Statistical physics of dyons and confinement

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1. What is "vacuum"?

- in Quantum Mechanics
- in Quantum Field Theory (QCD)

2. What is quantum theory at nonzero temperatures?

- in Quantum Mechanics
- in Yang–Mills theory (QCD)
- confinement and deconfinement

3. Saddle-point fields

- instantons
- monopoles, dyons
- instantons with non-trivial holonomy (KvBLL)

4. Statistical weight (or probability) of KvBLL instantons

5. Ensemble of dyons

6. Ground state: “confining” holonomy preferred

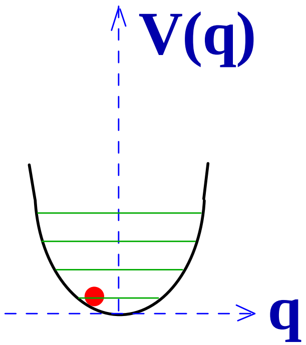
7. Correlator of Polyakov lines: “electric” string tension for k-strings

8. Average Wilson loop: “magnetic” string tensions

9. Comparison of observables with lattice results

Original results based on:
Phys. Rev. D76, 056001 (2007) [arXiv:0704.3181] by D.D. and **Victor Petrov**

1. What is "vacuum"? — It is the ground state of a quantum system.



Non-relativistic particle with mass m in a one-dimensional potential well $V(q)$.

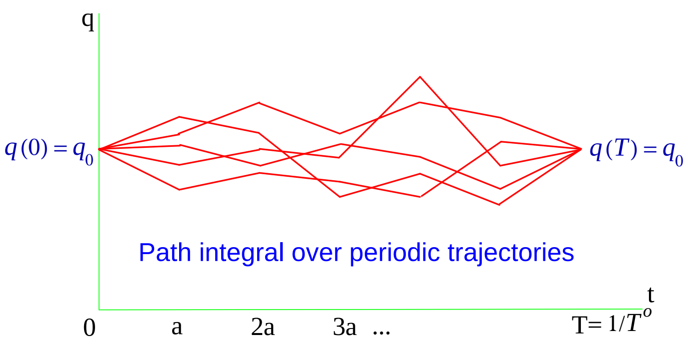
$$\text{Lagrangian} \quad L = \frac{m\dot{q}^2}{2} - V(q), \qquad \text{Energy} \quad H = \frac{m\dot{q}^2}{2} + V(q).$$

In Quantum Mechanics, to find the (quantized) energy levels E_n and the stationary wave functions $\psi_n(q)$ one solves the Schrödinger eqn:

$$\mathcal{H}\psi_n(q) = E_n\,\psi_n(q), \qquad \mathcal{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dq^2} + V(q).$$

Take non-zero temperature $T^{\rm o}$ and consider the **partition function** of the system [Feynman]:

$$\mathcal{Z} = \sum_n e^{-\frac{E_n}{kT^{\rm o}}} = \int dq_0 \int_{q(0)=q_0}^{q(T)=q_0} Dq(t)\, \exp\left(-\frac{1}{\hbar}\int_0^T dt\left[\frac{m\dot{q}^2(t)}{2} + V\left(q(t)\right)\right]\right), \quad T = \frac{\hbar}{kT^{\rm o}}.$$



Discretized action

$$S=\sum_n a\left[\frac{m}{2}\left(\frac{q(t_n)-q(t_{n-1})}{a}\right)^2+V\left(q(t_n)\right)\right].$$

Path integral can be understood as the limit of an infinite number of ordinary integrations – over the intermediate points $q_1\ldots q_N$:

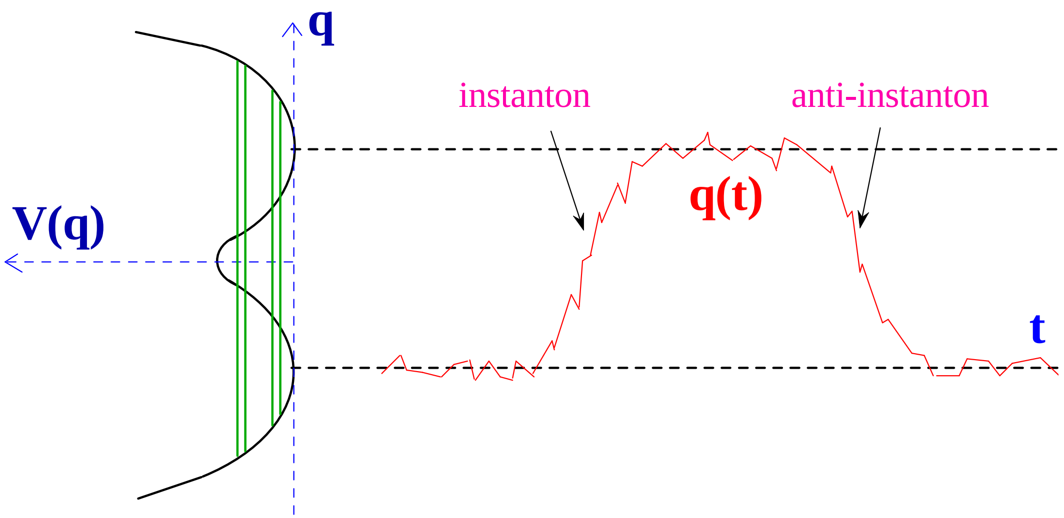
$$\mathcal{Z}=\mathcal{N}\lim_{N\to\infty}\prod_{n=1}^{N}\int dq_n\,e^{-\frac{S}{\hbar}}.$$

To cut out the lowest lying ground state, one has to take $T^{\mathrm{o}}\to 0,\quad T=\frac{\hbar}{kT^{\mathrm{o}}}\to\infty$.

This is how a typical "vacuum" trajectory for the quantum-mechanical particle in a potential well look like:

A quantum-mechanical particle in the vacuum experiences **zero-point oscillations**. This is why the ground-state energy is not zero but $E_0=\frac{\hbar\omega}{2}$.

Typical "vacuum" trajectory in case of the double-well potential:



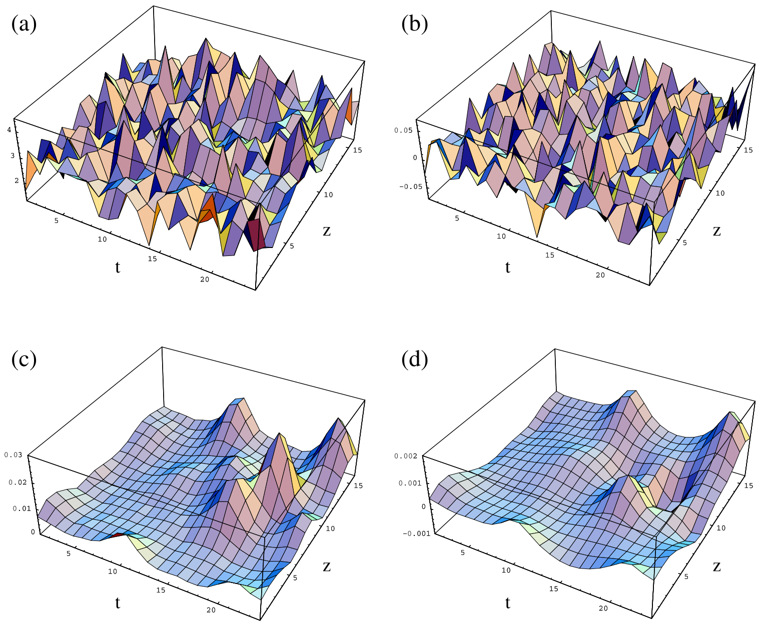
Instantons = classical trajectories with minimal action, satisfying the equation of motion

$$\frac{\delta S}{\delta q(t)} = 0 \qquad \text{or} \quad m\ddot{q} = -\frac{\partial V}{\partial q}$$

In the vacuum, one typically observes
zero-point oscillations on top of
classical trajectories of the fields

Quantum Mechanics is called a $(0+1)$-dimensional Quantum Field Theory (QFT).
In Yang–Mills theory the role of “coordinates” $q(t)$ is played by the amplitudes of the fields $A_i(\mathbf{x}, t)$ which depend on 3 space and 1 time coordinates.

