We can expand

$$S_{Dis}(N) = k_0 \sum_i \frac{(N_{i+1} - N_i)^2}{F(N_i)} - U(N_i)$$

to second order in $n_i = N_i - \bar{N}_i$

$$S_{Dis}(\bar{N}+n) = S_{Dis}(\bar{N}) + \frac{1}{2}\left[\sum_{i,j} n_i \hat{P}^{kin}_{ij} n_j + n_i \hat{P}^{Pot}_{ij} n_j\right] + O(n^3)$$

$$\hat{P}^{kin} = \sum_{i=1}^{N_T} k_i \hat{X}^{(i)}$$

$$\hat{X}^{(i)}_{jk} = \delta_{ij}\delta_{ik} + \delta_{i+1\,j}\delta_{i+1\,k} - \delta_{i+1\,j}\delta_{ik} - \delta_{ij}\delta_{i+1\,k}$$

$$\hat{P}^{Pot} = \sum_{i=1}^{N_T} u_i \hat{Y}^{(i)}$$

$$\hat{Y}^{(i)}_{jk} = \delta_{ij}\delta_{ik}$$

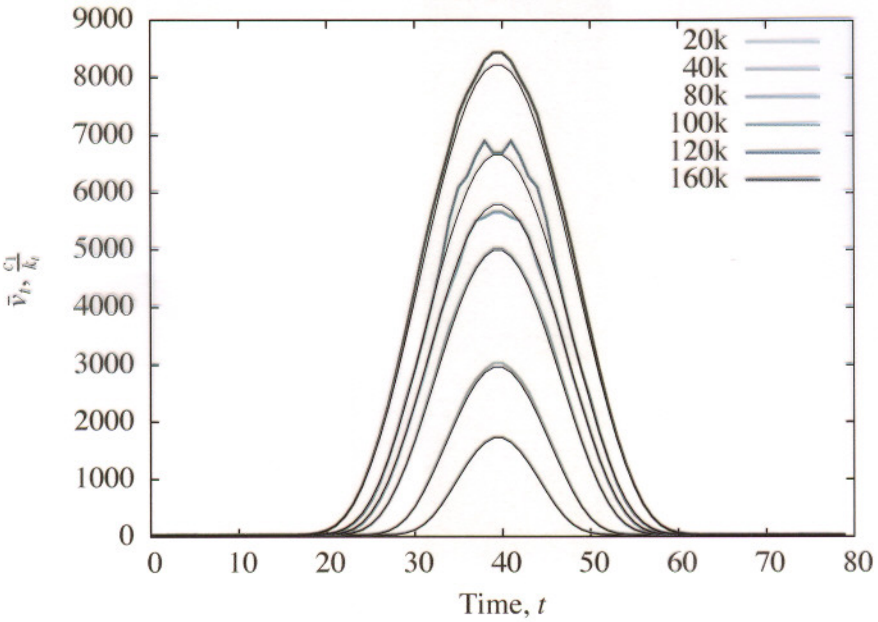
We now simply compare $\hat{P}$ coming from data to $k_0(\hat{P}^{kin} + \hat{P}^{pot})$. by a χ^2 fit of:

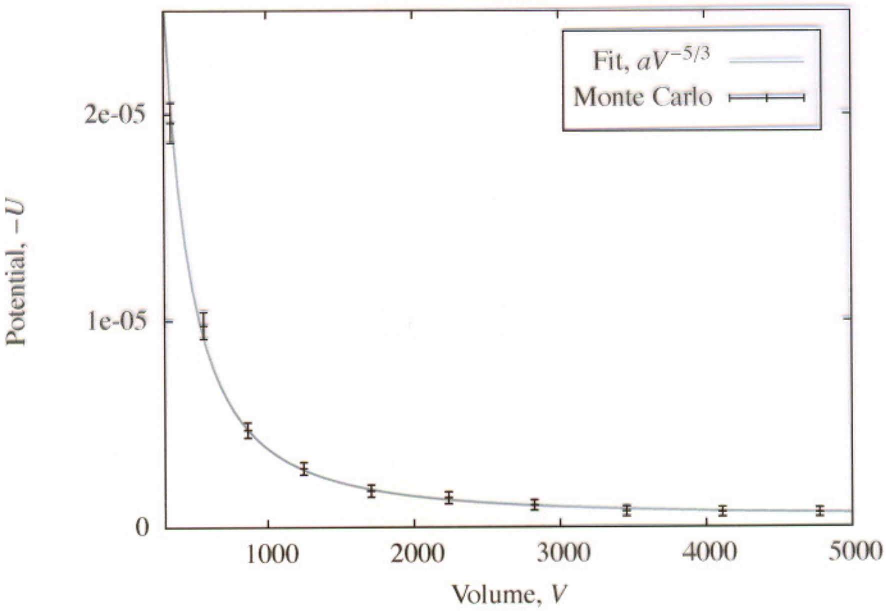
$$\mathrm{Tr}\left(\hat{P} - k_0\left(\hat{P}^{kin} + \hat{P}^{pot}\right)\right)^2$$

and we can now compare:

$$\bar{N}_3(i) \sim \frac{k_0}{k_i} \qquad a \cdot \frac{1}{\bar{N}(i)^{5/3}} \sim -u_i$$

