Twisted Mass, Overlap and Creutz Fermions: Cut-off Effects at Tree-level of Perturbation Theory

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The paper

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Seminar outline

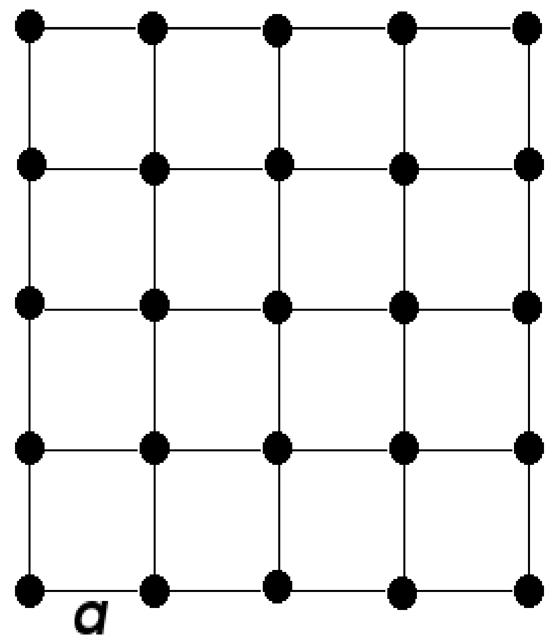
- Introduction – lattice field theory
- Wilson fermions
- Wilson Twisted Mass fermions. Maximal Twist
- Overlap fermions
- Creutz fermions
- Conclusion

The quantities under investigation – the pseudoscalar meson correlation function, mass & decay constant.

Lattice Field Theory

The main motivation:
Non-perturbative aspects of Quantum Field Theories
particularly: Quantum ChromoDynamics

asymptotic freedom	confinement
distances $\ll 1$ fm	distances ≈ 1 fm
quarks & gluons	hadrons & glueballs
perturbative	non-perturbative



Kenneth Wilson (1974)
Confinement of Quarks
Phys. Rev. D 10, 2445-2459

”Conventional” QFT

In ”conventional” Quantum Field Theory one calculates the vacuum expectation values:

$$\langle 0|T\{O_1(x_1)O_2(x_2)\ldots\}|0\rangle = \frac{\int \mathcal{D}A_\mu \mathcal{D}\psi \mathcal{D}\bar{\psi}\ O_1(x_1)O_2(x_2)\ldots e^{iS}}{\int \mathcal{D}A_\mu \mathcal{D}\psi \mathcal{D}\bar{\psi}\ e^{iS}}, \quad (1)$$

where:

$$S = \int d^4x\ \mathcal{L} \quad (2)$$

is the action.

Space-time has 3 spatial dimensions and 1 temporal dimension.

Minkowski metric:

$$g_{\mu\nu} = diag(1, -1, -1, -1). \quad (3)$$

„Euclidean” QFT

Before one goes to the lattice, one switches to the Euclidean metric:

$$g_{\mu\nu} = diag(1, 1, 1, 1) \tag{4}$$

by performing a Wick rotation:

$$x_0 \longrightarrow -ix_4. \tag{5}$$

We thus have:

$$\langle 0|T\{O_1(x_1)O_2(x_2)\ldots\}|0\rangle = \frac{\int \mathcal{D}A_\mu \mathcal{D}\psi \mathcal{D}\bar{\psi}\ O_1(x_1)O_2(x_2)\ldots e^{-S_E}}{\int \mathcal{D}A_\mu \mathcal{D}\psi \mathcal{D}\bar{\psi}\ e^{-S_E}}, \tag{6}$$

where:

$$S = i\int d^4x\ \mathcal{L}_{\mathcal{E}} \tag{7}$$

is the Euclidean action.

Expression (6) can be computed on the lattice.

„Naive" discretization of fermions

Consider the following action in EQFT:

$$S = \int d^4x \bar{\psi}(x) \left(\gamma_\mu \partial_\mu + m\right) \psi(x). \tag{8}$$

We have to discretize the derivative:

$$\partial_\mu \psi_\alpha(n) = \frac{1}{2}\left(\psi_\alpha(n+\hat{\mu}) - \psi_\alpha(n-\hat{\mu})\right) \equiv \frac{1}{2}(\nabla_\mu + \nabla^*_\mu)\psi_\alpha(n), \tag{9}$$

where ∇_μ and ∇^*_μ denote the forward and backward lattice derivative.

We can rewrite:

$$S = \sum_{n,m} \bar{\psi}_\alpha(n) K_{\alpha\beta}(n,m) \psi_\beta(m), \tag{10}$$

where:

$$K_{\alpha\beta}(n,m) = \sum_\mu \frac{1}{2} \left(\gamma_\mu\right)_{\alpha\beta} \left(\delta_{m,n+\hat{\mu}} - \delta_{m,n-\hat{\mu}}\right) + m\delta_{mn}\delta_{\alpha\beta}. \tag{11}$$

One can show that this leads to the following form of the fermion propagator:

$$S(p) = \frac{-i\gamma_\mu \sin p_\mu + \mathbb{1}m}{\sum_\mu \sin p_\mu^2 + m^2}, \tag{12}$$

where:

$$p_\mu = \frac{2\pi(n_\mu + \delta_\mu)}{L_\mu}, \tag{13}$$

$n_\mu = 0, 1, \ldots, L_\mu - 1$,
L_μ – spatial lattice extent in direction $\hat{\mu}$,
$\delta_\mu = 0$ for PBC, $\delta_\mu = 1/2$ for ABC.