The vacuum (= the ground state) is made of zero-point oscillations of the fields $A_i(\mathbf{x}, t)$ **on top of classical field configurations** $A^{\rm class}(\mathbf{x}, t)$:



left: action density, before and after smearing
right: so-called topological charge density

Computer simulations of the Yang–Mills vacuum [J.Negele et al.]

Top: snapshot of the full configuration, dominated by zero-point oscillations.
Bottom: Smearing kills the zero-point oscillations but reveals classical configurations of the gluon field, here: instantons and anti-instantons.

Yang–Mills theory at non-zero temperatures

Feynman's representation for the partition function:

$$\mathcal{Z} = \sum_n \langle n \left| e^{-\beta\mathcal{H}} \right| n\rangle \qquad \left[\beta = \frac{1}{T}\right]$$

$$= \sum_n \int dq\, \psi_n^*(q)\; e^{-\beta\mathcal{E}_n}\; \psi_n(q) = \int dq \int_{q(0)=q}^{q(\beta)=q} Dq(t)\; \exp\left(-\int_o^\beta dt \mathcal{L}_{\rm Euclid}[q,\dot{q}]\right).$$

In Yang-Mills theory the role of coordinates q is played by the spatial components of the gluon field $A_i^a(x)$:

$$\mathcal{Z} = \int DA_i(x) \int_{A_i(0,x)=A_i(x)}^{A_i(\beta,x)=A_i(x)} DA_i(t,x)\; \exp\left\{-\frac{1}{2g^2}\int_0^\beta dt \int d^3x \left[\left(\dot{A}_i^a\right)^2 + (B_i^a)^2\right]\right\},$$

$$B_i^a = \frac{1}{2}\epsilon_{ijk}\left(\partial_j A_k^a - \partial_k A_j^a + f^{abc}A_k^b A_k^c\right) \qquad \text{chromomagnetic field.}$$

However, in a gauge theory one sums not over all possible but only over **physical states**, *i.e.* satisfying Gauss' law. In the absence of external sources it means that only those states need to be taken into account that are invariant under gauge transformations:

$$A_i(x) \to [A_i(x)]^{\Omega(x)} = \Omega(x)^\dagger\, A_i(x)\,\Omega(x) + i\Omega(x)^\dagger\,\partial_i\Omega(x), \qquad \Omega(x) = \exp\{i\,\omega_a(x)t^a\}\,.$$

To restrict the summation to physical states, one projects to the physical *i.e.* gauge invariant states by averaging the initial and final configurations over gauge rotations [This is as if we would like to restrict the summation to spherically-symmetric states only]:

$$\mathcal{Z}_{\rm phys} \;=\; \sum_{\rm phys\ states} \langle n \left| e^{-\beta\mathcal{H}} \right| n\rangle = \sum_n \int d\Omega_{1,2} \int dq\, \psi_n^*(\Omega_1 q)\; e^{-\beta\mathcal{E}_n}\;\psi_n(\Omega_2 q)$$

$$= \int D\Omega_{1,2}(x) DA_i(x) \int_{A_i(0,x)=A_i(x)^{\Omega_1(x)}}^{A_i(\beta,x)=A_i(x)^{\Omega_2(x)}} DA_i(t,x)\;\exp\left\{-\frac{1}{2g^2}\int_0^\beta dt \int d^3x \left[\left(\dot{A}_i^a\right)^2 + (B_i^a)^2\right]\right\}.$$

Renaming the initial field $A_i^{\Omega_1(x)} \to A_i$ and introducing the relative gauge transformation $\Omega(x) = \Omega_2(x)\,\Omega_1^\dagger(x)$ one can rewrite this as

$$\mathcal{Z}_{\rm phys} = \int D\Omega(x) DA_i(x) \int_{A_i(x)}^{A_i(x)^{\Omega(x)}} DA_i(t,x)\; e^{-S[A_i]}\;.$$

A more customary form: integrate over strictly periodic gauge fields, but at the cost of introducing a non-zero $A_4(t,x)$, *e.g.* $i\exp(-it\omega^a t^a)\partial_t \exp(it\omega^a t^a)$. Then

$$\mathcal{Z}_{\rm phys} = \int DA_\mu\; \exp\left\{-\frac{1}{4g^2}\int d^4x\, F^a_{\mu\nu}F^a_{\mu\nu}\right\}, \qquad A_\mu(t,x) = A_\mu(t+\beta,x), \quad \beta = 1/T.$$

The eigenvalues of the **Polyakov line** are gauge invariant

$$L(x) = \mathrm{P}\,\exp\left(i\int_0^\beta dt\, A_4(t,x)\right) \qquad (=\Omega(x)) \qquad \text{“holonomy”}$$

One can choose the gauge where A_4 is time-independent, moreover, diagonal. In this gauge

$$L(x) \quad = \quad \exp(i\beta\, A_4(x)) = \begin{pmatrix} e^{2\pi i\mu_1} & 0 & 0 \\ 0 & e^{2\pi i\mu_2} & 0 \\ 0 & 0 & e^{2\pi i\mu_3} \end{pmatrix}, \qquad \mu_1+\mu_2+\mu_3=0,$$

$$\mu_1 \le \mu_2 \le \ldots \le \mu_N \le \mu_1 + 1.$$

“Trivial” holonomy: when the Polyakov line belongs to one of the elements of the center of the group Z_N, and $\mathrm{Tr}\, L \neq 0$:

1) $\mu_1=\mu_2=\mu_3=0 \quad \Longrightarrow \quad L = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

2) $\mu_1 = -\frac{2}{3},\ \mu_2=\frac{1}{3},\ \mu_3=\frac{1}{3} \quad \Longrightarrow \quad L = e^{\frac{2\pi i}{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

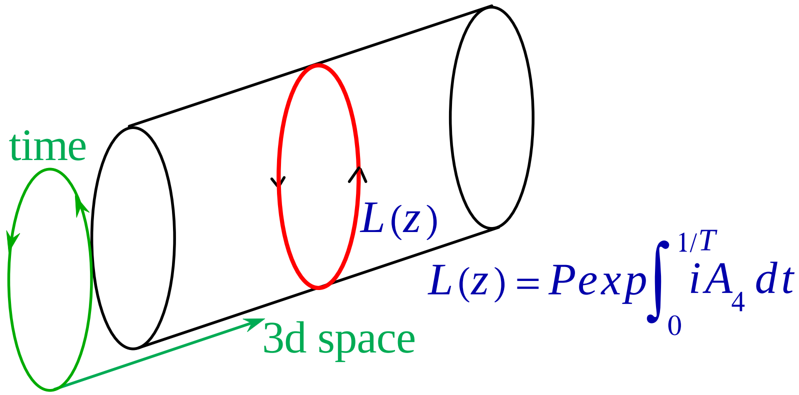
3) $\mu_1 = -\frac{1}{3},\ \mu_2=-\frac{1}{3},\ \mu_3=\frac{2}{3} \quad \Longrightarrow \quad L = e^{-\frac{2\pi i}{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