Data fit Data fit


20k
40k
80k
100k
120k
160k
9000
8000
7000
6000
5000
4000
3000
2000
1000
0
0
10
20
30
40
50
60
70
80
Time, t


Fit, $aV^{-5/3}$
Monte Carlo
2e-05
1e-05
0
1000
2000
3000
4000
5000
Potential, $-U$
Volume, V

Potential $U''(\bar{N}_3(i))$ quite difficult to extract. (Fig)

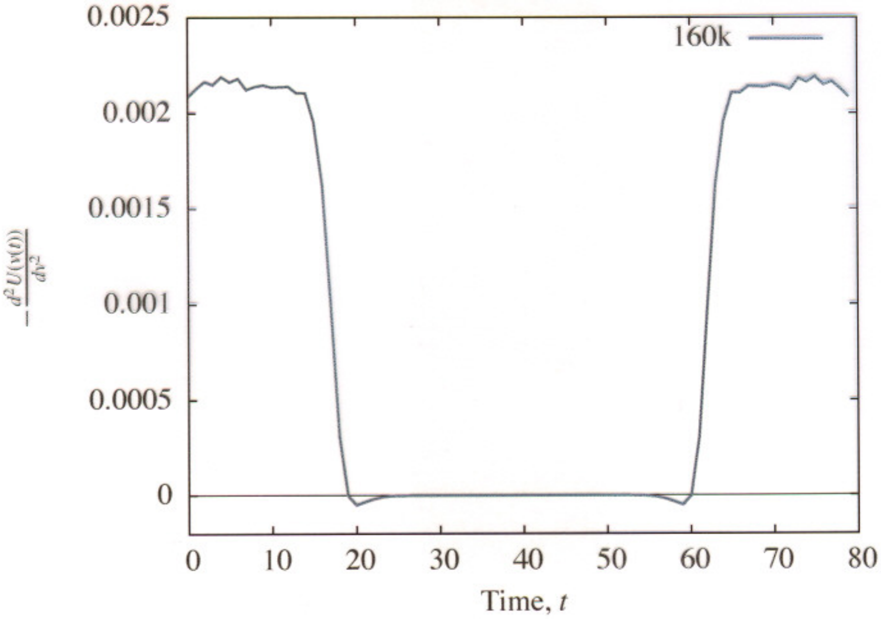
1. Potential term is alway subdominant to the kinetic term in the path integral for generic configurations.

ex $$S = \sum_{i=1}^{N} \Delta t \left[\left(\frac{x_{i+1}-x_i}{\Delta t} \right)^2 + \omega^2 x_i^2 \right]$$

$$(x_{i+1}-x_i)^2 \sim \Delta t \; : \qquad \frac{(x_{i+1}-x_i)^2}{(\Delta t)^2} \sim \frac{1}{\Delta t}$$

2. Stalk term becomes dominant when inverting $\hat{C}(i,j)$ and actually plays a rôle in the bulk even if they only have weak correlations with the bulk.

This is due to the existence of a zero mode for $\hat{C}(i,j)$ which we have to project out before inverting it:



0.0025
0.002
0.0015
0.001
0.0005
0
0 10 20 30 40 50 60 70 80
Time, t
$-\frac{d^2U(v(t))}{dv^2}$
160k

$$\sum_i N_3(i) = N_4 : \qquad \sum_i n_i = 0$$

$$e_i^{(0)} = \frac{1}{\sqrt{N_T}}$$

$$\sum_j \langle n_i n_j \rangle e_j^{(0)} \equiv \sum_j C_{ij} e_j^{(0)}$$

$$\|$$

$$\langle n_i \left(\sum_j n_j \right) \rangle \frac{1}{\sqrt{N_T}} = 0$$

So in fact discussion before was simplified:

Define $\hat{P}_0 = \hat{I} - |e^{(0)}\rangle\langle e^{(0)}|$.

$$\hat{P} = \hat{P}_0 \hat{C}^{-1} \hat{P}_0$$

$$\tilde{Y}^{(i)} = \hat{P}_0 Y^{(i)} \hat{P}_0$$

$$\tilde{Y}^{(i)}_{jk} = \underbrace{\delta_{ij}\delta_{ik}}_{Y^{(i)}_{jk}} - \frac{\delta_{ij}+\delta_{ik}}{N_T} + \frac{1}{N_T^2}$$