- This leads to an incorrect continuum limit, which is due to additional zeros of the sine at the corners of the Brillouin zone.

- Instead of 1 fermion we have 2^d of them!

- $2^d - 1$ of these fermions are lattice artifacts and have no counterpart in reality. This is the notorious *fermion doubling problem*.

- One can show that this problem is caused by the symmetric form of the lattice derivative.

Nielsen-Ninomiya theorem

Fermion doubling problem is very general.

Nielsen and Ninomiya showed in 1981 that it is impossible to have at the same time:

- locality,
- translational invariance,
- no doublers,
- chiral symmetry.

The N-N theorem is of topological origin:
chiral symmetry means that $D(p)$ is of the form $\gamma_\mu d_\mu(p)$. We can assign an index of +1 or -1 to every zero of the d_μ, and then the Hopf-Poincare index theorem states that the sum over the indices of the zeros of a vector field on a manifold is equal to the Euler characteristic of the manifold. For n-torus the Euler characteristic is 0, so zeros must come in pairs of opposite index.

Wilson discretization

The simplest way to overcome the fermion doubling problem is to add the so-called Wilson term to the fermion action.

The Wilson-Dirac operator is:

$$D_W = \frac{1}{2}\left(\gamma_\mu(\nabla_\mu^* + \nabla_\mu) - \nabla_\mu^*\nabla_\mu\right) + m. \tag{14}$$

The fermion propagator in momentum space:

$$S(p) = \frac{-i\gamma_\mu \sin p_\mu + \mathbb{1}\left(\sum_\mu\left(1-\cos p_\mu\right)+m\right)}{\sum_\mu \sin p_\mu^2 + \left(\sum_\mu\left(1-\cos p_\mu\right)+m\right)^2}. \tag{15}$$

In the continuum limit the doublers become infinitely heavy and decouple.

But: the Wilson term explicitly breaks chiral symmetry, even in the massless limit!

Correlation functions for pions

The interpolating fields describing the charged pions, π^+ and π^- are:

$$\mathcal{P}^{\pm}(x) \equiv \mathcal{P}^1(x) \mp i\mathcal{P}^2(x) \tag{16}$$

where $\mathcal{P}^a(x) = \bar{\psi}(x)\gamma_5\frac{\tau^a}{2}\psi(x)$, with $a = 1, 2, 3$, is the pseudoscalar density and τ^a are the standard Pauli matrices.

The quark propagator can be decomposed in terms of the gamma matrices as

$$\tilde{S}(p) = S_U(p)\mathbb{1} + \sum_\mu S_\mu(p)\gamma_\mu \tag{17}$$

in the case of overlap and Creutz fermions, while for twisted mass fermions an additional term proportional to γ_5 is present

$$\tilde{S}(p) = S_U(p)\mathbb{1} + \sum_\mu S_\mu(p)\gamma_\mu + S_5(p)\gamma_5. \tag{18}$$

Correlation functions for pions

With such a decomposition, the pseudoscalar correlation function can be written as:

$$C(t) = \frac{N_c N_d}{L^3 L_4^2} \sum_{p_4} \sum_{p_4'} \sum_{\vec{p}} \sum_{\mu} e^{i(p_4 - p_4')t} S_\mu(\vec{p}, p_4) S_\mu^*(\vec{p}, p_4'), \tag{19}$$

with $\mu = U, 1, 2, 3, 4$ or $\mu = U, 1, 2, 3, 4, 5$ depending on the kind of fermions that is being considered.

N_c – the number of colours,
N_d – the number of Dirac components,
$L_\mu = aN_\mu$ and N_μ – the number of lattice points in the $\hat{\mu}$-direction.

The expression in eq. (19) can be evaluated as it stands, or a time-momentum representation can be used obtained for a lattice with infinite time extension by performing the time integration over p_4 analytically.

Correlation function for proton

The local interpolating field describing the proton is given by

$$\mathscr{P}_\alpha(x) \equiv -\sqrt{2}\epsilon_{abc}\left[\vec{d}_a^T(x)\, C^{-1}\gamma_5 u_b(x)\right] u_{\alpha,c}(x). \tag{20}$$

The Greek (Latin) letters denote Dirac (colour) components,
u, d – the flavour content, C – charge conjugation matrix, $[\,]$ denotes spin trace.

The expression for the time dependence of the proton correlation function is:

$$C_{\mathscr{P}\bar{\mathscr{P}}}(t) = \frac{N_c N_d}{L^6} \sum_{\vec{p}} \sum_{\vec{q}} \Big\{ L_U(\vec{p},\vec{q},t) + L_4(\vec{p},\vec{q},t) \Big\}, \tag{21}$$

with the definitions:

$$L_U(\vec{p},\vec{q},t) \equiv S_U^u(-(\vec{p}+\vec{q}),t)\Big\{(N_d+1)\, S_U^u(\vec{p},t) S_U^u(\vec{q},t) + (N_d+3) \sum_{\mu=1}^{4} S_\mu^u(\vec{p},t) S_\mu^u(\vec{q},t), \Big\} \tag{22}$$

$$L_\mu(\vec{p},\vec{q},t) \equiv S_\mu^u(-(\vec{p}+\vec{q}),t)\Big\{(N_d+3)\, S_U^u(\vec{p},t) S_U^u(\vec{q},t) + (N_d+1) \sum_{\mu=1}^{4} S_\mu^u(\vec{p},t) S_\mu^u(\vec{q},t) \Big\}. \tag{23}$$

Scaling tests

- At tree-level a dimensionless quantity can be only a function of mL, a/L and am, where m indicates the quark mass.