"Maximally non-trivial" holonomy:

$$\mu_1 = -\tfrac{1}{3},\ \mu_2 = 0,\ \mu_3 = \tfrac{1}{3} \Longrightarrow L = \begin{pmatrix} e^{-\frac{2\pi i}{3}} & 0 & 0 \\ 0 & e^{\frac{0\,\pi i}{3}} & 0 \\ 0 & 0 & e^{\frac{2\pi i}{3}} \end{pmatrix}, \quad \mathrm{Tr}\, L = 0(!)$$

Physical meaning of the Polyakov line

$$< \mathrm{Tr}\ L(\mathbf{z}) >= e^{-m_{\mathrm{quark}}\,/\,T} \quad \begin{cases} = 0 & \text{below } T_c \\ \neq 0 & \text{above } T_c \end{cases}$$

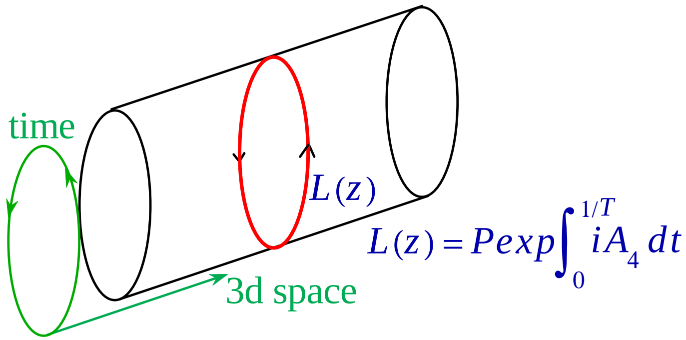


For confinement, it helps to have gluon configurations with a "maximally non-trivial" holonomy!

<u>Criteria of confinement</u> at nonzero temperature

1. Average Polyakov line

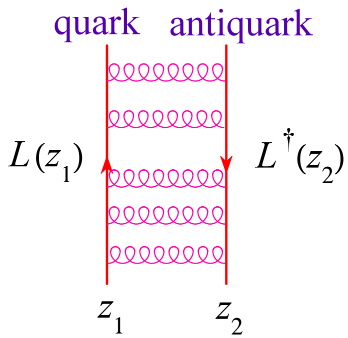
$$< \mathrm{Tr}\ L(\mathbf{z}) > \begin{cases} = 0 & \text{below } T_c \\ \neq 0 & \text{above } T_c \end{cases}$$



2. Linear rising potential between quarks

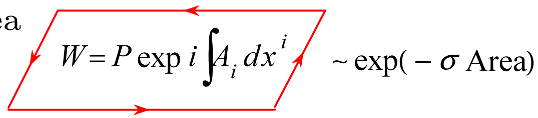
$$< \mathrm{Tr}\ L(\mathbf{z_1})\, \mathrm{Tr}\ L^\dagger(\mathbf{z_2}) >= e^{-V(z_1 - z_2)/T}$$

$$V(z_1 - z_2) = |\mathbf{z}_1 - \mathbf{z}_2|\,\sigma$$



3. Area law for Wilson loops in non-zero N-ality representations

$$< \mathrm{Tr}\ \mathrm{P}\ \exp\ i \oint A_i dx^i >= e^{-\sigma\ \mathrm{Area}}$$



4. Mass gap: No massless particles in the spectrum

5. No Stefan–Boltzmann law ($F \sim N^2T^4$); instead: $F = \mathcal{O}(N^0)$
but exponentially rising density of states (= the Hagedorn spectrum)

Saddle-point gluon fields –

– are those fields $A^a_\mu(\mathbf{x},t)$ that have relatively higher probability to occur in the vacuum; they satisfy the non-linear Maxwell eqn:

$$\frac{\delta S}{\delta A^a_\mu(x)} = 0 \qquad \text{or} \quad D^{ab}_\mu F^b_{\mu\nu} = 0$$

and have finite action $S = \int d^4x\, F^a_{\mu\nu} F^a_{\mu\nu} < \infty$.

Topological classification [Gross, Pisarski, Yaffe]

1. Topological charge

$$Q_{\rm T} = \frac{1}{16\pi^2}\int d^4x\, \epsilon^{\kappa\lambda\mu\nu}\, F^a_{\kappa\lambda}(x)\, F^a_{\mu\nu}(x)$$

2. Holonomy, more precisely the gauge-invariant eigenvalues of the Polyakov loop at spatial infinity:

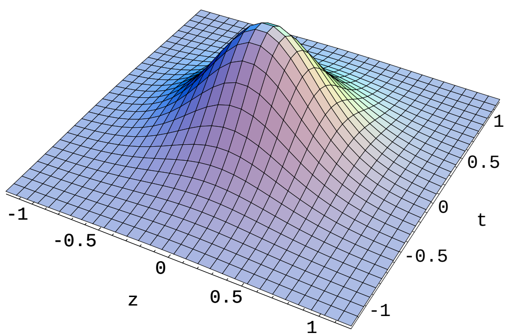
$$\text{eigenvalues of} \quad L = \mathcal{P}\exp\left(i\int_0^{1/T^{\rm O}} dt\, A_4(\mathbf{x},t)\right)$$

3. Magnetic charge, more precisely the gauge-invariant eigenvalues of the chromo-magnetic flux at spatial infinity:

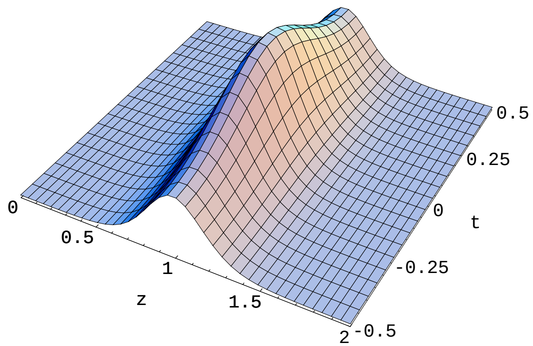
$$\mathbf{B} = \frac{\mathbf{r}}{|\mathbf{r}|^3} \times \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{etc.}$$

Standard Belavin–Polyakov–Schwarz–Tiupkin instantons

$Q_{\rm T} = \pm 1$, holonomy = trivial, magnetic charge(s) = 0. The gluon configuration is $O(3)$-symmetric at $T^{\rm o} \neq 0$ and $O(4)$-symmetric at $T^{\rm o} \to 0$. The action is $S = \frac{8\pi^2}{g^2}$.



instanton action density $F_{\mu\nu}^2$ at $T^{\rm o} = 0$



instanton action density $F_{\mu\nu}^2$ at $T^{\rm o} \neq 0$

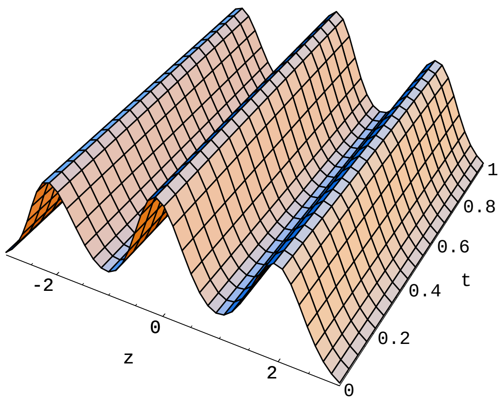
Magnetic monopoles, or BPS monopoles, or dyons