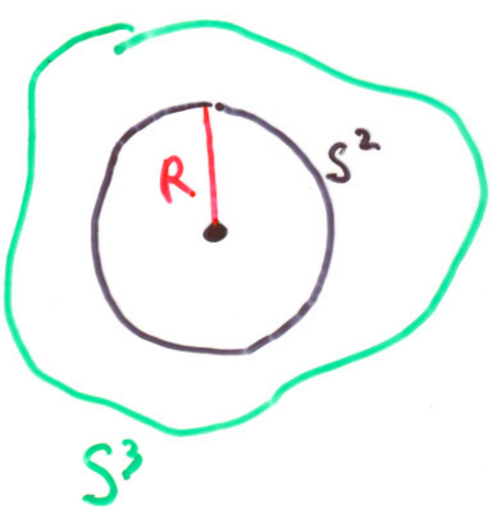
Can we obtain more geometric information ?

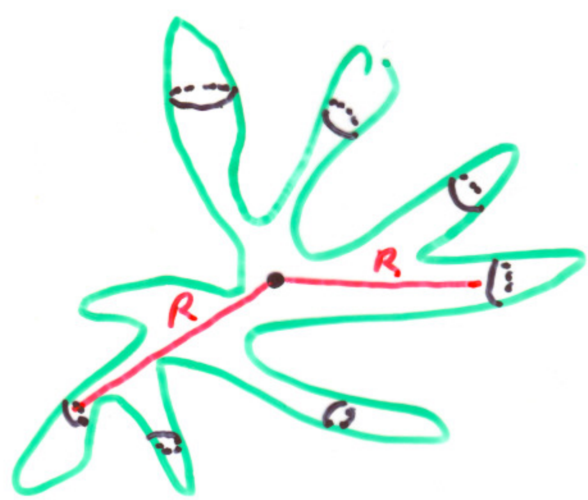
Is the average spatial geometry well described by standard S^3 geometry ?

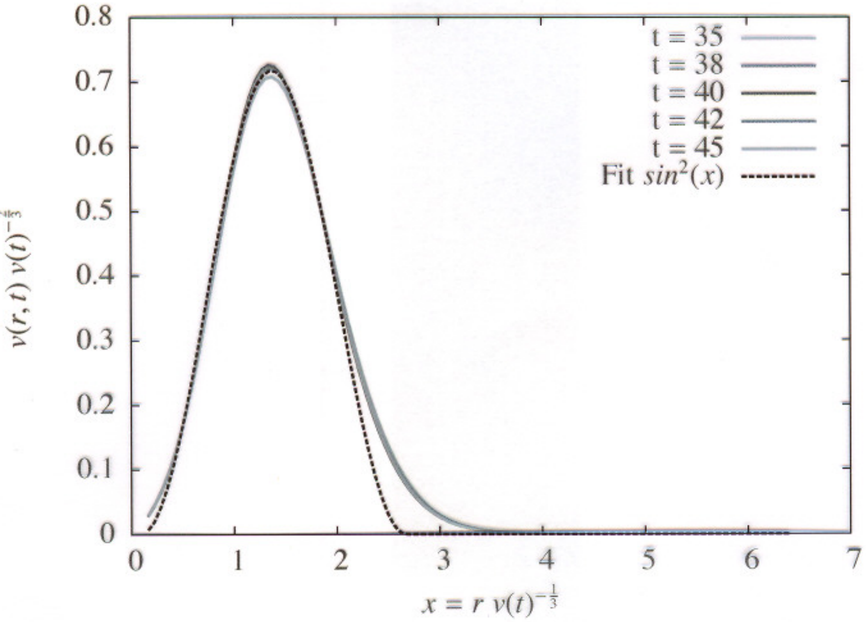
A spatial slice has topology of S^3

Is it homogeneous and isotropic ?
(in average)

$$\langle A(r, N_3) \rangle \sim N_3^{2/3} \sin^2\left(\frac{r}{c N_3^{1/3}}\right)$$







0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
1
2
3
4
5
6
7
t = 35
t = 38
t = 40
t = 42
t = 45
Fit $sin^2(x)$
$v(r,t)\ v(t)^{-\frac{2}{3}}$
$x = r\ v(t)^{-\frac{1}{3}}$

analysis of (quadratic) Quantum fluctuations around the average geometry.