- To perform the continuum limit one can fix mL to a certain value and the remaining dependence of the dimensionless quantity will be then in a/L.

- The continuum limit is then obtained sending $N = L/a$ to infinity.

- We set $a = 1$ and the $1/N$ and $1/N^2$ dependence of the dimensionless quantities under investigation will correspond to $\mathcal{O}(a)$ and $\mathcal{O}(a^2)$ scaling violations

- We consider:
  - the correlation function at a fixed physical time t/N,
  - the pseudoscalar decay constant f_{PS} ,
  - the pseudoscalar and proton masses M.

- This leads dimensionless quantities $N^3 C(t/N)$, NM and $N f_{\mathrm{PS}}$.

Mass average for Wilson fermions

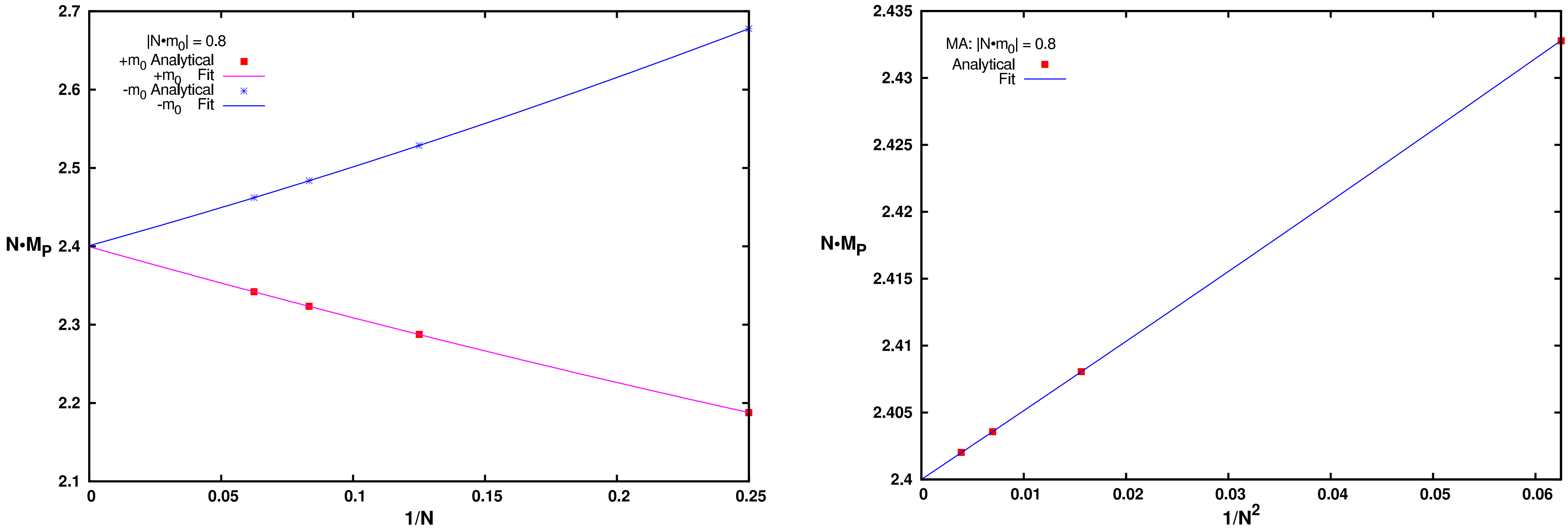


Figure 1: Left graph: the cutoff effects and the continuum limit of the proton mass obtained from two standard Wilson actions differing only in the sign of the quark mass, $|Nm_0| = 0.8$.
Right graph: the average of the proton masses obtained from the same two standard Wilson regularizations with quark masses $Nm_0 = \pm 0.8$.

The lines are fits of the data to the following functions:

$$y_1 = a_0 + a_1 \frac{1}{N} + a_2 \frac{1}{N^2}\ , \quad y_2 = b_0 + b_1 \frac{1}{N^2} + b_2 \frac{1}{N^4}. \qquad (24)$$

Wilson *twisted mass* discretization

An automatic $\mathcal{O}(a)$-improvement can be obtained if one adds an extra term to the action:

$$S = a^4 \sum_x \bar{\psi}\left(D_{Wilson} + i\mu_q\gamma_5\tau_3\right)\psi. \tag{25}$$

The expression for the Wilson twisted mass fermion propagator in the twisted basis, at tree-level of perturbation theory and in momentum space is given by

$$\widetilde{S}(p) = \frac{-i\mathring{p}_\mu\gamma_\mu \mathbb{1}_f + M(p)\mathbb{1}\mathbb{1}_f - i\mu_q\,\gamma_5\tau_3}{\sum_\mu \mathring{p}_\mu^2 + M(p)^2 + \mu_q^2}, \tag{26}$$

where

$$\mathring{p}_\mu = \frac{1}{a}\sin(ap_\mu), \quad \hat{p}_\mu = \frac{2}{a}\sin(\frac{ap_\mu}{2}), \quad M(p) = m_0 + \frac{ar}{2}\sum_\mu \hat{p}_\mu^2 \tag{27}$$

where $\mathbb{1}$ and $\mathbb{1}_f$ are the identity matrices in Dirac and flavour space.

The structure in colour space has not been written since it is just an identity matrix at tree level of PT.

The parameters m_0 and μ_q represent the untwisted and twisted quark masses, respectively. Maximal twist – in the case of tree-level of perturbation theory – is achieved by setting $m_0 = 0$. We then expect to have only $\mathcal{O}(a^2)$ lattice spacing effects in physical correlation functions.

