\Lambda^2_{\rm QCD}\right)} \quad \text{avec} \quad \Lambda_{\rm QCD} \sim 200\,\text{MeV}$$



The QCD running coupling

- The first Matsubara frequency : $Q = 2\pi T \simeq 2 \div 6$ GeV

$$g(4\,\mathrm{GeV}) \simeq 1.5 \implies \lambda \equiv g^2 N_c \simeq 7 \gg 1$$

... but $\alpha_s \equiv g^2/4\pi \simeq 0.25 \ll 1$

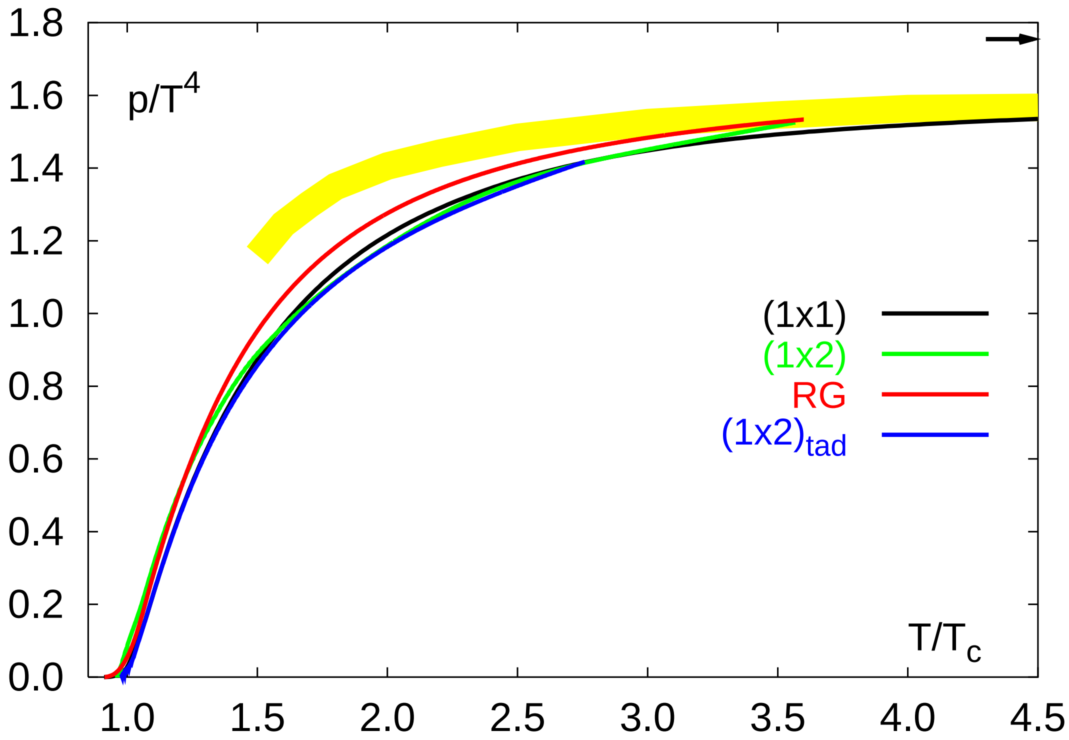
- For $\alpha_s \simeq 0.25$, perturbative QCD for vacuum processes (collider physics) works remarkably well !
- Medium effects can dramatically change the situation !
- What can we learn from the 'data' (RHIC/lattice QCD) ?
  - hydrodynamics : the most convincing evidence so far in favor of a strong–coupling like behaviour

    (elliptic flow, rapid thermalisation; cf. R. Peschanski)
  - thermodynamics (lattice) : inconclusive (see below)

    (unambiguous results, but ambiguous interpretation)
  - hard probes : unclear (in spite of some contrary claims !)

Lattice QCD at finite T

- Pressure from lattice QCD (Bielefeld Coll.)

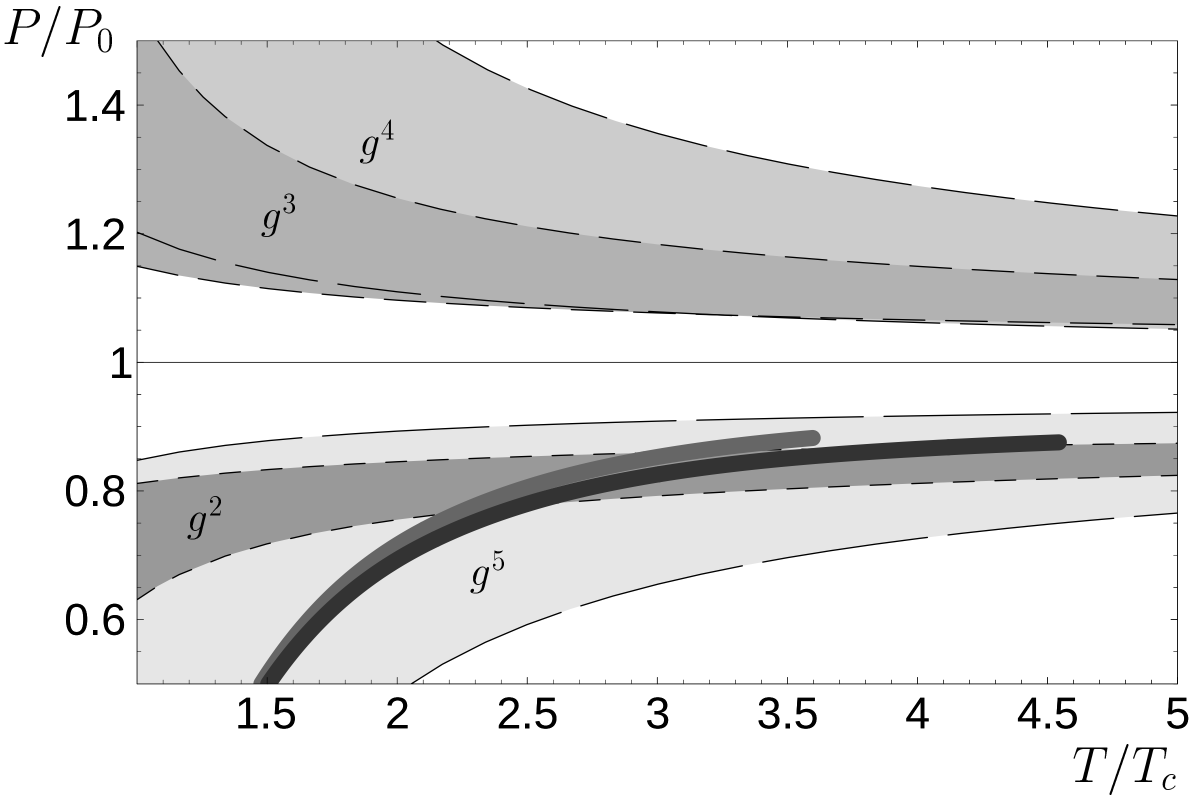


$P/P_0 \approx 0.85$ for $T = 3T_c$

- Is this deviation from P_0 small ? Or is it large ?
- AdS/CFT : $S/S_0 \to 3/4$ when $\lambda \to \infty$ ($\mathcal{N} = 4$ SYM)

Perturbation theory at finite T

- Perturbative expansion: a series in powers of g (not α_s !)



- No convergence until astronomically high temperatures ($T \sim 10^7$ GeV) !

Perturbation theory to order g^5

- Perturbative series in $g^2\phi^4$ scalar field theory :

$$P = \frac{\pi^2}{90}T^4\left[1 - \frac{15}{8}\left(\frac{g}{\pi}\right)^2 + \frac{15}{2}\left(\frac{g}{\pi}\right)^3 + \frac{135}{16}\left(\log\frac{\bar{\mu}}{2\pi T} + 0.4046\right)\left(\frac{g}{\pi}\right)^4 \right.$$
$$\left. - \frac{405}{8}\left(\log\frac{\bar{\mu}}{2\pi T} - \frac{4}{3}\log\frac{g}{\pi} - 0.9908\right)\left(\frac{g}{\pi}\right)^5 + \mathcal{O}(g^6\log g)\right],$$

(pure Yang–Mills, for definiteness: $N_f = 0$)

- QCD: Higher orders turns to be 'non–perturbative' (infinitely many diagrams contribute at a given order g^n with $n \geq 6$)

- Expansion in powers of g ! (rather than α_s/π)

Perturbation theory to order g^5

- Perturbative series in $g^2\phi^4$ scalar field theory :

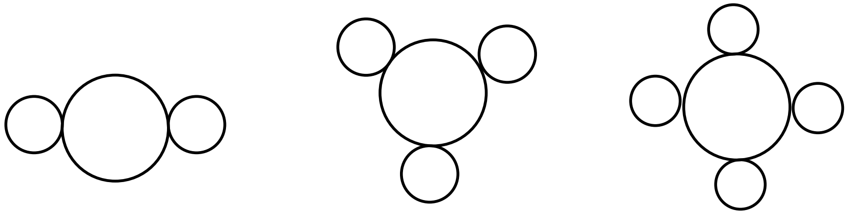
$$P = \frac{\pi^2}{90}T^4\Bigg[1 - 0.60\,g^2 + 0.24\,g^3 + 0.09\left(\log\frac{\bar\mu}{2\pi T} + 0.4046\right)g^4$$
$$- 0.16\left(\log\frac{\bar\mu}{2\pi T} - \frac{4}{3}\log\frac{g}{\pi} - 0.9908\right)g^5 + \mathcal{O}(g^6\log g)\Bigg]\,,$$

(pure Yang–Mills, for definiteness: $N_f = 0$)

- QCD: Higher orders turns to be ‘non–perturbative’ (infinitely many diagrams contribute at a given order g^n with $n \geq 6$)
- Expansion in powers of g ! (rather than α_s/π)
- Reasonable values for the coefficients, $c_i \lesssim \mathcal{O}(1)$, but $g > 1$

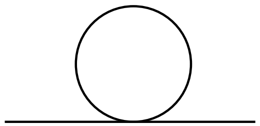
Ring diagrams

- Expansion in powers of g ! (rather than α_s/π) : ring diagrams



$$T\sum_n \int \frac{d^3k}{(2\pi)^3} \frac{1}{[(2\pi nT)^2 + k^2]^2} = \int \frac{d^3k}{(2\pi)^3} \frac{1}{k^3} \frac{1}{e^{\beta k} - 1} \sim \frac{T}{m_D} \sim \frac{1}{g}$$