(Fig)

$$C_{N_4}(i,i') \equiv \langle n(i)\, n(i') \rangle$$

$$n(i) \equiv N_3(i) - \overline{N_3}(i)$$

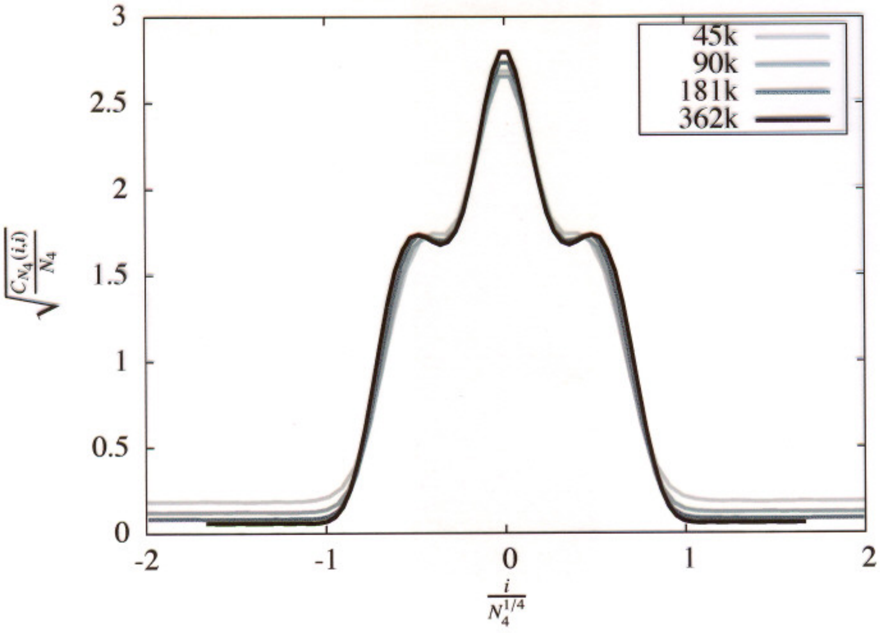
First observation

$$C_{N_4}(i,i') = N_4\, F\left(\frac{i}{N_4^{1/4}}, \frac{i'}{N_4^{1/4}}\right) \quad \text{(For bulk)}$$

F universal scaling function.

$$\langle n^2_{(i)} \rangle \sim N_4\, F\left(\frac{i}{N_4^{1/4}}, \frac{i}{N_4^{1/4}}\right)$$

(Fig.)



45k
90k
181k
362k
$\sqrt{\frac{C_{N_4}(i,i)}{N_4}}$
$\frac{i}{N_4^{1/4}}$

So: $\bar{N}_3(i) \sim N_4^{3/4}$

$|n(i)| \sim N_4^{1/2}$

Relative fluctuations go to zero in the infinite volume limit, for fixed bare coupling constants.

For the analysis of $F(t,t')$, $t \sim i/N_4^{1/4}$ it is convenient to make a decomposition in eigenmodes of $F(i,i')$ (or $C(i,i')$)

$$C_{ij} = \sum_{n=1}^{N} \lambda_n \, e_i^{(n)} e_j^{(n)}$$

We discard the zero mode corresponding to the constant $e_i^{(1)} = \frac{1}{\sqrt{N_T}}$

$$S(V_3) = \frac{1}{24\pi G} \int dt \sqrt{g} \left(\frac{g^{tt} \dot{V}_3^2(t)}{V_3(t)} + k_2 V_3^{1/3}(t) \right)$$

$$\int dt \sqrt{g}\, V_3(t) = V_4.$$

$$V_3(t) = V_3^{cl}(t) + x(t):$$

$$S(V_3) = S(V_3^{cl}) + \frac{1}{18G\pi} \frac{B}{V_4} \int dt\, x(t)\, \hat{H}\, x(t) + \cdots$$

$$\hat{H} = -\frac{d}{dt} \frac{1}{\cos^3 \frac{t}{B}} \frac{d}{dt} - \frac{4}{B^2 \cos^5(t/B)}$$

$$\int dt \sqrt{g}\, x(t) = 0$$

Denote eigenfunctions $f_n(t)$

Lowest eigenfunction ("even") has no zeros and does not satisfy the constraint. It is not really an eigenfunction of the system ($\hat{H}$ + constraint)

"Odd" eigenfunctions of $\hat{H}$ automatically satisfy the constraint. The lowest is a zero mode of $\hat{H}$

$$f_1(t) = \frac{4}{\sqrt{\pi B}} \sin\frac{t}{B} \cos^2\frac{t}{B} \approx \frac{dV^{a}_{c}(t)}{dt}$$