Topological charge = fractional, holonomy = arbitrary, magnetic charges quantized:

$$\mathbf{E} = \mathbf{B} = \frac{\mathbf{r}}{|\mathbf{r}|^3} \times \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

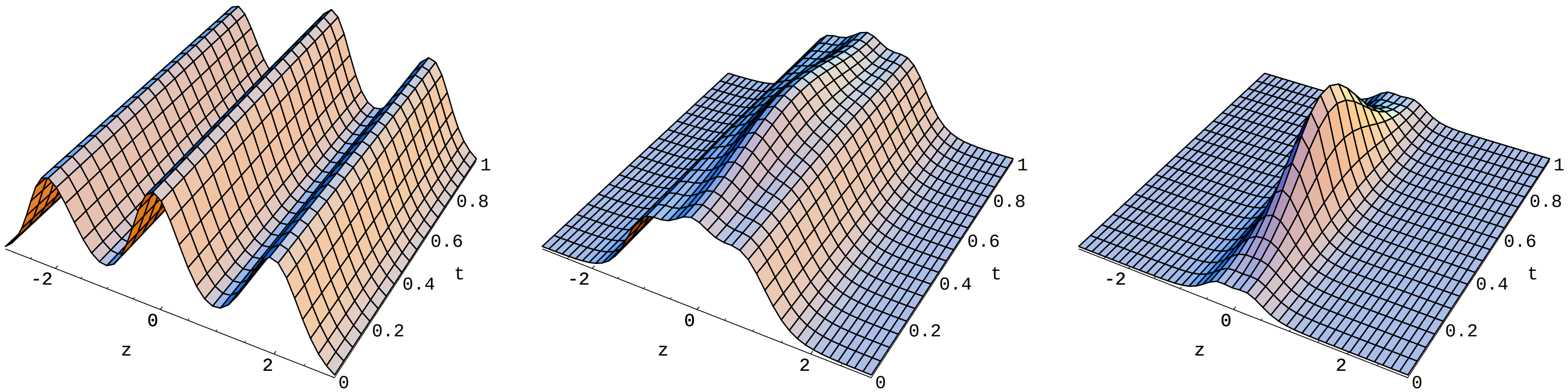
$$A_4(|\mathbf{x}| \to \infty) \to 2\pi T^{\rm o} \times \begin{pmatrix} \mu_1 & 0 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & \mu_3 \end{pmatrix}$$

Inside the dyons' cores of the size $\frac{1}{2\pi T^{\rm o}(\mu_i - \mu_j)}$ the field is highly non-linear, non-Abelian and, generally, time-dependent; the asymptotics is static and Abelian. The action density is time-independent everywhere:



Kraan–van Baal–Lee–Lu (KvBLL) instantons with non-trivial holonomy —

— generalize both standard instantons and magnetic monopoles. They can be considered as "made of" 3 dyons. The action density is static when dyons are far apart, and reminds the instanton when dyons merge together.



$Q_{\rm T} = \pm 1$, holonomy = arbitrary, full magnetic charge = 0.
The full classical action is exactly the same for all dyon configurations, $S = \frac{8\pi^2}{g^2}$

$$A_4(|\mathbf{x}| \to \infty) \to 2\pi T^{\rm o} \times \begin{pmatrix} \mu_1 & 0 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & \mu_3 \end{pmatrix}$$

The probability (or statistical weight) of the KvBLL instanton was computed *exactly* (!) by D.D., Gromov, Petrov and Slizovsky (2004), D.D. and Gromov (2005), Gromov (2006), Slizovsky (2007). The statistical weight depends on Λ, $T^{\rm o}$, $\{\mu_1, \mu_2 ...\}$, $|\mathbf{x}_{mn}|$. For the $SU(N)$ gauge group

$$W = \int d\mathbf{x}_1 ... d\mathbf{x}_N \, \mathrm{det} G \, f.$$

$$f = \frac{4\pi}{g^4} \frac{\Lambda_{\rm PV}^4}{T} c, \qquad \text{"fugacity"}$$

$$c = (\mathrm{Det}(-\triangle))^{-1}_{\rm reg,\, norm} \approx \exp\left(-VT^3 \, P^{\rm pert}(\mu)\right)$$

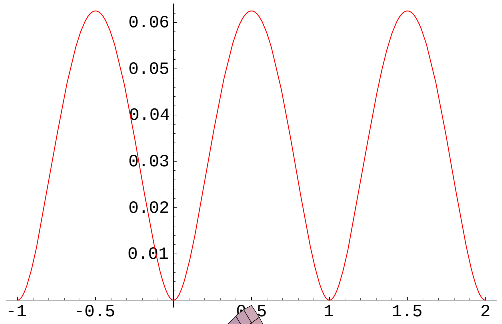
$$G_{mn}^{N\times N} = \delta_{mn}\left(4\pi\nu_m + \frac{1}{|\mathbf{x}_m - \mathbf{x}_{m-1}|} + \frac{1}{|\mathbf{x}_m - \mathbf{x}_{m+1}|}\right) - \frac{\delta_{m,n-1}}{|\mathbf{x}_m - \mathbf{x}_{m+1}|} - \frac{\delta_{m,n+1}}{|\mathbf{x}_m - \mathbf{x}_{m-1}|}$$

where $\nu_m = \mu_{m+1} - \mu_m$. [Conjectured by Lee, Weinberg and Yi, computed exactly by Kraan and DD and Gromov]

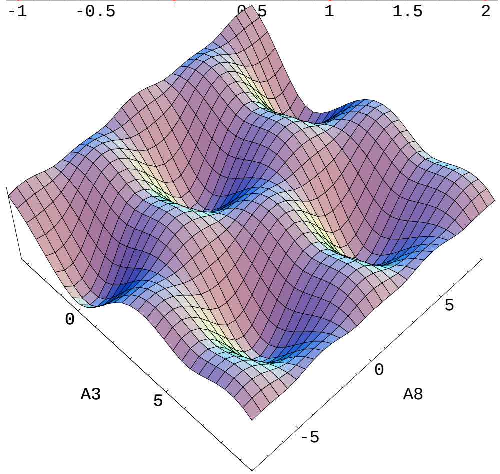
At trivial holonomy ($\mu_m = 0$) this measure reduces to the well-known standard instanton measure computed by 't Hooft (at $T=0$) and Gross, Pisarski and Yaffe (at $T \neq 0$).

Why "maximally non-trivial" holonomy is maximally non-trivial?

Effective (1-loop, 2-loop ...) quantum action as function of slowly varying $A_4(\mathbf{x})$ [Gross, Pisarski and Yaffe; D.D. and Oswald]:



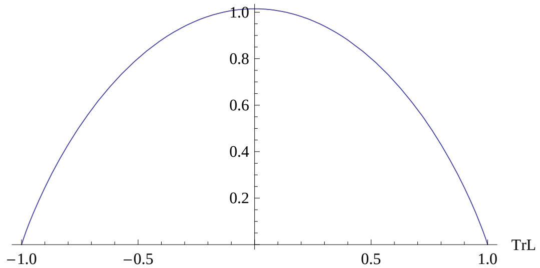
SU(2): Perturbative potential energy $P(\mu_1 - \mu_2)$. $\frac{1}{2}\mathrm{Tr}\, L = \pm 1$ correspond to the two minima: $\mu_1 - \mu_2 = 1, 0$; $\mathrm{Tr}\, L = 0$ corresponds to the **maximum** $\mu_1 - \mu_2 = \frac{1}{2}$.