- Strong sensitivity to infrared, due to Bose–Einstein statistics

$m_D^2 =$  $= g^2T^2$ Debye, or 'screening', mass

- Medium effects drastically affect pert. th. $\Longrightarrow$ resummations

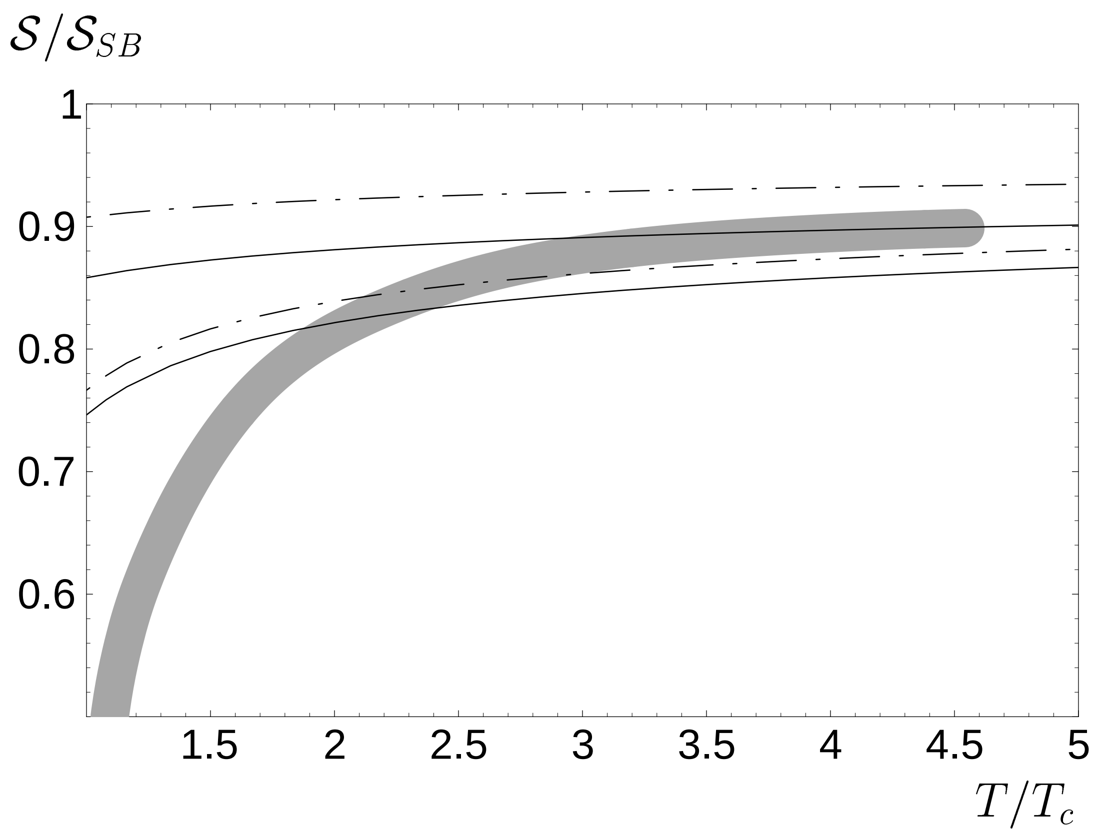
use 'dressed' propagators which include thermal masses

Motivation: Resummed perturbation theory

- Resummation of perturbation theory for QCD at finite T

  (J.-P. Blaizot, A. Rebhan, E. I, 2000)

- '2PI approximation' : thermal masses & screening

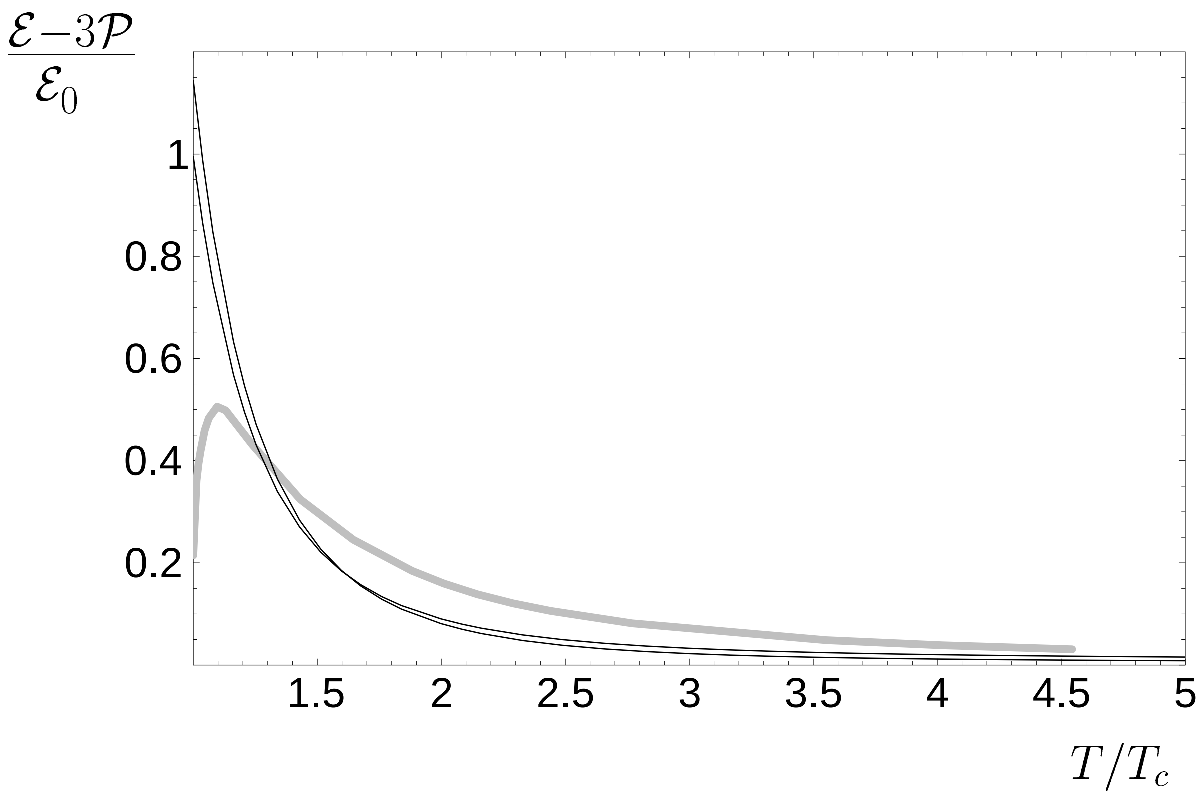


- Very good agreement ! ... but difficult to render systematic

Motivation: The trace anomaly

- Trace anomaly at finite T :

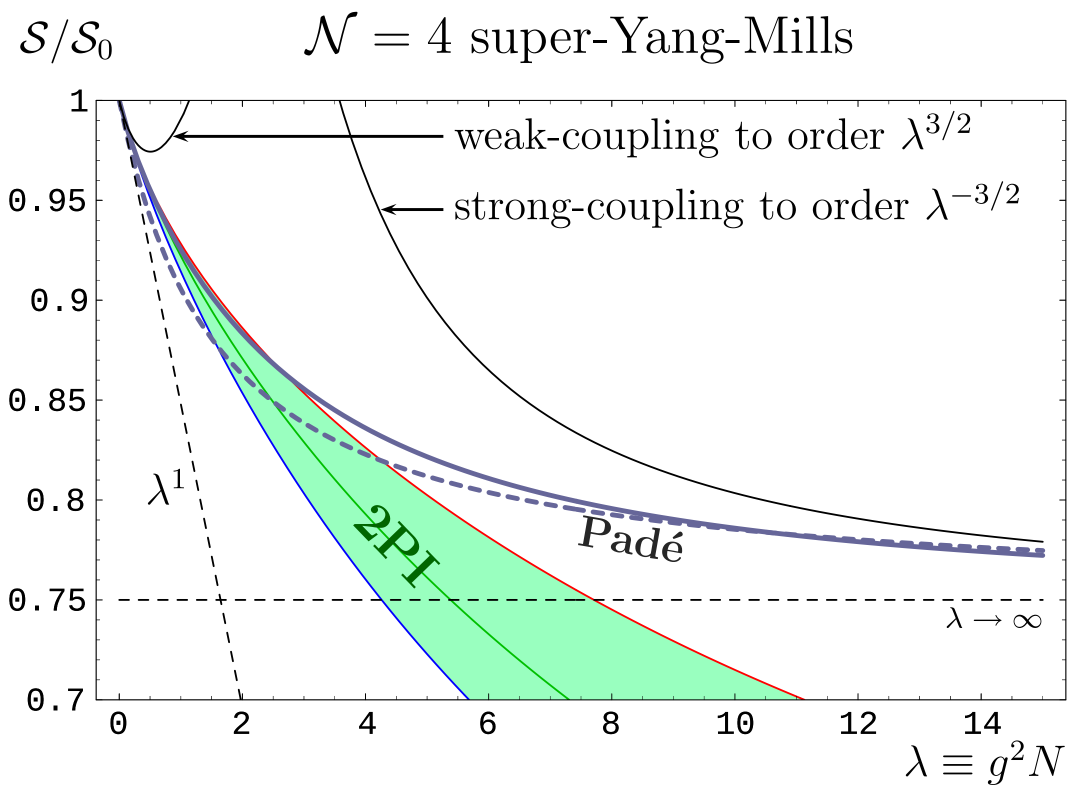
  Lattice QCD vs. resummed perturbation theory



$$\beta(g)\,\frac{\mathrm{d}P}{\mathrm{d}g} = \langle T^{\mu}_{\mu}\rangle = E - 3P$$