Overlap discretization

In 1982 Ginsparg and Wilson found a way to break the chiral symmetry on the lattice in a controlled way. As a result of renormalization group calculations their Dirac operator obeyed the following relation:

$$\gamma_5 D + D\gamma_5 = aD\gamma_5 D. \tag{28}$$

A particularly simple form of a Dirac operator obeying the G-W relation was given by H. Neuberger in 1997.

$$D_{overlap} = \frac{1}{a}\Big(1 - A(A^\dagger A)^{-1/2}\Big), \tag{29}$$

where:

$$A = 1 - aD_{Wilson}(p). \tag{30}$$

In 1998 Lüscher found that the Ginsparg-Wilson relation leads to a non-standard realization of chiral symmetry in the theory. The action is invariant under:

$$\psi \to e^{i\theta\gamma_5\left(1-\frac{aD}{2}\right)}\psi, \tag{31}$$

$$\bar{\psi} \to \bar{\psi}e^{i\theta\gamma_5\left(1-\frac{aD}{2}\right)}. \tag{32}$$

The Dirac operator no longer anticommutes with γ_5, so the conditions of the Nielsen-Ninomiya theorem do not apply.

Overlap discretization

The expression for the overlap propagator in momentum space at tree-level of perturbation theory is

$$\tilde{S}(p) = \frac{-i(1-\frac{ma}{2})F(p)^{-1/2}\mathring{p}_\mu\gamma_\mu + \mathcal{M}(p)\mathbb{1}}{(1-\frac{ma}{2})^2 F(p)^{-1}\sum_\mu \mathring{p}_\mu^2 + \mathcal{M}(p)^2} \tag{33}$$

where:

$$F(p) = 1 + \frac{a^4}{2}\sum_{\mu<\nu}\hat{p}_\mu^2\hat{p}_\nu^2 \tag{34}$$

$$\mathcal{M}(p) = \frac{1}{a}\left(1 + \frac{ma}{2} - \left(1 - \frac{ma}{2}\right)F(p)^{-1/2}\left(1 - \frac{a^2}{2}\sum_\mu \hat{p}_\mu^2\right)\right) \tag{35}$$

and $\mathbb{1}$ is the identity matrix in Dirac space.

Note that in the case of overlap fermions we only discuss one flavour.

Due to the existence of an exact lattice chiral symmetry, we again expect an $\mathcal{O}(a^2)$ scaling behaviour towards the continuum limit, if the correlation functions are computed with the proper improved operators.

Creutz discretization

In Dec'07 Creutz, motivated by the description of the graphene electronic structure in terms of the Dirac equation, generalized it to 4 dimensions to yield a strictly local fermion action describing two species and possessing an exact chiral symmetry.

The Creutz-Dirac operator can be written as:

$$D_{\mathrm{C}}(p) = i\sum_{\mu} \mathring{p}_{\mu}\,\bar{\gamma}_{\mu} - i\,\tfrac{a}{2}\sum_{\mu} \hat{p}_{\mu}^{2}\,\bar{\Gamma}_{\mu} + m_0\,\mathbb{1}, \tag{36}$$

where $\bar{\gamma}_{\mu}$ and $\bar{\Gamma}_{\mu}$ are some linear combinations of gamma matrices[1].

The expression for the fermion propagator:

$$\widetilde{S}_{\mathrm{C}}(p) = \frac{-i\sum_{\mu}\Big(\bar{s}_{\mu}(ap)+\bar{c}_{\mu}(ap)\Big)\,\gamma_{\mu}+m_0\,\mathbb{1}}{\sum_{\mu}\Big(\bar{s}_{\mu}(ap)+\bar{c}_{\mu}(ap)\Big)^{2}+m_0^{2}}. \tag{37}$$

We consider 2 values of the parameter C:

- $C = 3/\sqrt{10}$ – corresponds to the hypercubic lattice,
- $C = 3/\sqrt{14}$ – the case of a highly symmetric lattice built up of hexagonal chairs with an inter-bond angle of ≈ 104.5 degrees,

and also a modification of the action suggested by Borici.

[1] See the paper for details.

Comparing maximally twisted mass, overlap and Creutz fermions

- One interesting question is the *relative* size of cutoff effects when comparing maximally twisted mass, overlap and Creutz fermions.

- We have performed a scaling analysis for the correlation functions themselves at a fixed physical distance, the pseudoscalar mass and the pseudoscalar decay constant.

- Since all these lattice formulations are $\mathcal{O}(a)$-improved, we show all quantities investigated as a function of $1/N^2$.