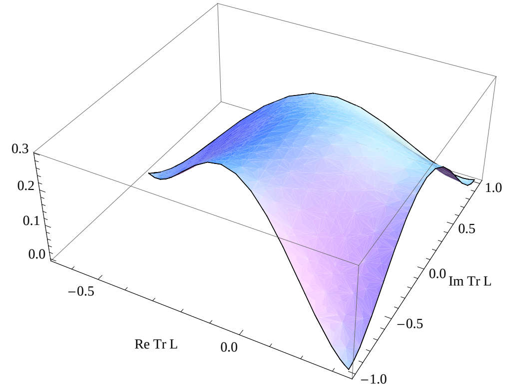


SU(3): Perturbative potential energy $P(\mu)$ as function of $\mu_1 - \mu_2,\ \mu_3 - \mu_2$ forms a double-periodic triangle lattice. At the minima, the Polyakov line $\in Z(3)$, corresponding to the deconfined phase. Confinement, $\mathrm{Tr}\, L = 0$, corresponds to the **maximum** of the perturbative energy.

$$P^{\mathrm{pert}} = V\ \frac{(2\pi)^2 T^3}{3} \sum_{m>n}^{N} (\mu_m - \mu_n)^2 [1 - (\mu_m - \mu_n)]^2 \Big|_{\mathrm{mod}1} .$$

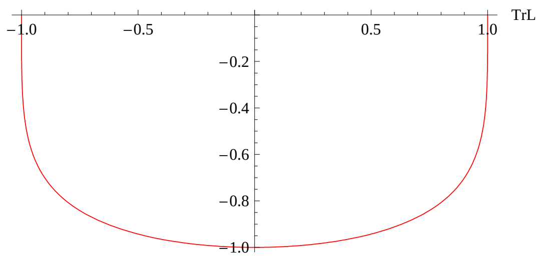
It has N minima (all with zero energy) at the trivial holonomy corresponding to N elements of the center of $SU(N)$. The large volume factor V seemingly prohibits any configurations with non-trivial holonomy!

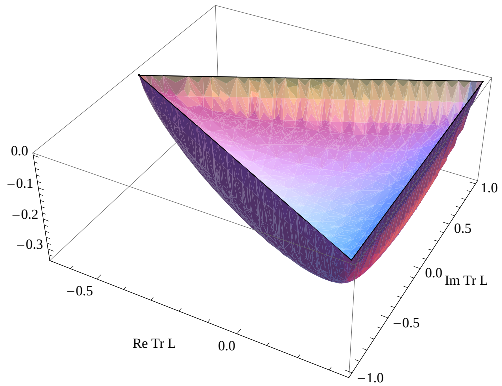




Perturbative potential energy as function of $\mathrm{Tr}\,L$: *left*: SU(2), *right*: SU(3). Confinement ($\mathrm{Tr}\,L = 0$) corresponds not to the minima but to the **maxima** of the potential energy. Both plots scale as $\sim T^4$.

However, the non-perturbative free energy of the ensemble of $\mathcal{O}(V)$ dyons has the minimum at $\mathrm{Tr}\,L = 0$:





At $T < T_c$ the dyon-induced free energy prevails and forces the system to pick the "maximally non-trivial" "confining" holonomy, $\mathrm{Tr}\,L = 0$. The phase transition will be 2^{nd} order in $SU(2)$ and 1^{st} order in $SU(3)$ and higher.

The above metric has been written for N dyons, all of *different* kind.

If there are K dyons of the *same* kind, the metric has been found by Gibbons and Manton (1995) from considering the classical equations of motion for K identical monopoles at large separations:

$$\tilde{G}_{ij} = \begin{cases} 4\pi\nu_m - \sum_{k\neq i} \frac{2}{|\mathbf{x}_i - \mathbf{x}_k|}, & i = j, \\ \frac{2}{|\mathbf{x}_i - \mathbf{x}_j|}, & i \neq j \end{cases}$$

Coulomb coefficients are determined by the scalar products of Cartan generators

$$C_m = \mathrm{diag}(0, 0, \ldots, \underbrace{1}_{m^{\mathrm{th}}\,\mathrm{place}}, -1, 0, \ldots, 0)$$

$$\mathrm{Tr}\,(C_m C_n) = \begin{cases} 2 & \text{same kind} \\ -1 & \text{nearest neighbour} \\ 0 & \text{non} - \text{nearest neighbour} \end{cases}$$

In the vacuum, there are K_1 dyons of the 1st kind ... K_N dyons of the Nth kind: one has to combine the known metric for same-kind and different-kind dyons.

Ensemble of dyons

Each dyon has 3 coordinates of the center, and 1 $U(1)$ phase (4 collective coordinates).

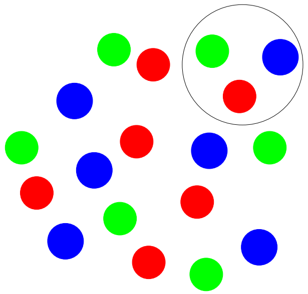
Partition function of the ensemble of N kinds of dyons for $SU(N)$:

$$\mathcal{Z} = \sum_{K_1 \ldots K_N} \frac{1}{K_1! \ldots K_N!} \prod_{m=1}^{N} \prod_{i=1}^{K_m} \int (d^3\mathbf{x}_{mi}\, f)\, \det G(\mathbf{x})\,.$$

$G(\mathbf{x})$ is a $(K_1 + \ldots + K_N) \times (K_1 + \ldots + K_N)$ metric tensor of the moduli space:

$$G_{mi,nj} = \delta_{mn}\delta_{ij}\left(4\pi(\mu_{m+1}-\mu_m)+\sum_k \frac{1}{|\mathbf{x}_{mi}-\mathbf{x}_{m-1,k}|}+\sum_k \frac{1}{|\mathbf{x}_{mi}-\mathbf{x}_{m+1,k}|}-2\sum_{k\neq i}\frac{1}{|\mathbf{x}_{mi}-\mathbf{x}_{mk}|}\right)$$

$$- \frac{\delta_{m,n-1}}{|\mathbf{x}_{mi}-\mathbf{x}_{m+1,j}|} - \frac{\delta_{m,n+1}}{|\mathbf{x}_{mi}-\mathbf{x}_{m-1,j}|} + 2\,\frac{\delta_{mn}}{|\mathbf{x}_{mi}-\mathbf{x}_{mj}|}\Bigg|_{i\neq j}\,.$$



3 kinds of the SU(3) gauge group. Same-kind dyons **repulse** each other, different-kind **attract** each other.

Coulomb coefficients are determined by the Cartan matrix:

$$C_{mn} = \mathrm{Tr}\, C_m C_n = \begin{cases} -1, & n = m-1 \\ 2, & n = m \\ -1, & n = m+1 \end{cases}$$

- the metric is **hyper-Kähler**, as it should be for the moduli space of any self-dual solutions
- for **different-kind** dyons it reduces to the exact metric at all separations, calculated by Kraan (2000) and Gromov and D.D. (2005)
- for **same-kind** dyons it reduces to the metric found by Gibbons and Manton (1995)
- **identity loss**: dyons of the same kind are indistinguishable, meaning that $\det G$ is symmetric under permutation of any pair of dyons $(i \leftrightarrow j)$ of the same kind m
- **factorization**: in the geometry when dyons fall into K well separated clusters of N dyons of all kinds in each, $\det G$ factorizes into a product of exact integration measures for K KvBLL instantons.