$\mathcal{N} = 4$ SYM plasma: weak vs. strong coupling

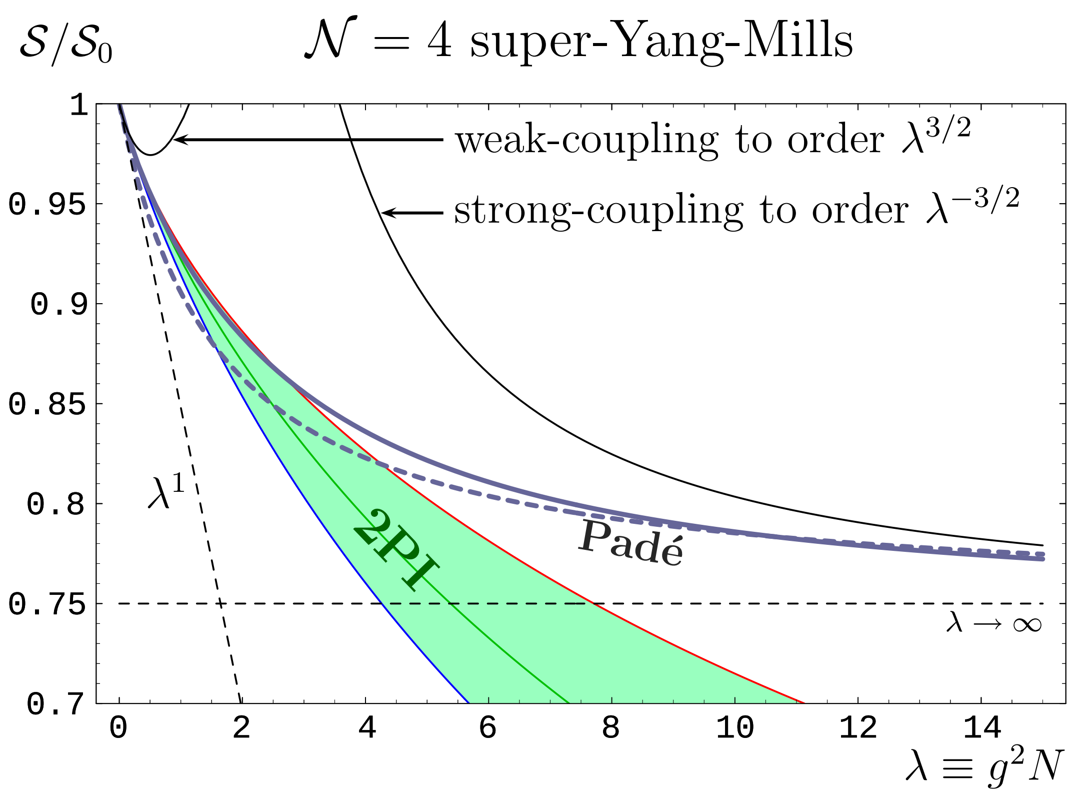
- Weak–coupling to $\mathcal{O}(\lambda^{3/2})$, strong–coupling to $\mathcal{O}(\lambda^{-3/2})$
- Very bad convergence either way !



- Unique Padé approximant *(J.-P. Blaizot, A. Rebhan, E. I., 06)*
- $S/S_0 = 0.85$ corresponds to intermediate coupling: $\lambda \simeq 4$

$\mathcal{N} = 4$ SYM plasma: weak vs. strong coupling

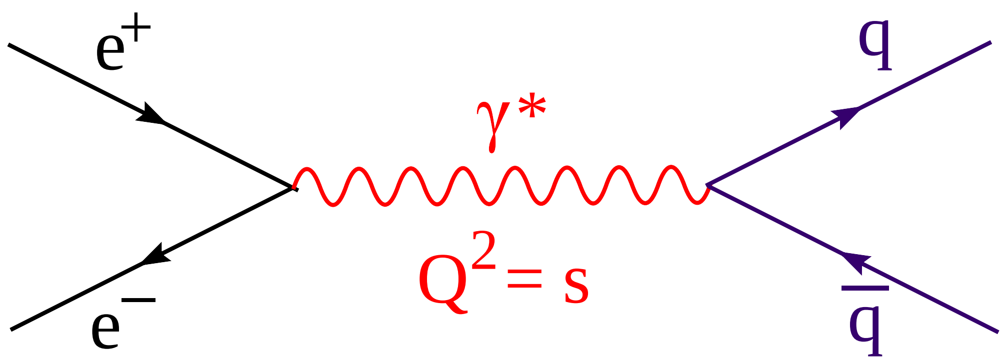
- Resummed perturbation theory does a good job in the domain where $S/S_0 = 0.85$



- $\mathcal{N} = 4$ SYM plasma: A convenient theoretical laboratory to study weak vs. strong coupling methods

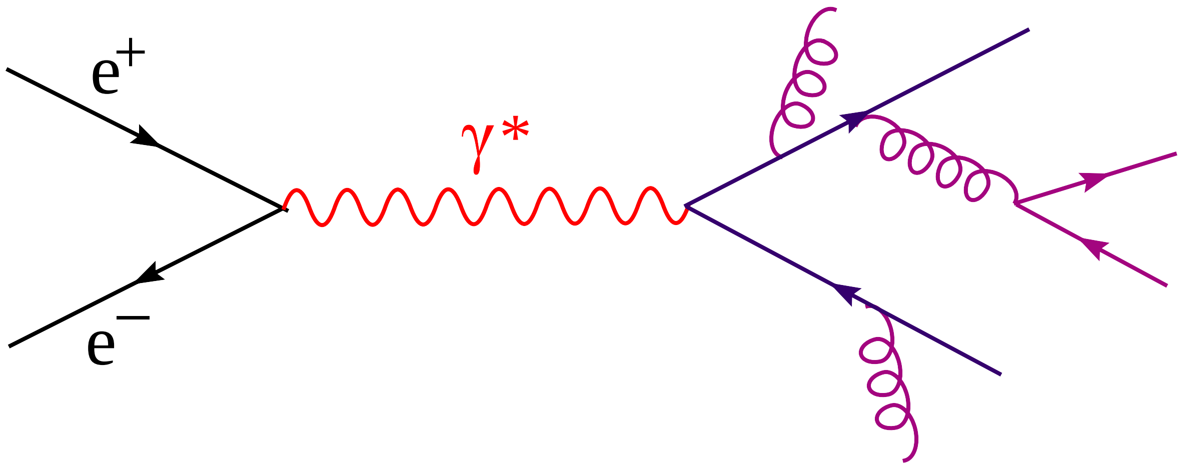
e^+e^- annihilation

- Lowest–order in perturbative QCD: $e^+e^- \rightarrow \gamma^* \rightarrow q\bar{q}$



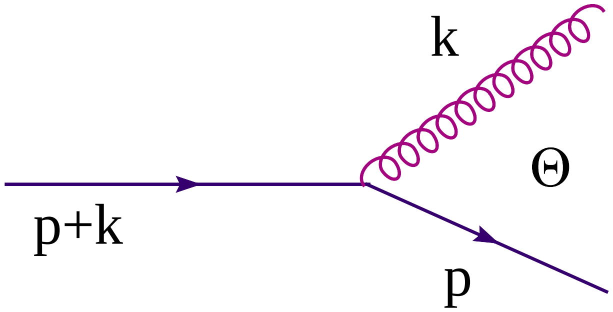
- A time–like current $(Q^2 = s > 0)$ decaying into a $q\bar{q}$ pair
- Center of mass frame : a pair of back–to–back 'jets'
- Bare partons cannot appear in the final state (confinement)
- The structure of the final state is determined by
    - parton branching
    - hadronisation

Parton branching: time–like cascade



- ‘Formation time’ (it takes some time to emit a gluon !)

Parton branching: time–like cascade



$$p \cdot k = kE\,(1 - \cos\Theta)$$

$$k_\perp \approx k\,\Theta$$

- Same as the lifetime of the virtual parent quark $(p+k)$

$$t_{\rm form} \sim \frac{1}{M_{\rm virt}}\frac{E}{M_{\rm virt}} \sim \frac{E}{(p+k)^2} \sim \frac{E}{kE\Theta^2} \sim \frac{k}{k_\perp^2}$$

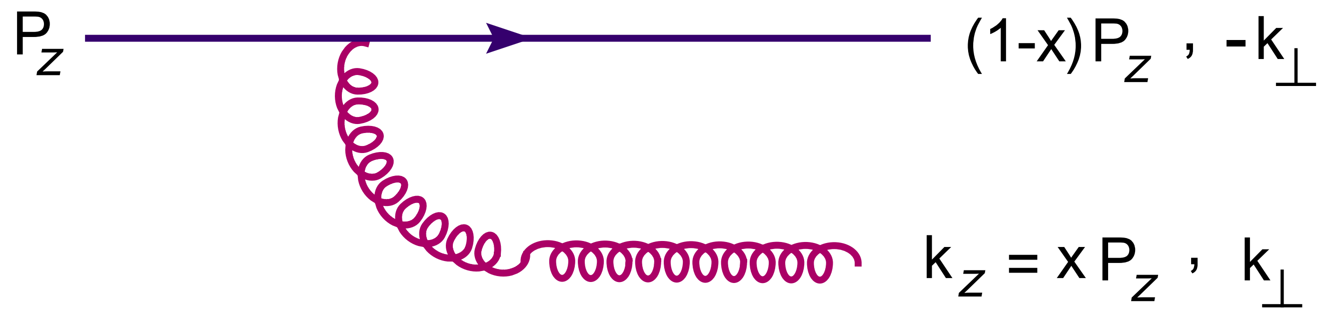
- Hadronisation time : $t_{\rm hadr} \sim t_{\rm form}$ with $k_\perp \sim \Lambda_{\rm QCD}$
- Partonic picture applies over a relatively large period of time:

$$t_{\rm coll} \sim \frac{1}{\sqrt{s}} < t < t_{\rm hadr} \sim \frac{\sqrt{s}}{\Lambda_{\rm QCD}^2}$$

- The strength of the coupling plays no role for this argument !

Parton branching at weak coupling

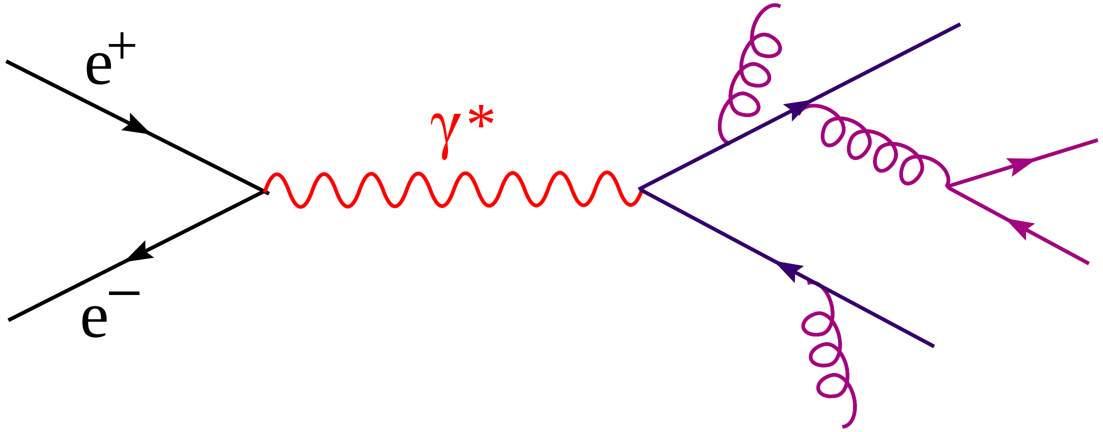
- Gluon emission to lowest order in perturbative QCD:



$$\mathrm{d}\mathcal{P}_{\rm Brem} \sim \alpha_s(k_\perp^2)\, N_c \,\frac{\mathrm{d}^2 k_\perp}{k_\perp^2}\,\frac{\mathrm{d}x}{x}$$