The pion mass

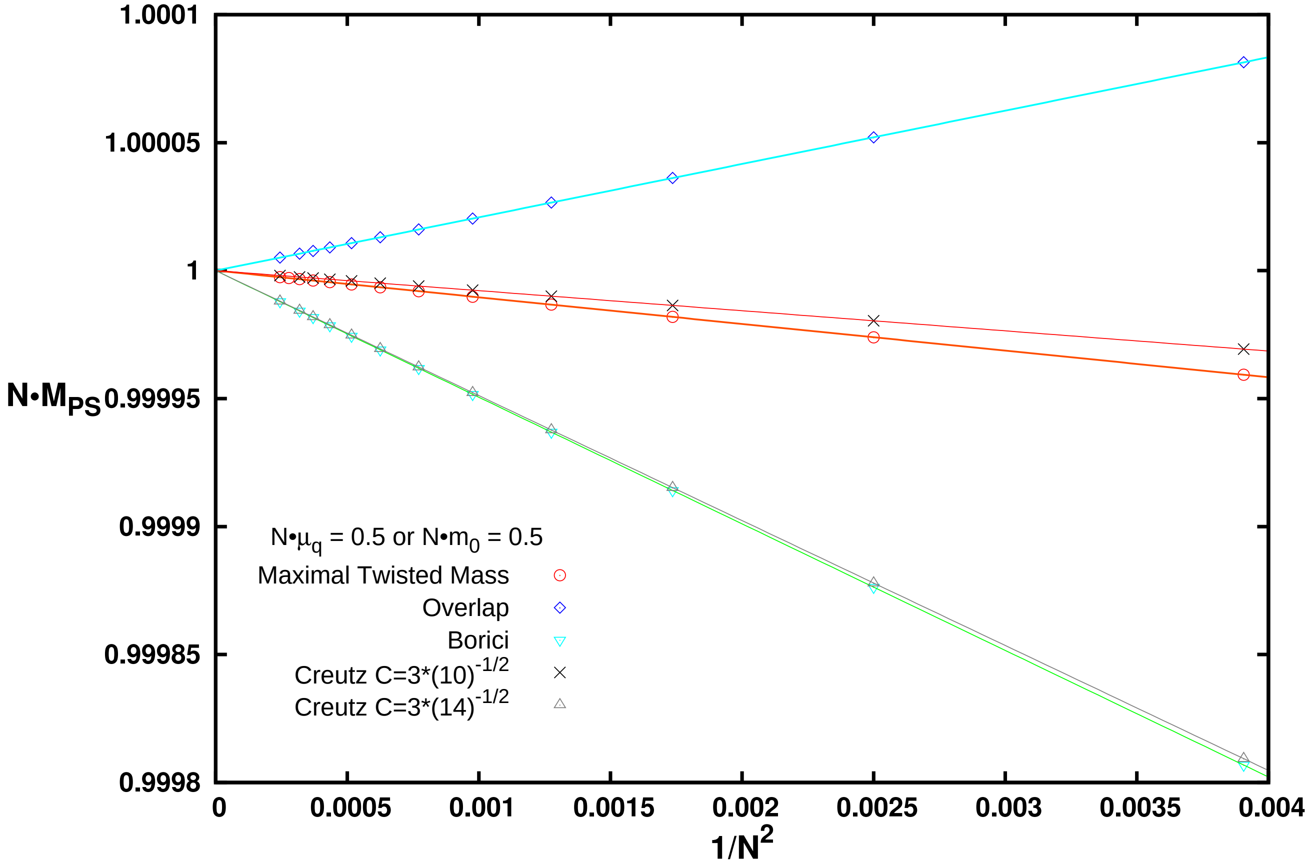


Figure 2: The cutoff effects and the continuum limit of the pseudoscalar mass.

Fit coefficients for the pseudoscalar mass: $Nm_{ps} = a + b\frac{1}{N^2} + c\frac{1}{N^4}$:

Nm_{ps}	a	b	c
MTM	1	-0.0104167	0.000292154
OVERLAP	1	0.0208333	0.000783943
BORICI	1	-0.0494786	0.00558893
CREUTZ - $\sqrt{10}$	1	-0.00781168	-0.010171
CREUTZ - $\sqrt{14}$	1	-0.0488288	0.00287405