Fugacity (from the one-loop renormalization, computed exactly by D.D., Gromov, Petrov and Slizovskiy for $SU(2)$, and Gromov and Slizovskiy for general $SU(N)$)

$$f = \frac{4\pi}{g^4} \frac{\Lambda_{\rm PV}^4}{T} c, \qquad c = (\mathrm{Det}(-\triangle))^{-1}_{\rm reg,\, norm} \overset{T\to 0}{\approx} 1.$$

Therefore, the free energy and correlation functions in the ensemble will depend on the Yang–Mills scale parameter Λ, temperature T and holonomy $\{\mu_m\}$.

The partition function is very unusual: the ensemble is governed not by $\exp(-U_{\rm int})$ but by a determinant whose dimension is equal to the number of particles! One can write $\det G = \exp(\mathrm{Tr}\,\log G)$ but then $U_{\rm int}$ will depend on many-body forces!

The dyon ensemble can be presented **exactly** as a 3d QFT!

Two tricks.

- "Fermionization" [Berezin]:

$$\det G = \int \prod_A d\psi_A^\dagger \, d\psi_A \, \exp\left(\psi_A^\dagger \, G_{AB} \, \psi_B\right)$$

- "Bosonization" [Polyakov]:

$$\exp\left(\sum_{m,n} \frac{Q_m Q_n}{|\mathbf{x}_m - \mathbf{x}_n|}\right) = \int D\phi \, \exp\left(-\int d\mathbf{x}(\partial_i\phi\partial_i\phi + \rho\phi)\right)$$

$$= \exp\left(\int \rho \frac{1}{\triangle} \rho\right), \qquad \rho = \sum Q_m \, \delta(\mathbf{x} - \mathbf{x}_m)$$

Here the "charges" Q are anticommuting Grassmann variables but one can easily integrate out $\psi, \psi^\dagger$.

One needs $2N$ boson fields v_m, w_m to reproduce diagonal elements of G and $2N$ anticommuting ghost fields $\chi_m^\dagger$, χ_m to reproduce non-diagonal elements of G.

The partition function of the dyon ensemble can be identically written as a quantum field theory [D.D. and Petrov (2007)]:

$$\mathcal{Z} = \int D\chi^\dagger\, D\chi\, Dv\, Dw\, \exp\int d^3x \left\{\frac{T}{4\pi}\left(\partial_i\chi_m^\dagger\partial_i\chi_m + \partial_i v_m\partial_i w_m\right)\right.$$
$$\left. + \quad f\left[(-4\pi\mu_m + v_m)\frac{\partial\mathcal{F}}{\partial w_m} + \chi_m^\dagger\,\frac{\partial^2\mathcal{F}}{\partial w_m\partial w_n}\,\chi_n\right]\right\}$$
$$\mathcal{F} = \sum_{m=1}^{N} e^{w_m - w_{m+1}} \qquad \text{(periodic, or “affine” Toda lattice)}$$

N boson fields v_m are Abelian electric potentials
N boson fields w_m are the **dual** Abelian potentials
$2N$ anticommuting ghost fields $\chi_m^\dagger$, χ_m are in fact needed to support the holomorphic (hyper-Kähler) properties of the dyons' metric.

The $3d$ quantum field theory is exactly solvable!

$$\int Dv_m \quad \longrightarrow \quad \delta\left(-\frac{T}{4\pi}\partial^2 w_m + f\frac{\partial\mathcal{F}}{\partial w_m}\right)$$

This δ-function restricts possible fields w_m over which one still has to integrate. Integration over small fluctuations about $\bar{w}_m$ solving the δ-function gives a Jacobian

$$\int Dw_m\,\delta\left(-\frac{T}{4\pi}\partial^2 w_m + f\frac{\partial\mathcal{F}}{\partial w_m}\right) \quad \longrightarrow \quad \det{}^{-1}\left(-\frac{T}{4\pi}\partial^2\delta_{mn} + f\frac{\partial^2\mathcal{F}}{\partial w_m\partial w_n}\bigg|_{w=\bar{w}}\right)$$

which is immediately canceled by the ghost determinant in the same background $\bar{w}_m$:

$$\int D\chi_m^\dagger D\chi_m \quad \longrightarrow \quad \det\left(-\frac{T}{4\pi}\partial^2\delta_{mn} + f\frac{\partial^2\mathcal{F}}{\partial w_m\partial w_n}\bigg|_{w=\bar{w}}\right)$$

Boson loops cancel exactly ghost loops (like in supersymmetry!) $\Longrightarrow$ the dyons ensemble is basically governed by a classical field theory, no quantum corrections!!

Ground state: "confining" holonomy preferred

To find the ground state we examine the fields' potential energy

$$\mathcal{P} = -4\pi f \sum_m \nu_m \, e^{w_m - w_{m+1}}, \qquad \nu_m = \mu_{m+1} - \mu_m.$$

Stationary point in w_m:

$$e^{w_1 - w_2} = \frac{(\nu_1\nu_2\nu_3\ldots\nu_N)^{\frac{1}{N}}}{\nu_1}, \quad e^{w_2 - w_3} = \frac{(\nu_1\nu_2\nu_3\ldots\nu_N)^{\frac{1}{N}}}{\nu_2}, \quad \text{etc.} \quad \Longrightarrow$$

$$\mathcal{P} = -4\pi f \, N \, (\nu_1\nu_2\ldots\nu_N)^{\frac{1}{N}}, \qquad \nu_1 + \nu_2 + \ldots + \nu_N = 1,$$

which has the minimum at

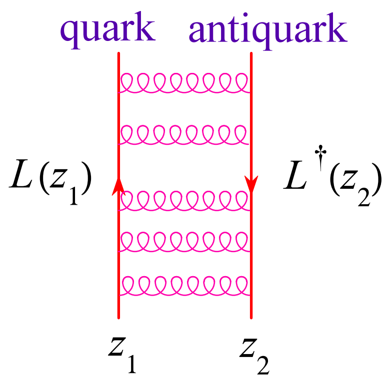
$$\nu_1 = \nu_2 = \ldots = \nu_N = \frac{1}{N}, \qquad \mathcal{P}^{\min} = -4\pi f.$$

Equal ν's correspond to the "confining" holonomy! The free energy at the minimum is

$$F^{\min} = \mathcal{P}^{\min} V = -4\pi f V = -\frac{16\pi^2}{g^4}\Lambda^4 \frac{V}{T} = -\frac{16\pi^2}{g^4}\Lambda^4 \, V^{(4)}$$

and there are no corrections to this result!