- Phase–space enhancement for the emission of
  - collinear $(k_\perp \to 0)$
  - and/or soft $(x \to 0)$ gluons
- Generic for a theory with dimensionless coupling and massless vector bosons

Jets



- 'Multi–jet event' : large emission angle & $x \sim \mathcal{O}(1)$

$$k_\perp \sim k \sim \sqrt{s} \implies \mathcal{P}_{\mathrm{Brem}} \sim \alpha_s(s) \ll 1$$

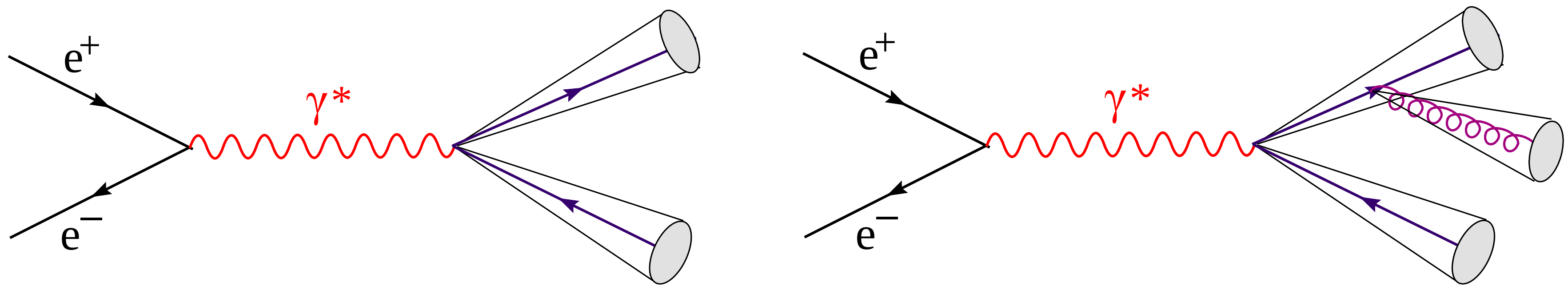
small probability for emitting an extra gluon jet !

- 'Intra–jet activity' : collinear and/or soft gluons

$$\Lambda_{\mathrm{QCD}} \ll k_\perp \ll k \ll \sqrt{s} \implies \mathcal{P}_{\mathrm{Brem}} \sim \alpha_s \ln^2 \frac{\sqrt{s}}{\Lambda_{\mathrm{QCD}}} \sim \mathcal{O}(1)$$

modifies particle multiplicity but not the number of jets

Final state



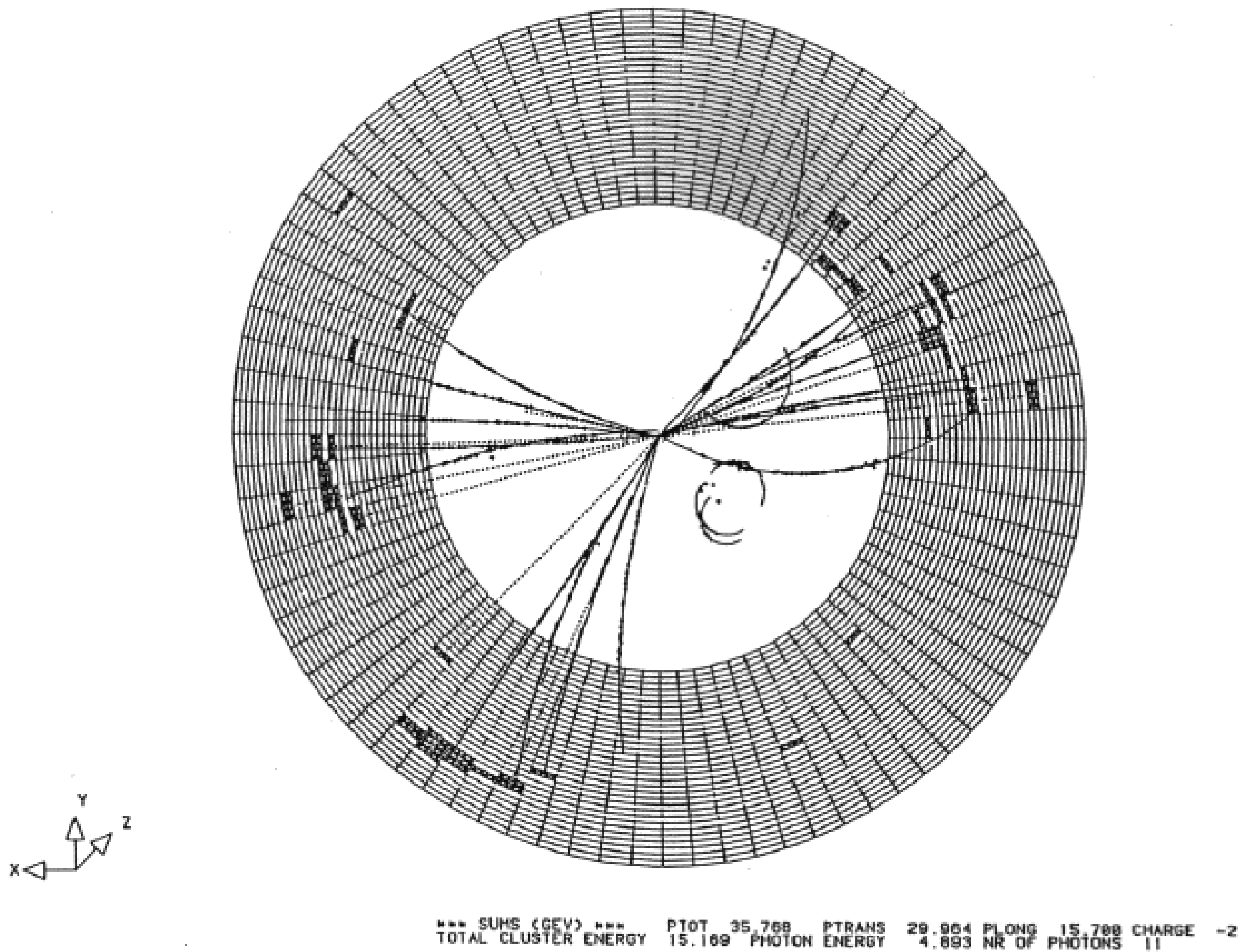
- Few, well collimated, jets
- e^+e^- cross–section computable in perturbation theory

$$\sigma(s) = \sigma_{\mathrm{QED}} \times \left(3\sum_f e_f^2\right)\left(1 + \frac{\alpha_s(s)}{\pi} + \mathcal{O}(\alpha_s^2(s))\right)$$

σ_{QED} : cross–section for $e^+e^- \to \mu^+\mu^-$

- No logs: collinear and infrared singularities mutually cancel

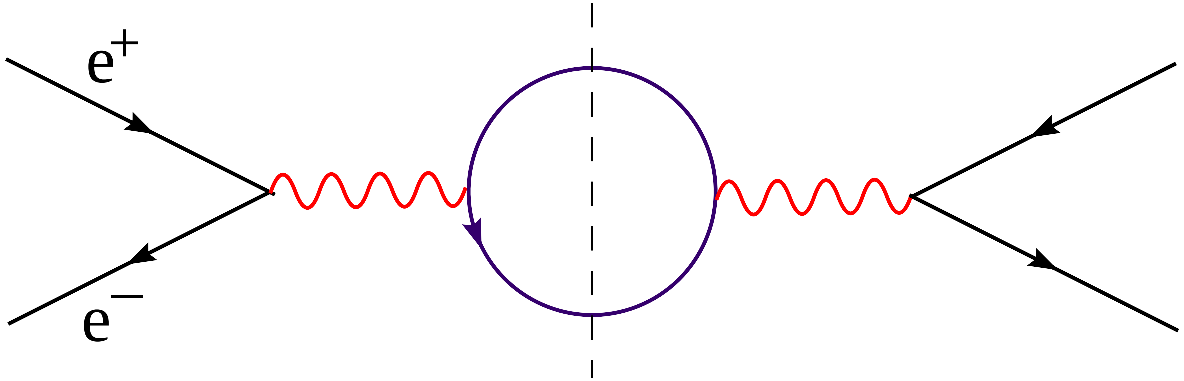
3–jet event at OPAL (CERN)



Current–current correlator

- Total cross–section given by the optical theorem

$$\sigma(e^+e^-) = \frac{1}{2s}\,\ell^{\mu\nu}\,\mathrm{Im}\,\Pi_{\mu\nu}(q)$$



$$\Pi_{\mu\nu}(q) \equiv \int \mathrm{d}^4x\,\mathrm{e}^{-iq\cdot x}\,i\theta(x_0)\,\langle 0\,|\,[J_\mu(x), J_\nu(0)]\,|\,0\rangle$$

$$J^\mu = \sum_f e_f\,\bar{q}_f\,\gamma^\mu\,q_f$$

- Vacuum polarization tensor for time–like momenta $(q^\mu q_\mu > 0)$

Current–current correlator

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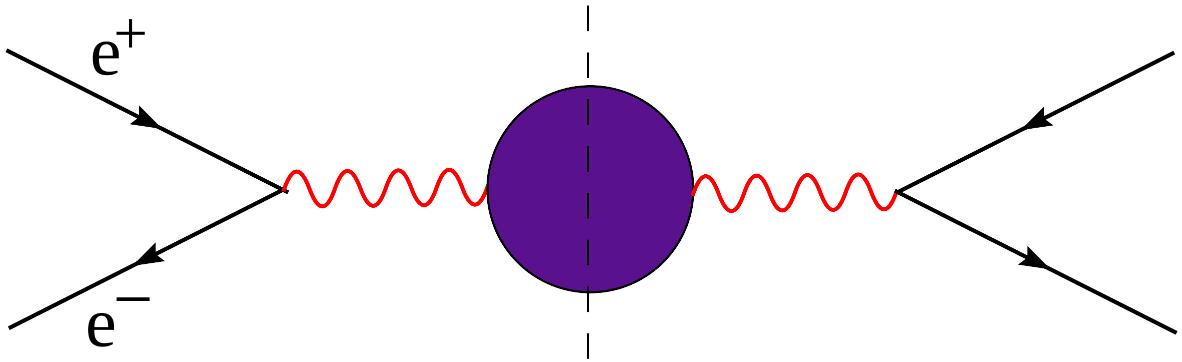