Thus translation-mode $\sim V^{a}_{c}(t) \sim \cos^3 t/B$

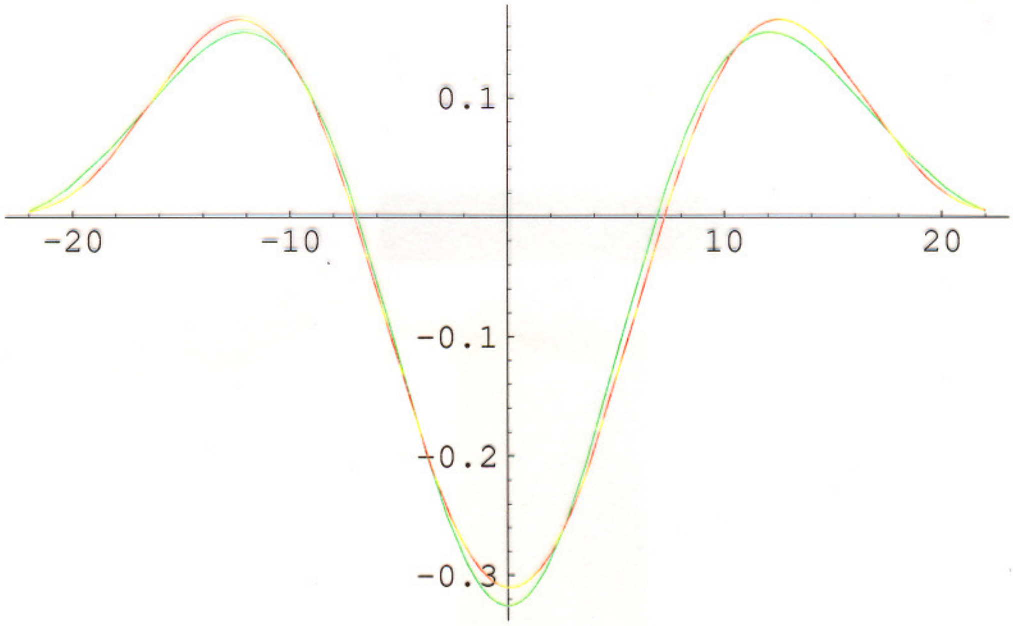
But in data we have tried to fix translational mode, so it should also be discarded.

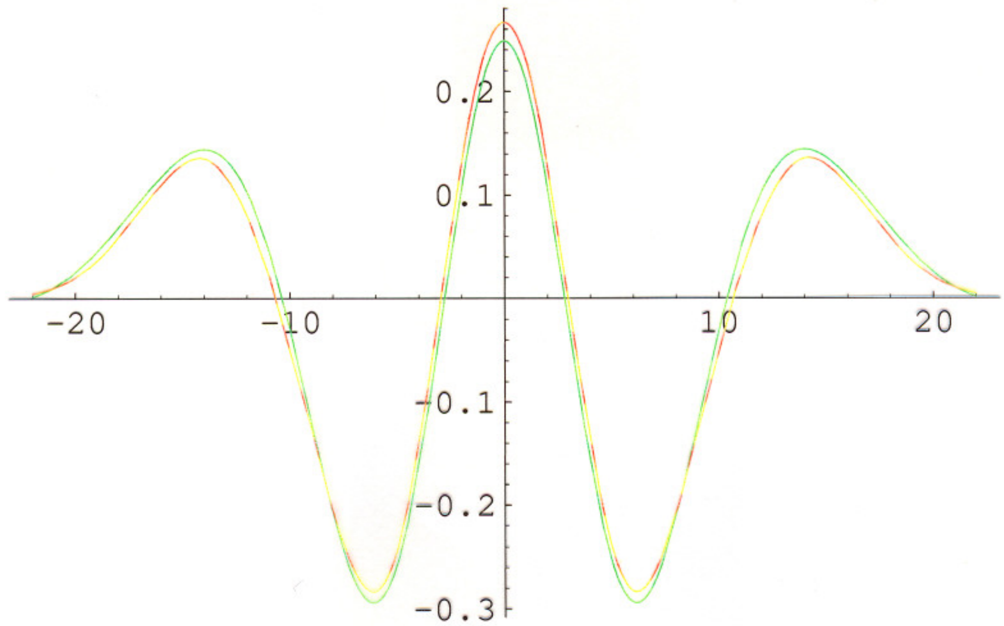
Lowest real mode: $f_2(t)$

Compare to : $e_2(t)$

(Fig)

Even modes very well reproduced.


0.1
-20
-10
10
20
-0.1
-0.2
-0.3


0.2
0.1
-20
-10
10
20
-0.1
-0.2
-0.3

Odd modes e_{2n+1}, $n \geq 1$ are less well reproduced by f_{2n+1}, $n \geq 1$

In fact $e_{2n+1}(t)$ seem to contain a component of the zero mode $f_1(t)$ of $\hat{H}$

Reason: We have only managed to fix the CM up to one lattice spacing.