The pion correlation function

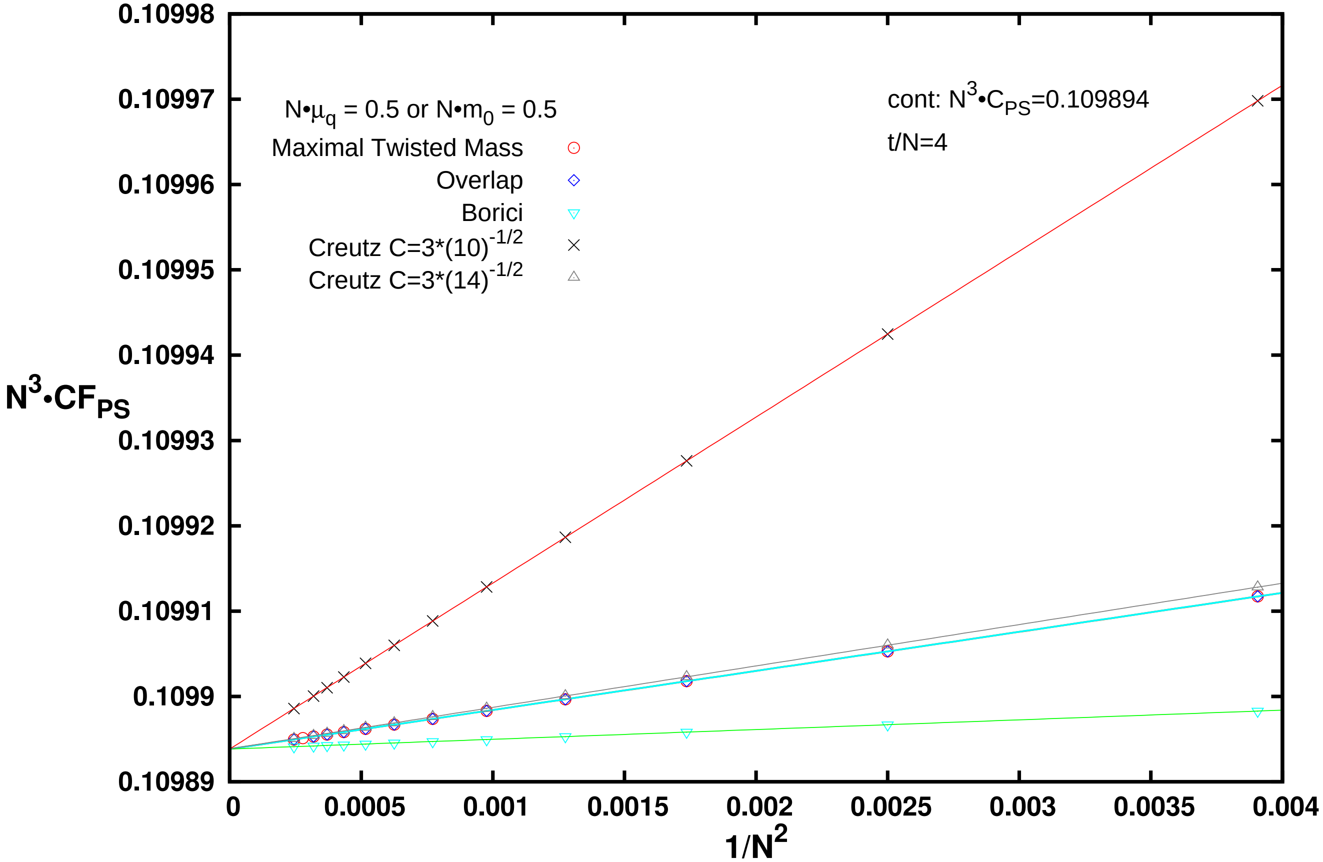


Figure 3: The cutoff effects and the continuum limit of the pseudoscalar correlator.

Fit coefficients for the pseudoscalar correlation function: $N^3 C_{ps} = a + b\frac{1}{N^2} + c\frac{1}{N^4}$:

$N^3 C_{ps}(t/N = 4)$	a	b	c
MTM	0.109894	0.00457891	$-3.33302 \cdot 10^{-5}$
OVERLAP	0.109894	0.0045789	0.000181822
BORICI	0.109894	0.00114427	-0.00135812
CREUTZ - $\sqrt{10}$	0.109894	0.0194625	-0.00286602
CREUTZ - $\sqrt{14}$	0.109894	0.00486428	-0.002942

The pion decay constant

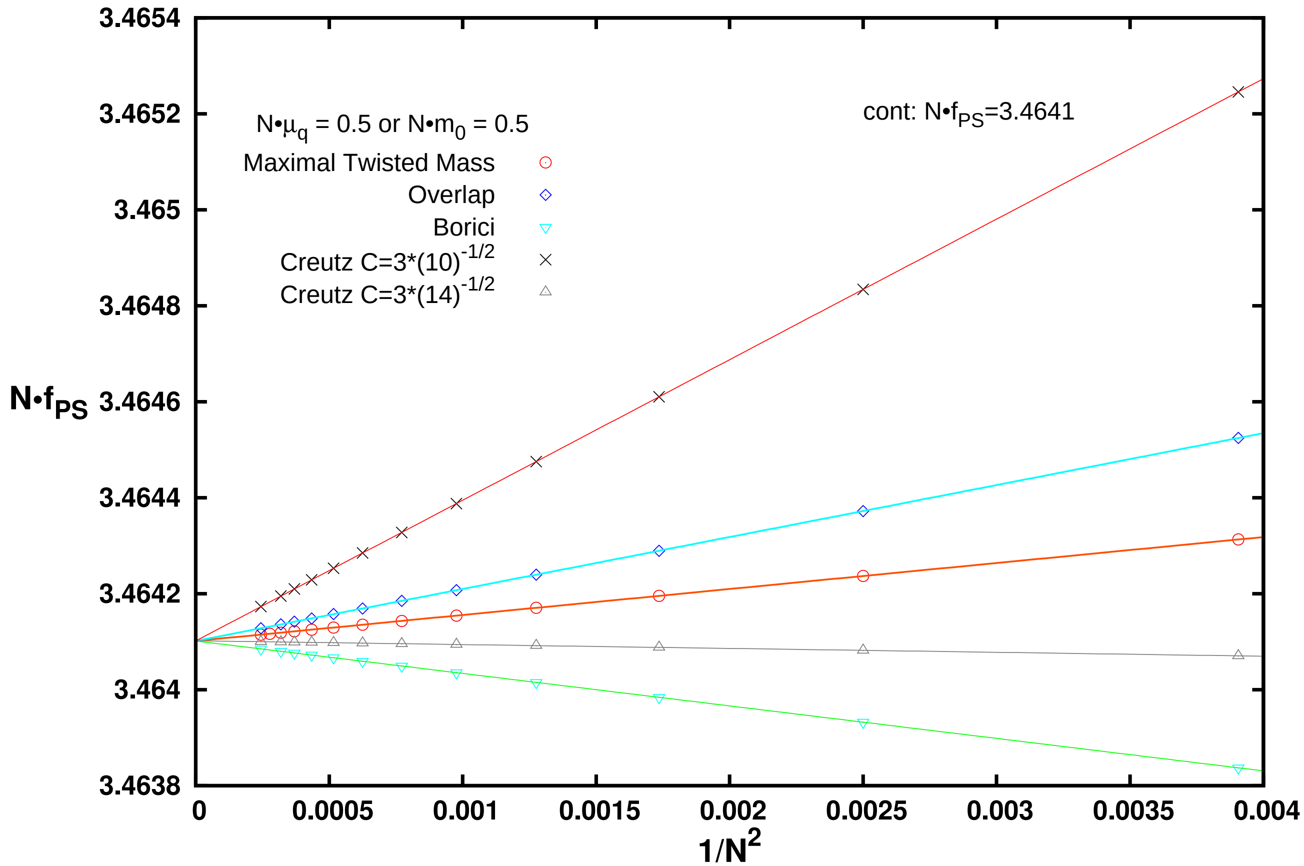


Figure 4: The cutoff effects and the continuum limit of the pion decay constant.