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$$J^\mu \;=\; \sum_f e_f\,\bar{q}_f\,\gamma^\mu\,q_f$$

- Valid to leading order in $\alpha_{\rm em}$ but all orders in α_s

Current–current correlator

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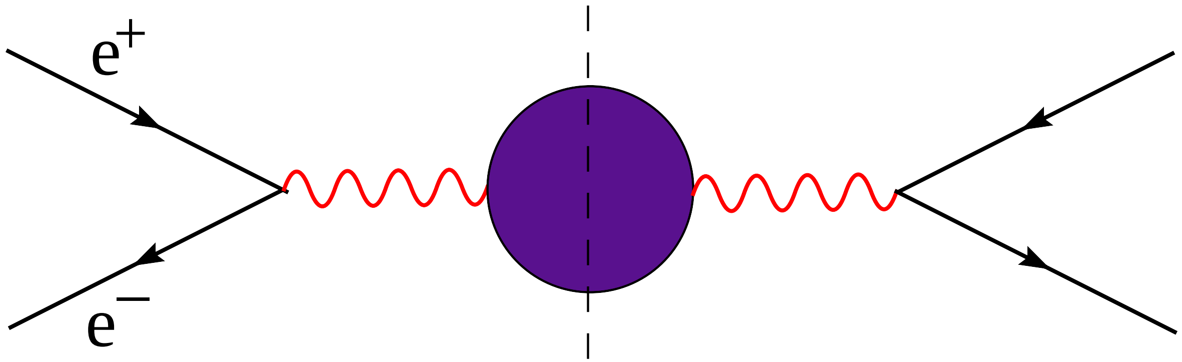
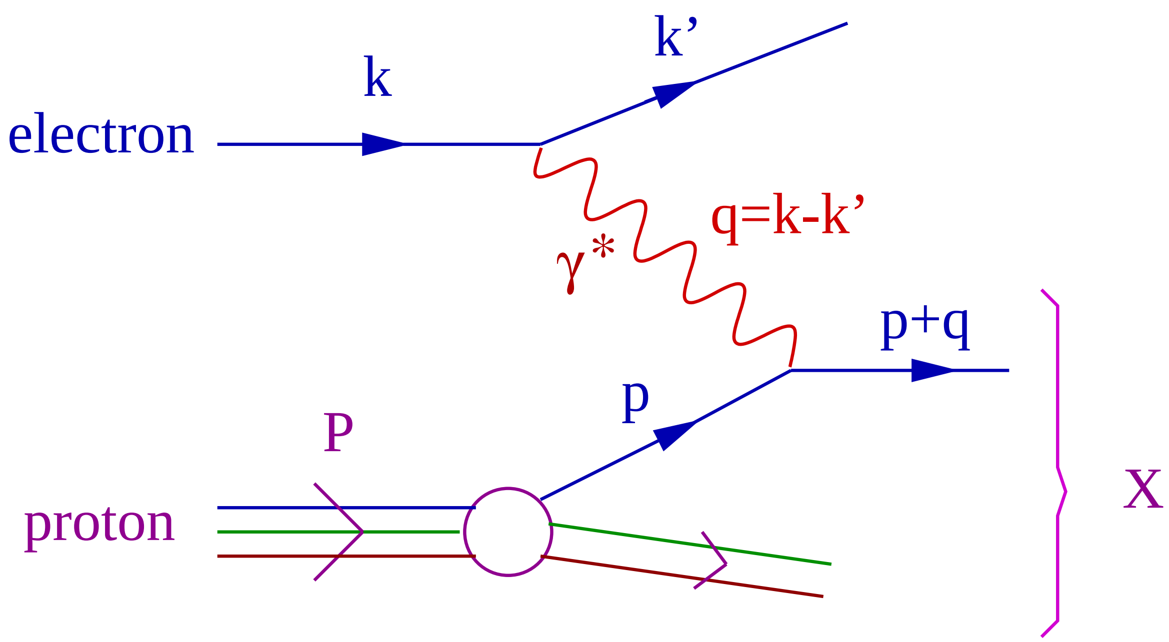


$$\Pi_{\mu\nu}(q) \;\equiv \int \mathrm{d}^4x\,\mathrm{e}^{-iq\cdot x}\, i\theta(x_0)\,\langle 0\,|\,[J_\mu(x), J_\nu(0)]\,|\,0\rangle$$

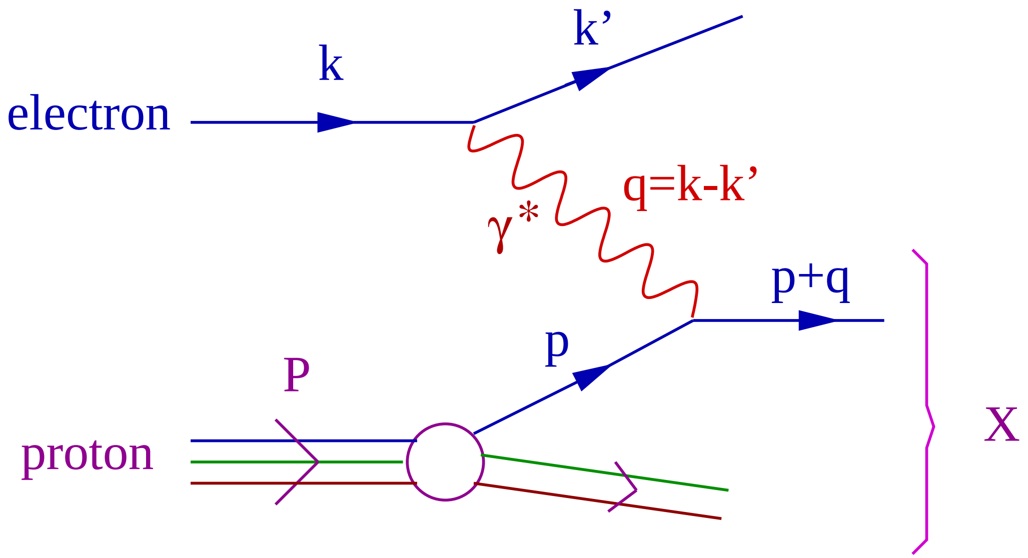
- Inclusive calculation (a ‘black box’)
- No specific information about the structure of the final state (‘how many jets, how they are distributed’)

electron (k) + proton (P) $\longrightarrow$ electron (k') + X (P_X)



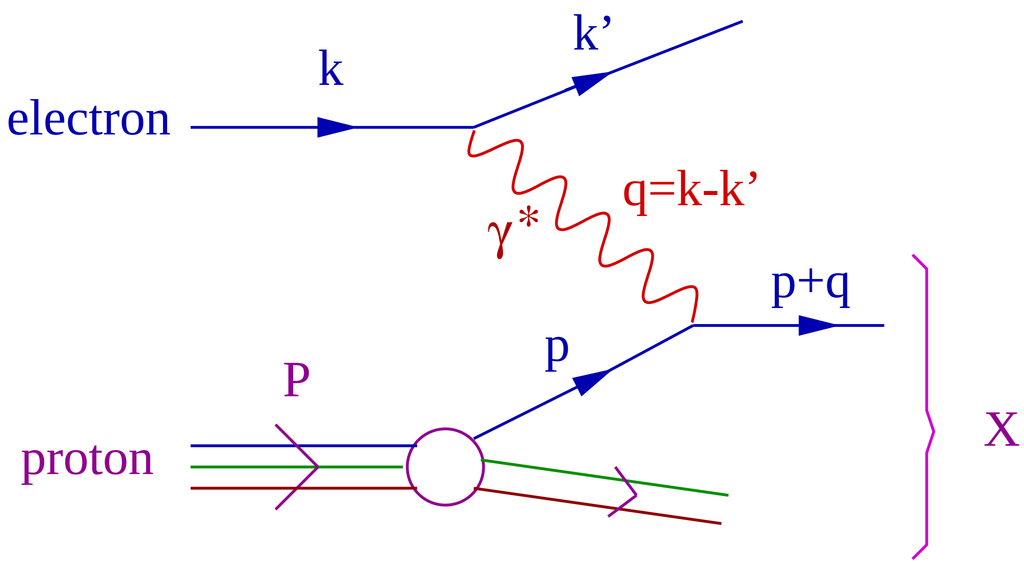
- 'Inclusive cross–section' : One allows for all the possible final states X of the hadronic system
- $q^\mu = k^\mu - k'^\mu \Longrightarrow q^2 \equiv q^\mu q_\mu < 0$: space–like photon

Kinematics and resolution scales



- Two independent kinematical invariants :
  - γ^* virtuality : $Q^2 \equiv -q^\mu q_\mu \geq 0$
  - Bjorken's x : $0 < x \equiv \frac{Q^2}{2P\cdot q} \simeq \frac{Q^2}{s+Q^2} < 1$
- ... with a direct physical interpretation :
  - the virtual photon resolution in transverse space ...
  - and, respectively, longitudinal momentum.

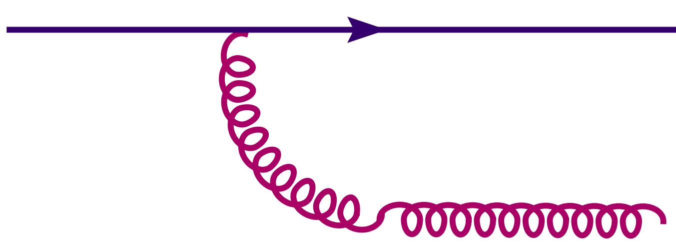
Kinematics and resolution scales



- Two independent kinematical invariants :
  - γ^* virtuality : $Q^2 \equiv -q^\mu q_\mu \geq 0$
  - Bjorken's x : $0 < x \equiv \frac{Q^2}{2P\cdot q} \simeq \frac{Q^2}{s+Q^2} < 1$
- Parton picture: γ^* absorbed by a quark excitation with
  - transverse size $\Delta x_\perp \sim 1/Q$
  - and longitudinal momentum $p_z = xP$

Infinite momentum frame

- ‘Partons’ are virtual (off–shell) excitations: they can radiate
- Parton picture makes sense only in a frame where the proton is moving very fast (‘infinite momentum frame’, or IMF)
  - ▷ parton lifetime is amplified by Lorentz time dilation