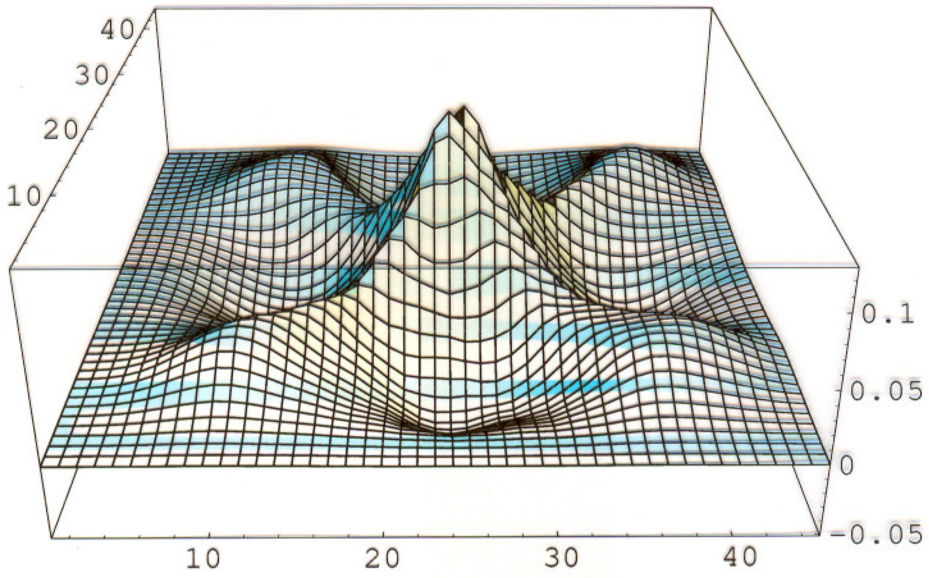
$$S(V_3) = S(V_3^{cl}) + \frac{1}{18\pi G}\,\frac{B}{V_4}\int dt\; x(t)\,\hat{H}\,x(t),$$

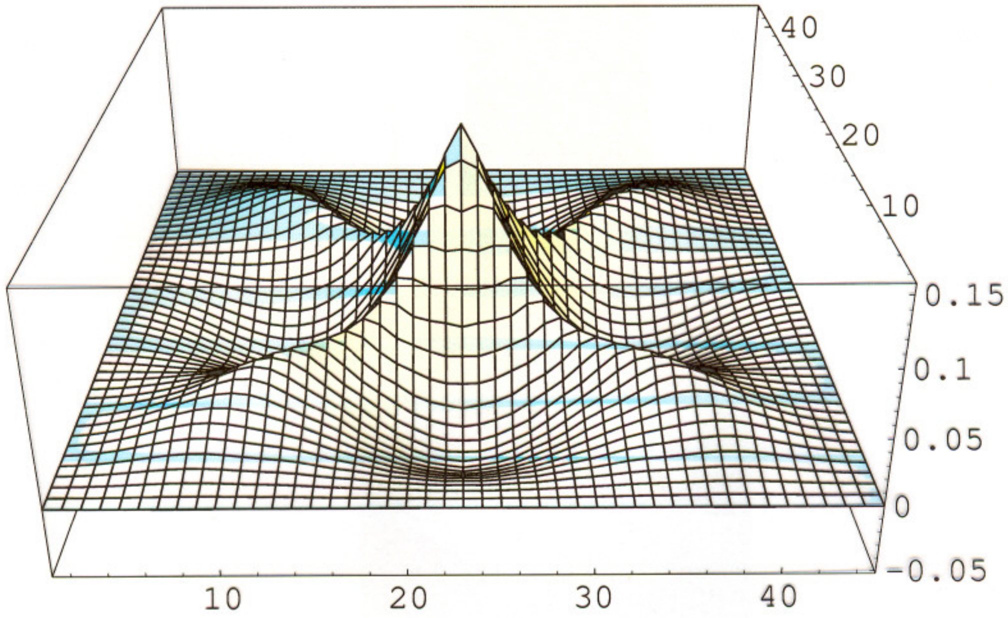
$$C^{(\text{theory})}(t,t') = \langle\, x(t)\, x(t')\,\rangle \sim \left(\hat{H}^{-1}\right)(t,t')$$

$$\hat{H}^{-1}(t,t') = \sum_{n\geq 2}^{\infty} \frac{f_n(t)\, f_n(t')}{\lambda_n}$$

We can now compare $C^{\text{theory}}(t,t')$ and $F(t,t')$.

(Fig.)


40
30
20
10
0.1
0.05
0
-0.05
10
20
30
40


40
30
20
10
0.15
0.1
0.05
0
-0.05
10
20
30
40

Size of our (Planckian) universe and flow of coupling constants.

$N_4 = 160.000$:

$N_4^{1/4} = 20$

convert 4-simplex to hypercubes : $20 \to 10$

our universe is like 10^4 hypercube :

comparable to the Lattice QCD universe when fermions are included.

when fermions not included : 20^4 standard
$(32)^4$ large.