Heavy quark potential from Polyakov lines' correlator



Polyakov lines serve as a source for the Abelian electric fields v_m:

$$L(\mathbf{z}) = \mathrm{P}\ \exp\ i\int_0^{1/T} dt\, A_4(\mathbf{z}) = \mathrm{diag}\left(\exp\left(2\pi i\mu_m - \frac{i}{2}v_m(\mathbf{z})\right)\ldots\right)$$

The correlation function of two Polyakov lines in the fundamental representation

$$\left\langle \mathrm{Tr}\,L(\mathbf{z}_1)\mathrm{Tr}\,L^\dagger(\mathbf{z}_2)\right\rangle = \sum_{m_1,n_1} e^{2\pi i(\mu_{m_1}-\mu_{n_1})}\int Dw_m \exp\left(\int d\mathbf{x}\,\frac{4\pi f}{N}\mathcal{F}(w)\right)$$

$$\cdot\prod_m \delta\left(-\frac{T}{4\pi}\partial^2 w_m + f\frac{\partial\mathcal{F}}{\partial w_m} - \frac{i}{2}\,\delta(\mathbf{x}-\mathbf{z_1})\delta_{mm_1} + \frac{i}{2}\,\delta(\mathbf{x}-\mathbf{z_2})\delta_{mn_1}\right) \quad \left(\text{from } \int Dv_m\ldots\right)$$

$$\cdot\mathrm{det}\left(-\frac{T}{4\pi}\partial^2\delta_{mn} + f\frac{\partial^2\mathcal{F}}{\partial w_m\partial w_n}\right) \quad (\text{from ghosts}).$$

One has to find $w_m(\mathbf{x})$ from the δ-function and plug it into the action $\int\mathcal{F}(w)$. There will be no quantum corrections to this classical calculation.

At large separations $\mathbf{z}_1-\mathbf{z}_2$, $w_m(\mathbf{x})$ is small, and one can linearize the equation on $w_m(\mathbf{x})$.

$$\mathcal{F}(w) = \sum_m e^{w_m - w_{m+1}} \approx N + \frac{1}{2} w_m \mathcal{M}_{mn} w_n ,$$

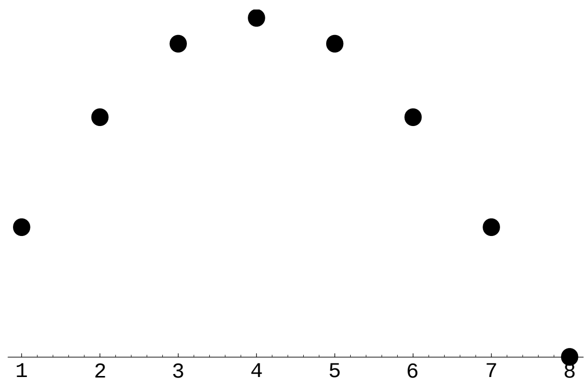
$$\frac{\partial \mathcal{F}}{\partial w_m} \approx \mathcal{M}_{mn} w_n ,$$

$$\mathcal{M} = \begin{pmatrix} 2 & -1 & 0 & \ldots & 0 & -1 \\ -1 & 2 & -1 & \ldots & 0 & 0 \\ 0 & -1 & 2 & -1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ -1 & 0 & 0 & \ldots & -1 & 2 \end{pmatrix}$$

Cartan matrix $\mathcal{M}_{mn} = \mathrm{Tr}\, C_m C_n$.

Its eigenvalues are

$$\mathcal{M}^{(k)} = \left(2 \sin \tfrac{\pi k}{N}\right)^2 , \qquad k = 1, \ldots, N.$$



The result for the correlation function:

$$\left\langle \mathrm{Tr}\, L(\mathbf{z}_1) \mathrm{Tr}\, L^\dagger(\mathbf{z}_2) \right\rangle = \mathrm{const.}\ \exp\left(-|\mathbf{z}_1 - \mathbf{z}_2|\, M\, 2 \sin\frac{\pi}{N} \right),$$

where

$$M^2 = \frac{4\pi f}{T} = \frac{16\pi^2 \Lambda^4}{g^4 T^2} = \mathcal{O}(N^2).$$

This should be compared with the standard definition of the heavy-quark potential

$$\left\langle \mathrm{Tr}\, L(\mathbf{z}_1) \mathrm{Tr}\, L^\dagger(\mathbf{z}_2) \right\rangle = C\ \exp\left(-\frac{V(\mathbf{z}_1 - \mathbf{z}_2)}{T} \right)$$

from where we deduce the linear heavy-quark potential at large separations:

$$V(\mathbf{z}_1 - \mathbf{z}_2) = |\mathbf{z}_1 - \mathbf{z}_2|\, MT\, 2 \sin\frac{\pi}{N} = \sigma\, |\mathbf{z}_1 - \mathbf{z}_2|, \qquad C = \mathcal{O}(N^0),$$

with the 'string tension'

$$\sigma = MT\, 2 \sin\frac{\pi}{N} = \frac{\Lambda^2}{\lambda} \frac{N}{\pi} \sin\frac{\pi}{N}, \qquad \lambda = \frac{g^2 N}{8\pi^2} = \mathcal{O}(N^0) \quad \text{'t Hooft coupling.}$$

The string tension is independent of T and stable at large N, as expected.

N-ality and k-strings

All IREP's of $SU(N)$ can be put into N classes with respect to confinement:

- those that appear in adjoint $\otimes$ adjoint $\otimes \ldots$
- those that appear in (rank-k antisymmetric rep) $\otimes$ adjoint $\otimes$ adjoint $\otimes \ldots$
  These are called N-ality $= k$ representations, $k = 1, \ldots, N-1$.

All adjoint sources can be screened by an appropriate number of gluons. Therefore, all N-ality $= k$ sources must have asymptotically the same string tension $\sigma(k, N)$. Its behaviour with k and N is of fundamental importance as it discriminates between various confinement mechanisms.

We obtain

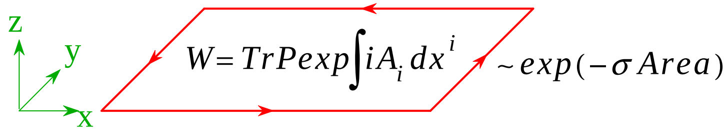
$$\left\langle \mathrm{Tr}\, L_k(\mathbf{z}_1) \mathrm{Tr}\, L_k{}^\dagger(\mathbf{z}_2) \right\rangle = \mathrm{const.}\ \exp\left(-|\mathbf{z}_1 - \mathbf{z}_2|\, M\, 2 \sin\frac{\pi k}{N} \right),$$

hence

$$\sigma(k, N) = \frac{\Lambda^2}{\lambda} \frac{N}{\pi} \sin\frac{\pi k}{N}$$

known as "the sine regime"; it has been encountered in certain supersymmetric models.

Magnetic string tension from average Wilson loops



Averaging Wilson loops over the dyon ensemble we find the area law, where the string tension is determined from solitons of the so-called periodic Toda lattice ($m = 1 \ldots N$), with a source along the surface of the loop:

$$-\partial^2 w_m + M^2 \left(e^{w_m - w_{m+1}} - e^{w_{m-1} - w_m} \right) = -2\pi i \, \delta_{m m_1} \, \delta'(z) \, \theta(x, y \in \mathrm{Area}), \quad M^2 = \frac{4\pi f}{T}.$$

We have found solutions (the **'pinned solitons'**) for all N and k. In all cases the 'magnetic' string tension coincides with the 'electric' one (computed from Polyakov lines):

$$\sigma(k, N) = \frac{\Lambda^2}{\lambda} \frac{N}{\pi} \sin \frac{\pi k}{N}.$$

Lorentz symmetry is restored at $T \to 0$ by the dyon ensemble, despite its $3d$ formulation!