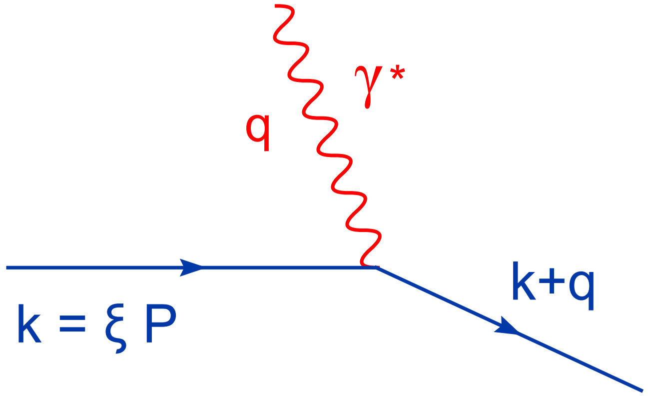


$$\Delta t \sim \frac{1}{\Delta E} = \frac{k_z}{k_\perp^2} = \frac{xP}{k_\perp^2} \gg \frac{1}{k_\perp}$$

- The ‘daughter’ gluon has a large lifetime too so long as $xP \gg k_\perp$ (always true in the IMF since $P \to \infty$)

Partons in DIS

- The absorption of the virtual photon in the proton IMF :



$$P^\mu = (P, 0, 0, P)$$
$$k^\mu \approx (\xi P, k_\perp, \xi P)$$

$$(k+q)^2 \approx 0 \quad \Longrightarrow \quad -Q^2 + 2\xi P\cdot q \approx 0$$
$$\Longrightarrow \quad \xi = \frac{Q^2}{2P\cdot q} = x$$

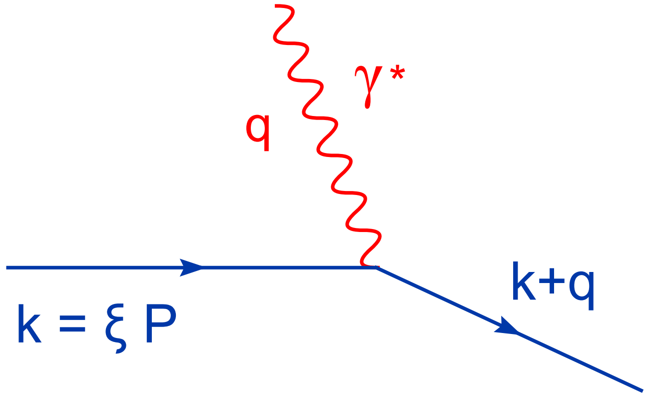
- The parton lifetime should be larger than the collision time

$$\Delta t_{\rm part} \sim \frac{2xP}{k_\perp^2} \; > \; \Delta t_{\rm col} \sim \frac{2xP}{Q^2}$$

$\Longrightarrow$ The photon 'sees' all the partons having $k_\perp^2 < Q^2$

Partons in DIS

- The absorption of the virtual photon in the proton IMF :



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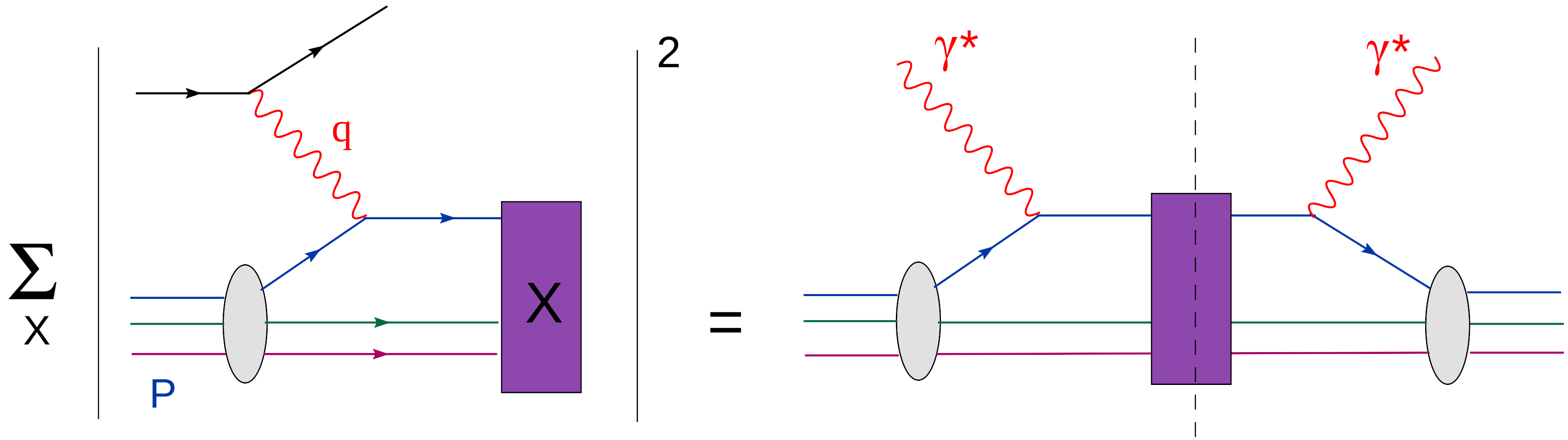
- By the uncertainty principle, such partons are localized
  - within a longitudinal extent $\Delta z \sim 1/xP$
  - within an area $\Delta\Sigma \sim 1/Q^2$ in the transverse plane

The proton structure function

- Differential cross section for virtual photon absorbtion :

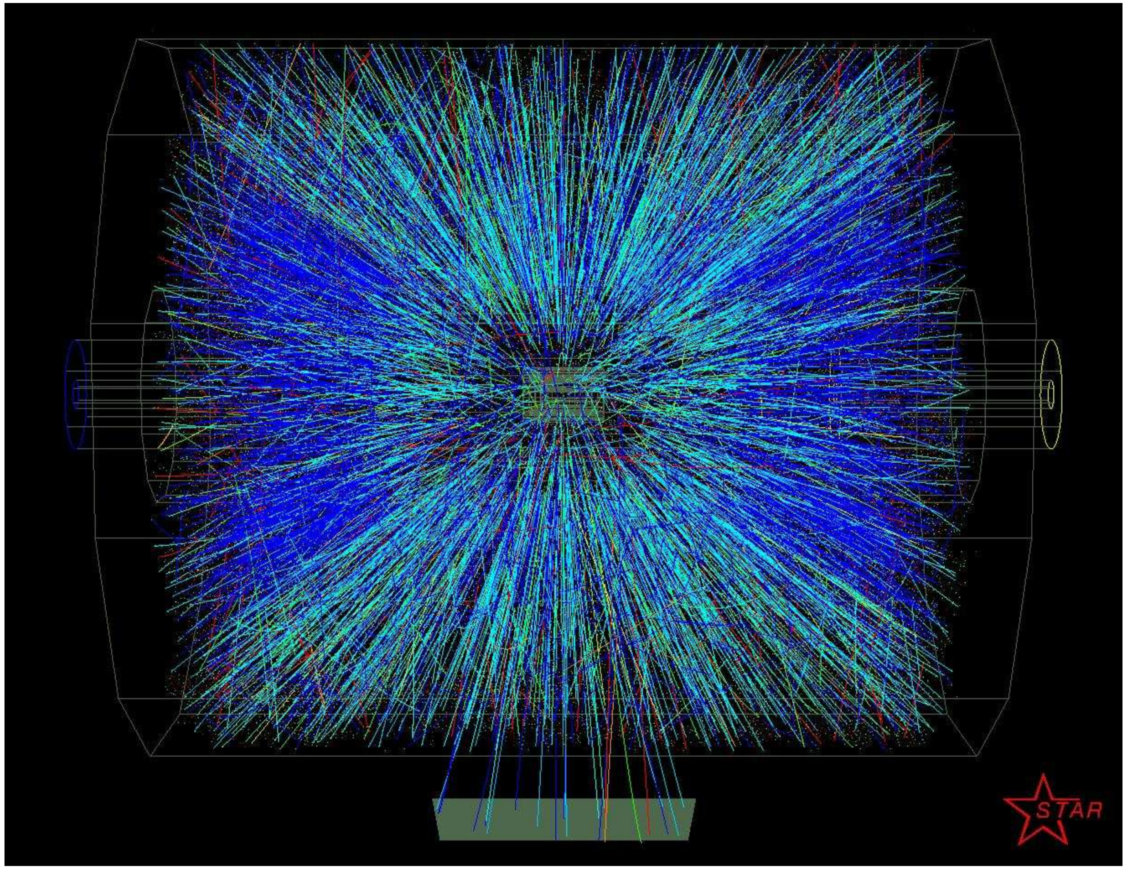
$$\sigma_{\gamma^* p}(x, Q^2) = \frac{4\pi^2 \alpha_{\rm em}}{Q^2} F_2(x, Q^2)$$



$$F_2(x, Q^2) = \sum_f e_f^2 \left[x q_f(x, Q^2) + x \bar{q}_f(x, Q^2) \right]$$

- $q_f(x, Q^2)$: number density of quarks of flavor f with longitudinal momentum fraction x and transverse size $1/Q$

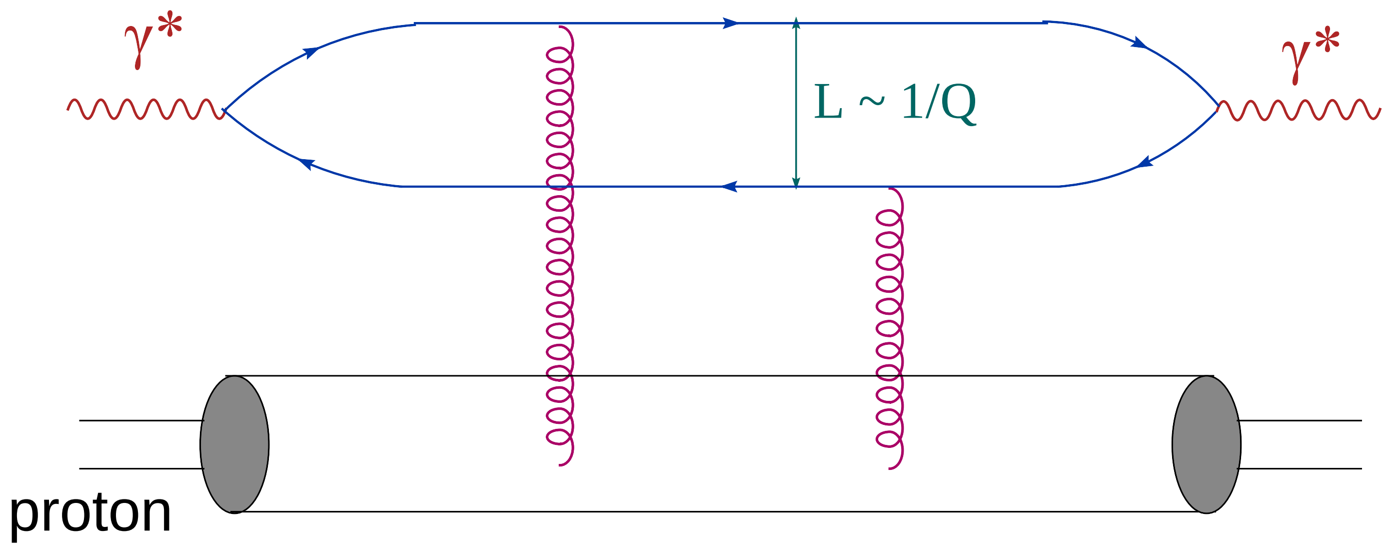
Partons at RHIC



- Partons are actually ‘seen’ (liberated) in the high energy hadron–hadron collisions
  - central rapidity: small–x partons
  - forward/backward rapidities: large–x partons