Estimate of G.

$$\int dt \sqrt{g_{tt}}\, V_3(t) = V_4 = C_4\, a^4 N_4 = \frac{8\pi^2}{3} R^4$$

$$\sum_i \Delta i \, N_3(i) = N_4$$

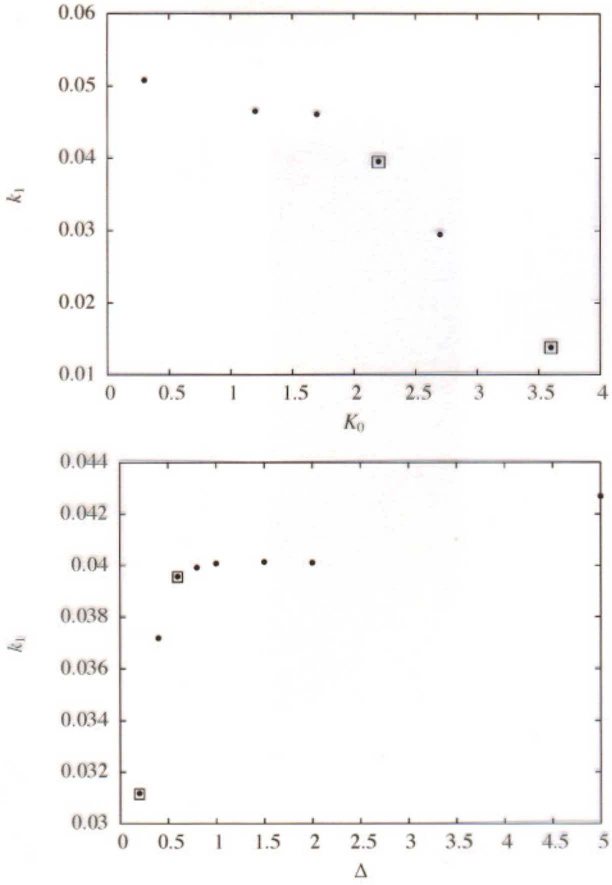
$$\frac{1}{24\pi}\,\frac{1}{G} \int dt \sqrt{g_{tt}} \left[\frac{g^{tt}\, \dot{V}_3^2(t)}{V_3(t)} + k_2 V^{1/3}(t) \right]$$

$$k_0 \sum_i \Delta i \left[\frac{\left(N_3(i+1) - N_3(i)\right)^2}{N_3(i)} + \tilde{k}_2\, N_3^{1/3}(i) \right]$$

$$G = \frac{1}{k_0}\,\frac{C_4}{24\pi}\,\frac{1}{g_{tt}}\, a^2 \quad , \quad \sqrt{g_{tt}} = \frac{R}{B}$$

$$\frac{R}{B} \sim \frac{\sqrt{C_4 \;\; 3/8\pi^2}\;\; \cancel{\sqrt[4]{N_4}}}{s_0\, \cancel{\sqrt[4]{N_4}}}$$

Important point: $\boxed{G \sim \frac{1}{k_0}}$ (Fig)


0.06
0.05
0.04
0.03
0.02
0.01
k_1
0
0.5
1
1.5
2
2.5
3
3.5
4
K_0
0.044
0.042
0.04
0.038
0.036
0.034
0.032
0.03
k_1
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5
Δ

Reference point:

$\kappa_0 = 2.2$, $\Delta = 0.6$; $N_4 = 362.000$

$\sqrt{G} = 0.5\, a$, $R \approx 13\, a$

$\boxed{\text{A universe of 27 Planck length.}}$

When we increase κ_0 : $k_0 \searrow$

$\sqrt{G} \sim \frac{1}{\sqrt{k_0}}$ increases :

For $\kappa_0 = 3.6$: $\sqrt{G} = 0.8\, a$

Good reasons to believe that $k_0 \to 0$ at the phase boundary.

But we do not know yet.

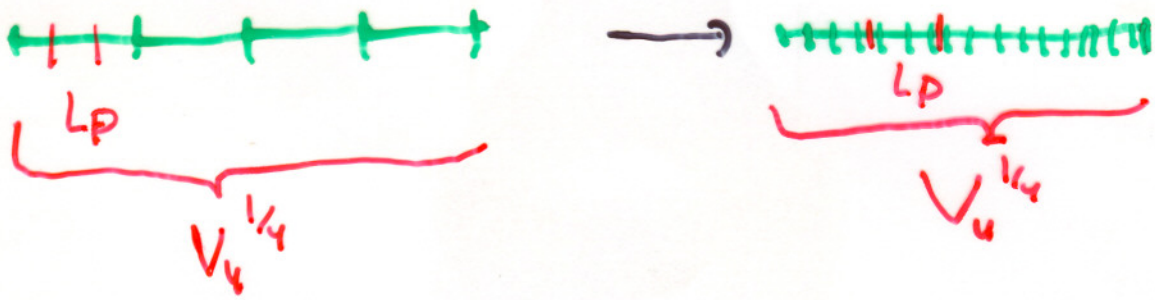
Entering the sub-Planckian regime:

Adjust the bare couplings such that

$$V_4 = C_4(\kappa_0, \Delta) \cdot N_4 a^4$$

$$G = \tilde{C}_4(\kappa_0, \Delta) \frac{1}{k_0(\kappa_0, \Delta)} a^2$$

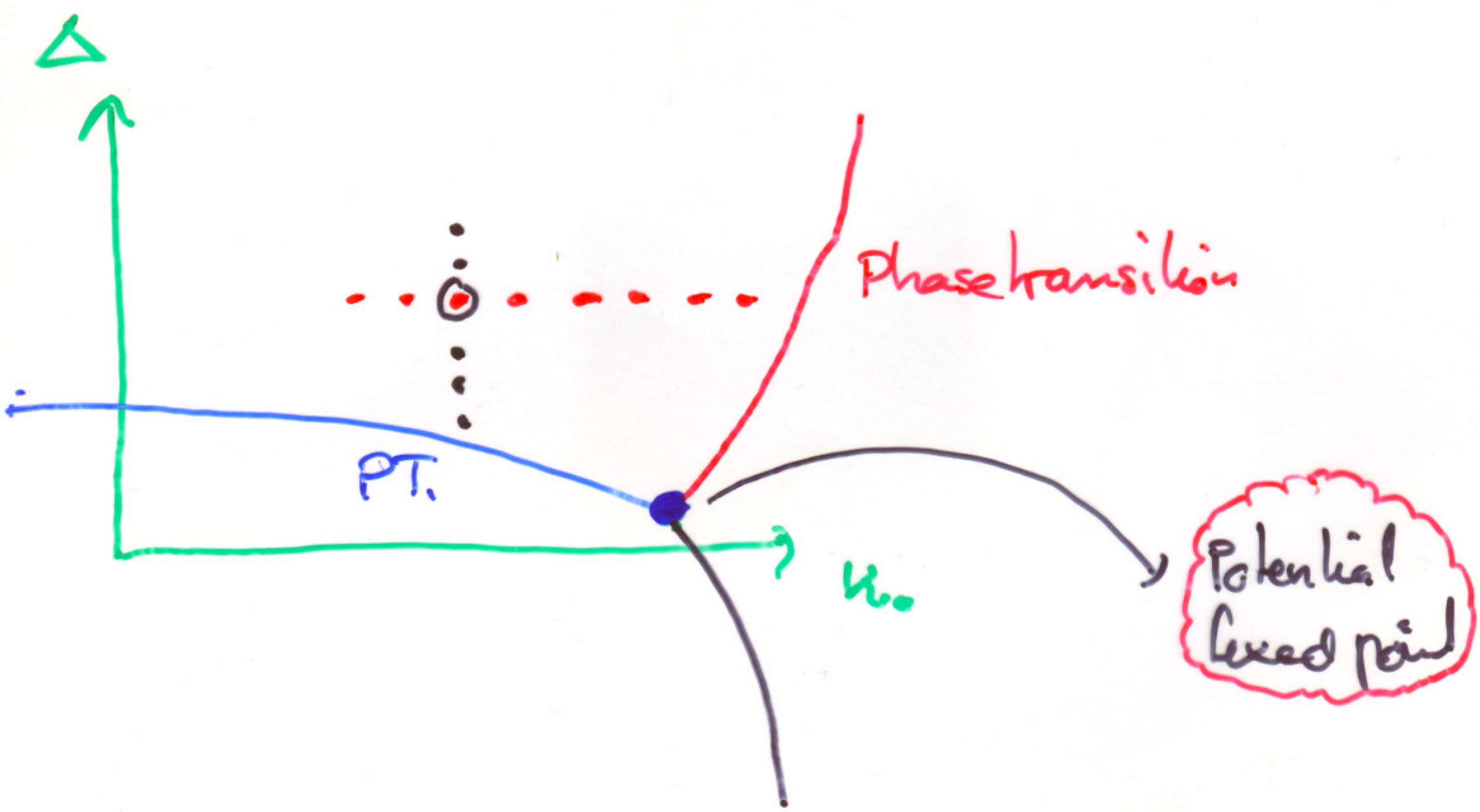
remain constant: $K_0(\kappa_0, \Delta) \sim \frac{1}{\sqrt{N_4}}$



L_P, $V_4^{1/4}$ fixed lattice finer

Moving to subplankian regime we expect complications: quadratic approximation might break down and more terms might have to be included in the effective action.

Potentially good: Might allow us to make contact to RG approach.



Highest priority: Include matter

Allows test of interpretation of G
and potentially allows a test of
various cosmological models of
inflation.

Next highest priority: Sub-Planckian region
and fixed point.

(going to be demanding).