Fit coefficients for the pseudoscalar decay constant: $N f_{ps} = a + b\frac{1}{N^2} + c\frac{1}{N^4}$:

$N f_{ps}$	a	b	c
MTM	3.4641	0.0541266	−0.000815548
OVERLAP	3.4641	0.108253	0.00554908
BORICI	3.4641	−0.0676637	−0.00486739
CREUTZ - $\sqrt{10}$	3.4641	0.293217	−0.0770494
CREUTZ - $\sqrt{14}$	3.4641	−0.00790885	−0.0367598

Discretizations comparison

- No clear picture of a particularly good or bad fermion discretization emerges.
- We find that indeed all three kinds of lattice fermions show the expected $\mathcal{O}(a)$-improvement.
- However, the relative size of the $\mathcal{O}(a^2)$ effects depends pretty much on the considered observable.
- If at all, one could say that maximally twisted mass fermions show uniformly small $\mathcal{O}(a^2)$ cutoff effects.
- On the other hand, it is somewhat amazing that Creutz fermions which break a number of important discrete symmetries do not suffer from very large $\mathcal{O}(a^2)$ cutoff effects.
- From our scaling analysis it is not possible to exclude a certain type of lattice fermion. Only scaling tests for the interacting theory will reveal the size of actual scaling violations of the considered observable.

Effects from non-optimal tuning

- We have also addressed a question of effects when tuning is performed non-optimally.

- We study:

  1. the cutoff effects when there is an $\mathcal{O}(a)$ error in tuning to maximal twist.
  2. the case when the quark masses of two lattice fermion formulations are not exactly matched. This case is relevant for so-called mixed action simulations.

- Ad 1. Out of maximal twist

  – We study a situation when we allow an $\mathcal{O}(a)$ error in setting the untwisted quark mass to zero.
  – In order to realize this situation at tree-level of PT we 'force' these effects by simply fixing the twisted mass to be the physical quark mass and the untwisted mass is set to be proportional to $\frac{1}{N}$, as:

$$N\mu_q = \alpha \quad \text{and} \quad Nm_0 = \frac{\beta}{N} \backsim O(a), \tag{38}$$

  where α is kept fixed and β is a measure parametrizing the amount of violation of the maximal twist setup.

Out of maximal twist

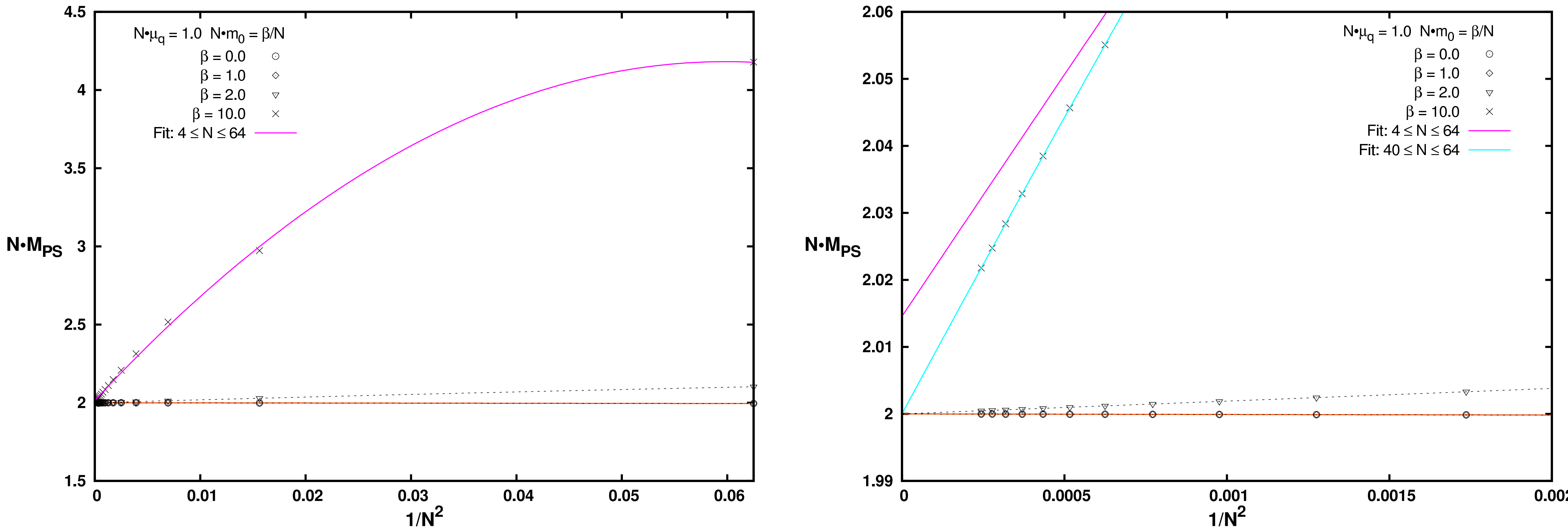


Figure 5: Left graph: Behaviour of the pion mass as a function of $\frac{1}{N^2}$, for lattices with size $4 \leq N \leq 64$. The twisted quark mass is set to $N\mu_q = 1.0$ and the untwisted quark mass is zero up to $\mathcal{O}(a)$ cutoff effects i.e. $Nm_0 = \frac{\beta}{N}$ with $\beta = 0.0, 1.0, 2.0, 10.0$.
Right graph: a zoom of the graph on the left with an additional fit for the analytical data corresponding to $\beta = 10.0$ which considers only large lattices $40 \leq N \leq 64$.