DIS: Dipole picture

- DIS in the proton rest frame $\Longrightarrow$ γ^* has a high energy ω



▷ γ^* fluctuates into a $q\bar{q}$ pair which then scatters off the proton

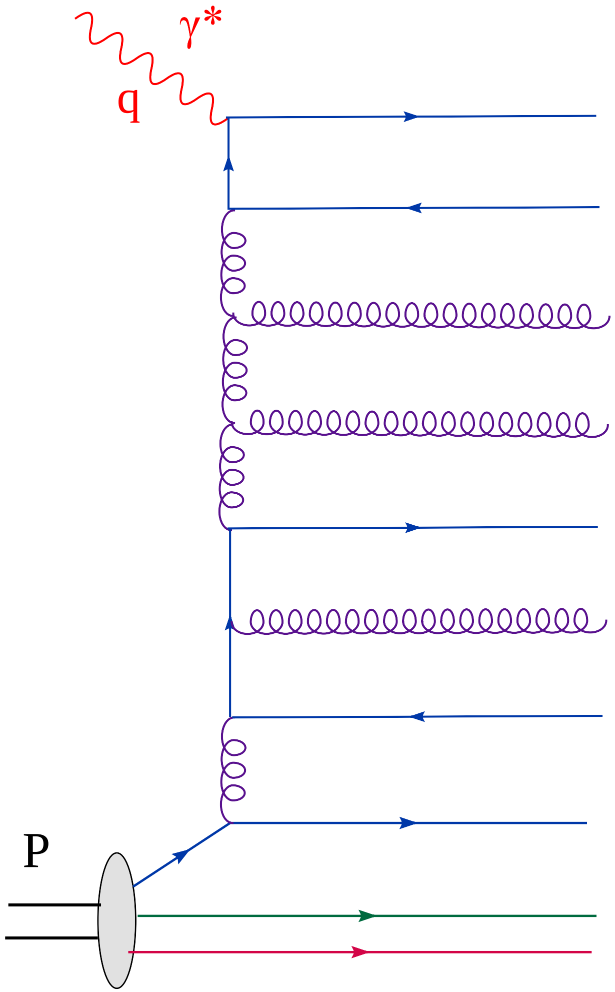
- Long lived ‘color dipole’ fluctuation, or ‘meson’

$$\Delta t \sim \frac{\omega}{Q^2} \gg R_p$$

- The transverse size of the ‘meson’ : $L \sim 1/Q$

Parton evolution in pQCD

- Parton branching within space–like cascades



$$\mathrm{d}\mathcal{P}_{g\to g} \sim \alpha_s N_c \frac{\mathrm{d}^2 k_\perp}{k_\perp^2} \frac{\mathrm{d}x}{x}$$

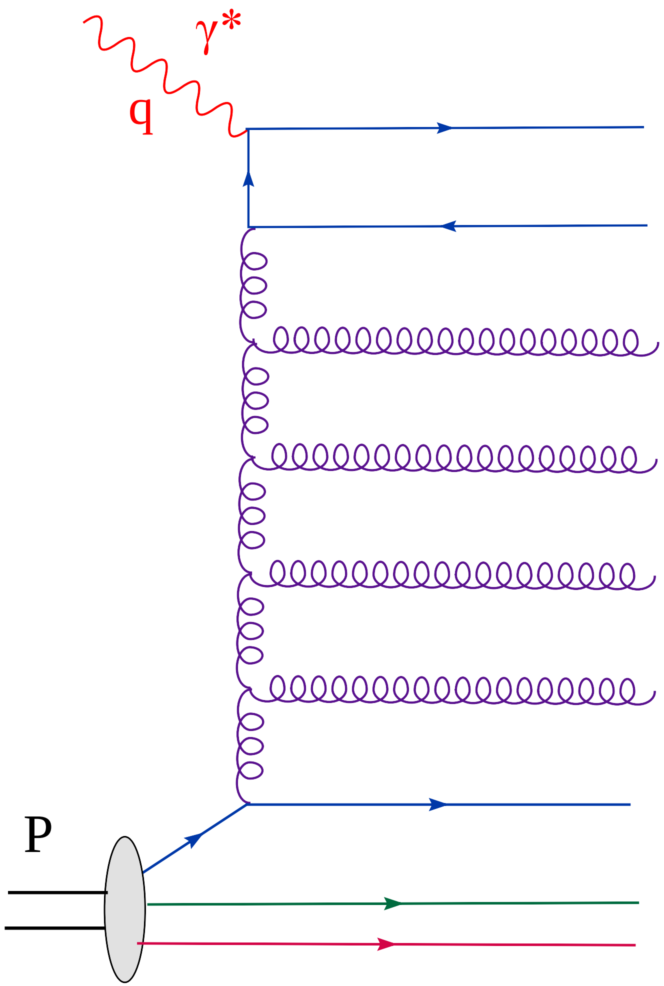
$$\mathrm{d}\mathcal{P}_{q\to g} \sim \alpha_s C_F \frac{\mathrm{d}^2 k_\perp}{k_\perp^2} \frac{\mathrm{d}x}{x}$$

$$\mathrm{d}\mathcal{P}_{q\to q} \sim \alpha_s C_F \frac{\mathrm{d}^2 k_\perp}{k_\perp^2} \frac{\mathrm{d}x}{1-x}$$

- Strong ordering in $k_\perp$ and/or x (gluons only)

High energy: BFKL evolution

- $s \gg Q^2 \implies x \simeq Q^2/s \ll 1:$ gluon cascades dominate



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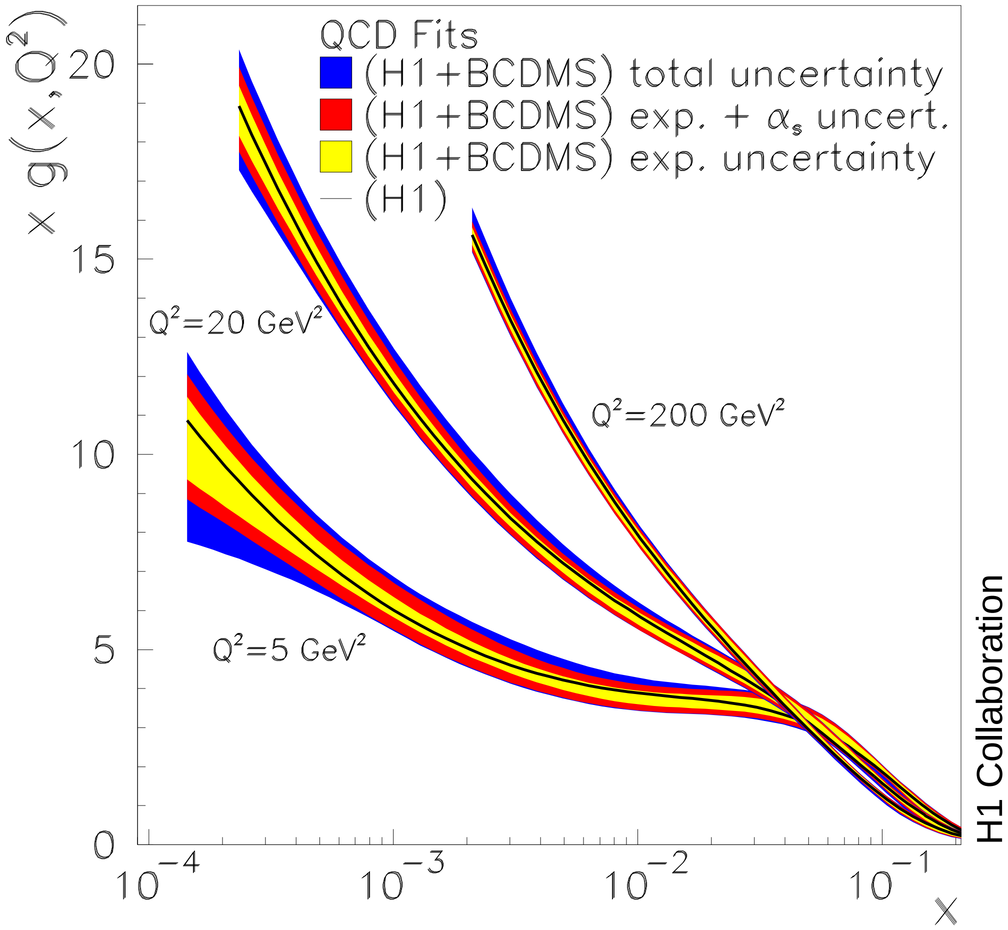
$$\mathrm{d}\mathcal{P}_{q\to q} \sim \alpha_s C_F \frac{\mathrm{d}^2 k_\perp}{k_\perp^2} \frac{\mathrm{d}x}{1-x}$$