An unexpected finding: There is a continuous set of solitons characterized by a phase, leading to a continuous set of string profiles but all with the **same** string tension. Therefore, there is an extra Goldstone mode living on the string (on top of the usual long-wave deformations), and the effective action is more complicated than the standard Nambu–Goto one. *Non-critical string?*

k-string profile is given by the solution of the Toda equations, with a source depending on k:

$$w^{(k)}_{m,m+1}(z) = \begin{cases} \ln \dfrac{\left[1+\gamma\kappa^{k(m-1)}E^{(k)}(z)\right]\left[1+\gamma\kappa^{k(m+1)}E^{(k)}(z)\right]}{\left[1+\gamma\kappa^{k\,m}E^{(k)}(z)\right]^2}, & z>0, \\ \ln \dfrac{\left[1+\gamma^*\kappa^{*\,k(m-1)}E^{(k)}(-z)\right]\left[1+\gamma^*\kappa^{*\,k(m+1)}E^{(k)}(-z)\right]}{\left[1+\gamma^*\kappa^{*\,k\,m}E^{(k)}(-z)\right]^2}, & z<0, \end{cases}$$

$$\begin{cases} m = 1, \ldots, N, \\ k = 1, \ldots, N-1, \end{cases}$$

$$E^{(k)}(z) = \exp(-M\sqrt{\mathcal{M}^{(k)}}\, z), \quad \sqrt{\mathcal{M}^{(k)}} = 2\sin\frac{\pi k}{N}, \quad \kappa = \exp\left(\frac{2\pi i}{N}\right), \; \gamma = e^{i\alpha}.$$

The phase α is arbitrary: the solution depends on α but the string tension does not! It is a new “Goldstone mode” of the string.

Thermodynamics of the deconfinement phase transition

In the confinement phase, the free energy is

$$\frac{F}{V} = -N^2\frac{\Lambda^4}{2\pi^2\lambda^2} + T^4\,\frac{\pi^2}{45}\left(N^2-\frac{1}{N^2}\right) - T^4\,\frac{\pi^2}{45}\left(N^2-1\right)$$

(dyon-induced) (perturbative energy at maximum) (Stefan–Boltzmann)

$\mathcal{O}(N)$ gluons are canceled from the free energy, as it should be in the confining phase!

The 1st order confinement-deconfinement phase transition is expected at

$$T_c^4 = \frac{45}{2\pi^4}\,\frac{N^4}{N^4-1}\,\frac{\Lambda^4}{\lambda^2}\,.$$

T_c is stable in N as expected.

Robust quantities (both from the theoretical and lattice viewpoints) are those measured in units of the string tension $\sigma = \frac{\Lambda^2}{\lambda}\frac{N}{\pi}\sin\frac{\pi}{N}$:

	$N=3$	4	6	8
$T_c/\sqrt{\sigma}$, theory	0.6430	0.6150	0.5967	0.5906
$T_c/\sqrt{\sigma}$, lattice	0.6462(30)	0.6344(81)	0.6101(51)	0.5928(107)

Lattice data are from Lucini, Teper and Wenger (2003).

Topological susceptibility (lattice data are from Lucini and Teper (2001)):

$$\frac{(<Q_T^2>)^{\frac{1}{4}}}{\sqrt{\sigma}} = \begin{cases} 0.439, & \text{theory} \\ 0.434(10), & \text{lattice} \end{cases} \qquad \text{for } N = 3.$$

Why does the semiclassical picture work so suspiciously well?

Contribution from a classical saddle point (here: dyon), schematically

$$\underbrace{M^4 \exp\left(-\frac{2\pi}{\alpha_s}\right)}_{\Lambda^4}\left[1 + c_1\frac{\alpha_s N}{2\pi} + c_2\left(\frac{\alpha_s N}{2\pi}\right)^2 + \ldots\right]$$

What is α_s? It is the running coupling constant whose argument is the maximal scale in the problem, i.e. $\max\left(\mathrm{density}^{\frac{1}{3}}, \mathrm{temperature}\right)$. At $T \approx T_c \approx \Lambda$

$$\frac{1}{\lambda} \equiv \frac{2\pi}{\alpha_s\, N} = \frac{11}{3}\, \ln\left(\frac{4\pi T}{\Lambda e^{\gamma_{\mathrm{E}}}}\right)\Bigg|_{T\approx T_c\approx\Lambda} \approx 7.$$

Therefore, the loop expansion is a series in $\approx \frac{1}{7}$.

[Similar arithmetics is met in Wilson's ϵ expansion for anomalous dimensions in critical phenomena: the expansion parameter is formally $\mathcal{O}(1)$, however in reality the loop expansion is a series in $\frac{1}{2\pi}$. It gives accurate results from the first couple of terms.]

Summary

1. The statistical weight of gluon field configurations in the form of N kinds of dyons has been computed exactly to 1-loop

2. Statistical physics of the ensemble of interacting dyons is governed by an exactly solvable 3d QFT

3. The ensemble of dyons self-organizes in such a way that all criteria of confinement are fulfilled

> Non-trivial holonomy allows the existence of dyons
> Dyons request the holonomy to be maximally non-trivial !

4. All quantities computed are in good agreement with lattice data

5. A simple picture of a semi-classical vacuum based on dyons works well!

Effective 1-loop action for slowly varying eigenvalues of the Polyakov line [D.D. and Oswald (2004)]:

$$\text{Eigenvalues of } L = \mathrm{P}\,\exp\left(i\int dx^4\, A_4(\mathbf{x},x^4)\right) = \left(e^{2\pi i\mu_1},\ldots,e^{2\pi i\mu_N}\right)$$

$$\begin{aligned} S_{\rm eff}^{1-\rm loop} &= \sum_{m>n}^{N}\int d^3x \\ &\times\left\{(\partial_i\nu_{mn})^2\frac{11}{12}T\left[2\,\log\left(\frac{4\pi T}{\Lambda\, e^{\gamma_E}}\right)+H(\nu_{mn})\right]\right. \\ &\left.+\frac{(2\pi)^2T^3}{3}\nu_{mn}^2(1-\nu_{mn})^2\right\},\quad \nu_{mn}=\mu_m-\mu_n, \\ H(\nu) &= \left[\psi(\nu)+\psi(1-\nu)+2\gamma_E\right]_{\rm mod\ 1} \end{aligned}$$

Since $\psi(\nu)\approx -1/\nu$ at small ν, the gradient term becomes negative near "trivial" holonomy, which signals its instability even in pure perturbation theory.

The constructed $4KN \times 4KN$ metric tensor $g_{A\alpha,B\beta}$ is hyper-Kähler. It means that there exist three "complex structures" $I(a)$, $a = 1, 2, 3$, (all three are $4KN \times 4KN$ matrices) such that

$$I(a)g = gI(a)^T \quad (\text{“T” means transposed}) \tag{1}$$

and which satisfy the Pauli algebra,

$$I(a)I(b) = \epsilon^{abc} I(c) - \delta^{ab}\mathbf{1}. \tag{2}$$

Related to $I(a)$, there are three Kähler symplectic 2-forms

$$\omega(a) = \Omega(a)_{B\beta,C\gamma}\, dy_B^\beta \wedge dy_C^\gamma, \quad \Omega(a) = -\Omega(a)^T, \tag{3}$$

where

$$\Omega(a) = I(a)g. \tag{4}$$

The 2-forms $\omega(a)$ are closed:

$$d\omega(a) = 0 \quad \text{or} \quad \frac{\partial}{\partial y_A^\alpha}\Omega(a)_{B\beta,C\gamma}\, dy_A^\alpha \wedge dy_B^\beta \wedge dy_C^\gamma = 0. \tag{5}$$

Explicitly, the three Kähler forms $\omega(a)$ are

$$\omega(a) = 2(d\psi_A + \mathbf{W}_{AA'} \cdot d\mathbf{x}_{A'}) \wedge dx_A^a - G_{BC}\, \epsilon^{abc}\, dx_B^b \wedge dx_C^c. \tag{6}$$