Unmatched quark masses

- We study the continuum limit and the size of the cutoff effects of lattice quantities constructed from ratios of physical observables computed on the lattice from two different regularizations i.e. Wilson twisted mass fermions at maximal twist and overlap fermions.

- In particular, we want to study the situation when both quark masses are not exactly fixed to the same value but differ up to $\mathcal{O}(a^2)$ effects.

- The reason for studying such setup is that in real simulations using a mixed but $\mathcal{O}(a)$-improved action both masses can be fixed to the same value only up to $\mathcal{O}(a^2)$ effects.

- In order to realize non-matched quark masses, we fix the twisted quark mass exactly at $N\mu_q = 0.5$ and allow for an $\mathcal{O}(a^2)$ error in setting the overlap quark mass:
$$Nm = 0.5 - v/N^2. \tag{39}$$
We vary the parameter v from $v = 0$ to $v = 4.0$.

Unmatched quark masses

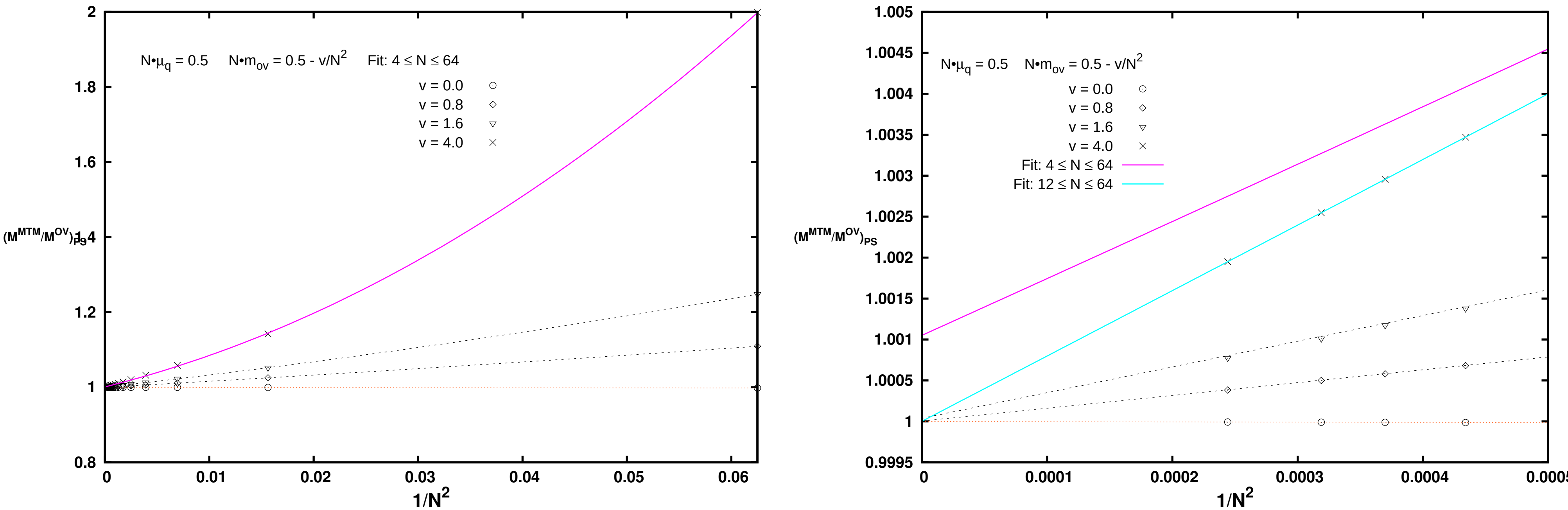


Figure 6: The cutoff effects and continuum limit of the ratio of the pseudoscalar mass computed for maximally twisted mass and overlap fermions. In both graphs $N\mu_q = 0.5$, $Nm = 0.5 - 0.4/N^2$ and $t/N = 4$. The left graph shows the full range of lattice sizes considered while the right graph represents a zoom.

Conclusions

- We have performed a scaling test in the lattice spacing towards the continuum limit for three kinds of lattice fermions (WTM at max. twist, overlap, Creutz).
- Our setup has been tree-level of pertubation theory.
- We looked at the pseudoscalar correlation function at a fixed time and the corresponding pseudoscalar mass and decay constant.
- We have verified the automatic $\mathcal{O}(a)$ improvement for WTM at maximal twist and showed the mechanisms of mass average.
- The relative comparison of all three kind of lattice fermions did not result in a clear picture in the sense that one lattice fermion shows consistently smaller or bigger $\mathcal{O}(a^2)$ lattice artefacts than the other.
- We found that the sizes of $\mathcal{O}(a^2)$ lattice artifacts depend on the considered observable with perhaps the exception of maximally twisted mass fermions which shows a rather uniform behaviour with small $\mathcal{O}(a^2)$ effects.
- Finally, we studied the situation when parameters are tuned non-optimally.
- Our conclusion of these studies is that when the corresponding $\mathcal{O}(a^2)$ error is too large, the continuum limit becomes not reliable unless the lattice spacing is small enough.