- Rapid rise in the gluon distribution at high–energy/small–x

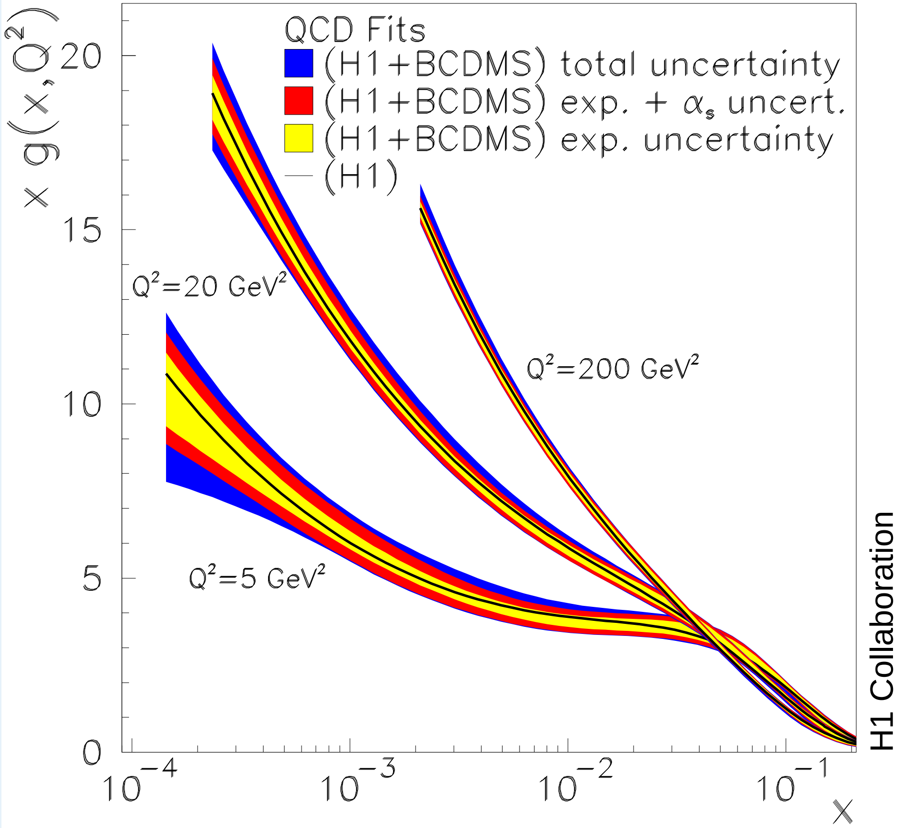
Gluons at HERA

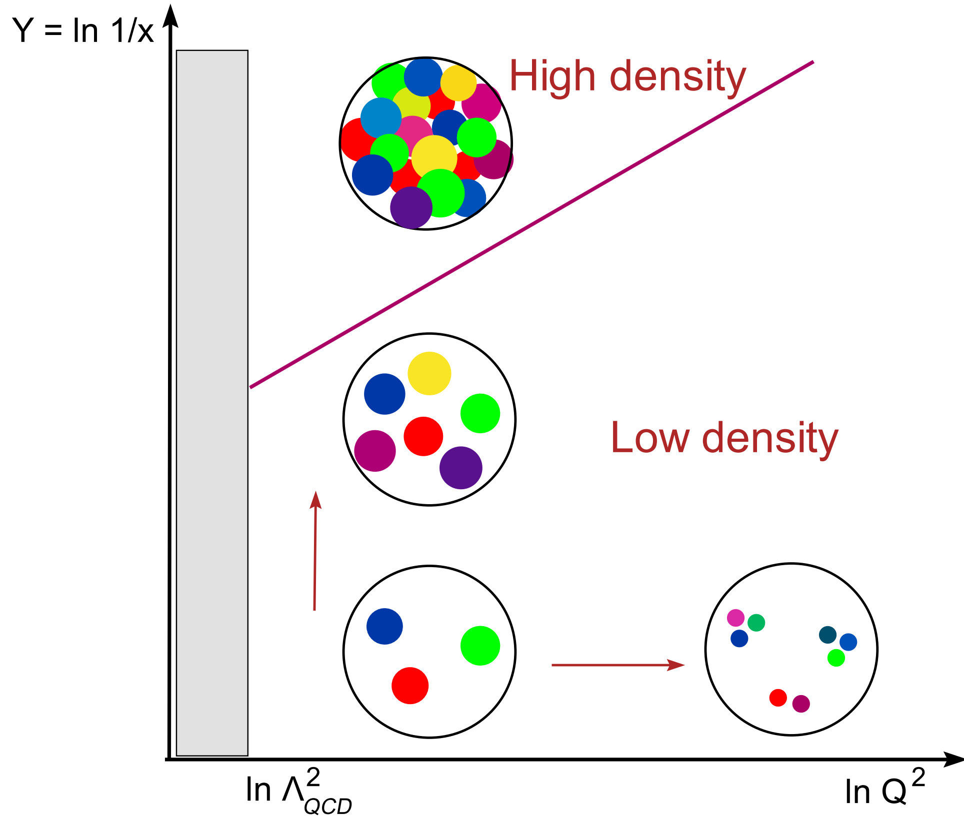
$xG(x, Q^2) \approx$ # of gluons with transverse area $\sim 1/Q^2$ and $k_z = xP$



$\triangleright$ Rapid rise with $1/x$: $xG(x, Q^2) \sim 1/x^{\lambda}$ with $\lambda = 0.2 \div 0.3$

Gluons at HERA





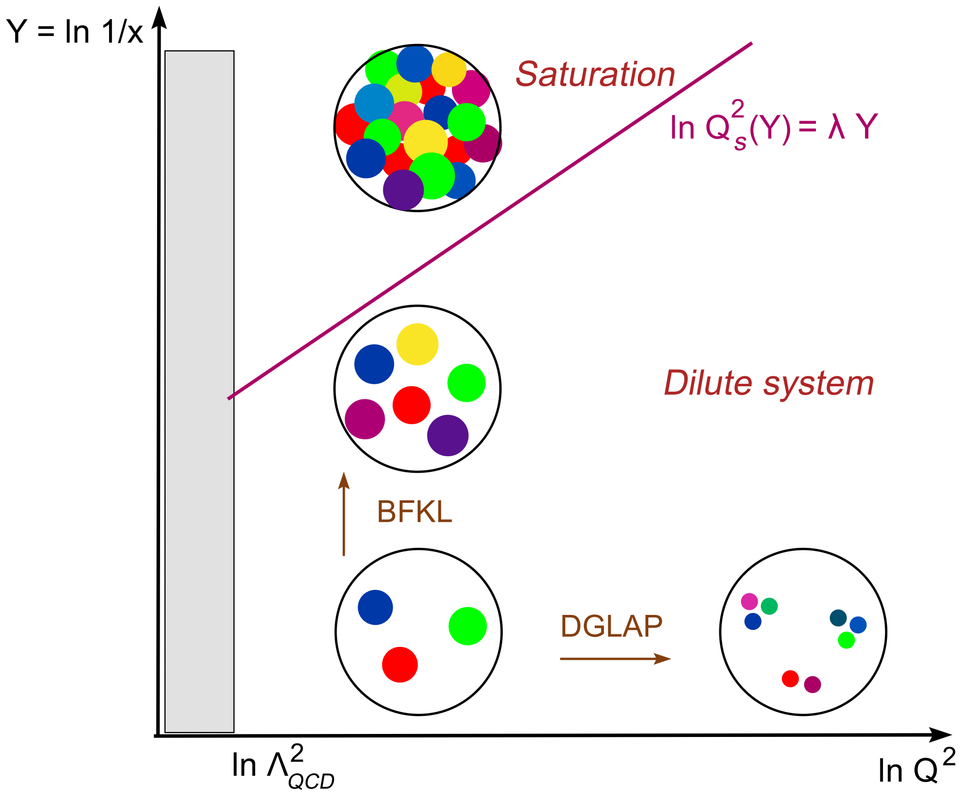
▷ High–Q^2 evolution : The parton density is decreasing

▷ Small–x evolution: An evolution towards increasing density

The Saturation Momentum

- Onset of non–linear physics : $n(x, Q^2) \sim 1/\alpha_s$

  $n(x, Q^2)$: the gluon occupation number

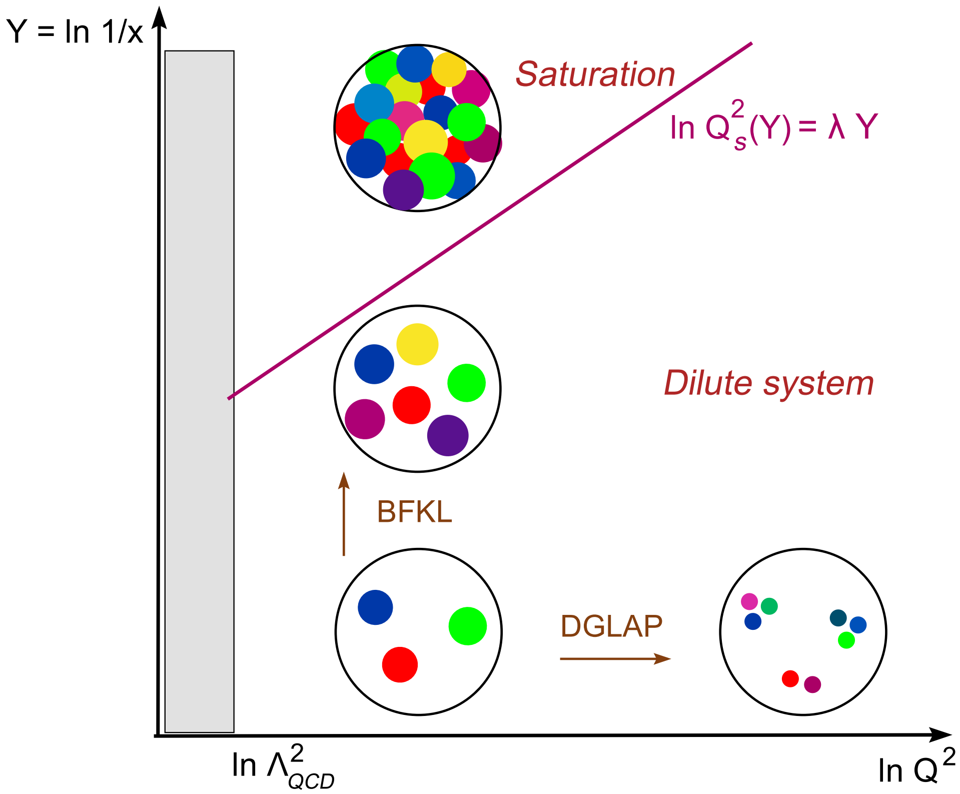


- The gluons must be numerous enough (small x) and large enough (low Q^2) to strongly overlap with each other.

The Saturation Momentum

- Onset of non–linear physics : $n(x, Q^2) \sim 1/\alpha_s$

  $n(x, Q^2)$: the gluon occupation number



- For given (small) x, the gluon transverse momenta must be small enough: $Q^2 \lesssim Q_s^2(x) \sim \Lambda^2 x^{-\lambda}$

Geometric Scaling at HERA

(Staśto, Golec-Biernat and Kwieciński, 2000)

$$\sigma(x,Q^2) \approx \sigma(\tau) \quad \text{with} \quad \tau \equiv Q^2/Q_s^2(x), \quad Q_s^2(x) = (x_0/x)^\lambda \,\text{GeV}^2\,, \quad \lambda \simeq 0.3$$

$x \leq 0.01$

$Q^2 \leq 450$ GeV^2

$Q_s^2 \sim 1$ GeV^2

for $x \sim 10^{-4}$