Scattering Amplitudes and Strings on AdS - II

Luis Fernando Alday

Utrecht University

Cracow School of Theoretical Physics
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BDS ansatz for MHV gluons scattering amplitudes of planar $\mathcal{N}=4$ SYM.

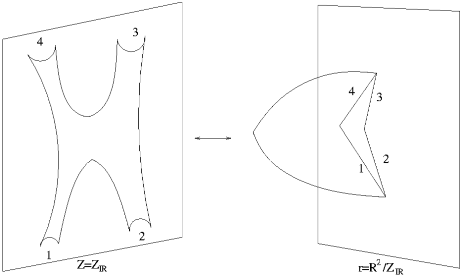
4 point amplitude

$$\mathcal{A} = \mathcal{A}_{tree}\left(\mathcal{A}_{div,s}\right)^2\left(\mathcal{A}_{div,t}\right)^2 \exp\left\{\frac{f(\lambda)}{8}(\ln s/t)^2 + const\right\}$$

$$\mathcal{A}_{div,s} = \exp\left\{-\frac{1}{8\epsilon^2} f^{(-2)}\left(\frac{\lambda\mu^{2\epsilon}}{s^\epsilon}\right) - \frac{1}{4\epsilon} g^{(-1)}\left(\frac{\lambda\mu^{2\epsilon}}{s^\epsilon}\right)\right\}$$

Prescription

Scattering amplitude at strong coupling? Use AdS/CFT!



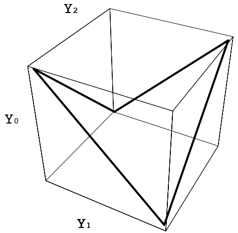
$$\mathcal{A}_n \sim e^{-\frac{\sqrt{\lambda}}{2\pi} A_{min}}$$

Four point amplitude at strong coupling

Let's see how the prescription works for $n = 4$:

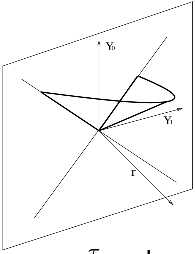
Consider $k_1 + k_3 \rightarrow k_2 + k_4$

$$s = -(k_1 + k_2)^2, \qquad t = -(k_1 + k_4)^2$$



Need to find the minimal surface ending on such sequence of light-like segments.

- Warm up: Try to find the solution near one of the cusps.



The surface can be embedded in AdS_3

$$ds^2 = \frac{-dy_0^2 + dy_1^2 + dr^2}{r^2}$$

$$y_0 = e^{\tau}\cosh\sigma, \quad y_1 = e^{\tau}\sinh\sigma, \quad r = e^{\tau} w(\tau)$$

$$\Downarrow$$

$$S_{NG} \sim \int d\sigma \int d\tau \frac{\sqrt{1-(w(\tau)+w'(\tau))^2}}{w(\tau)^2}$$

Solution for the cusp (Kruczenski)

$$w = \sqrt{2} \to r = \sqrt{2}\sqrt{y_0^2 - y_1^2} = \sqrt{2y^+y^-}, \quad y^{\pm} = y^0 \pm y^1$$

- S_{NG}: Poincare coordinates (r, y_0, y_1, y_2) and parametrize the surface by its projection to (y_1, y_2) plane.
- Action for two fields $r(y_1, y_2), y_0(y_1, y_2)$. *E.g.* if $s = t$ the fields live on a square parametrized by y_1, y_2.

$$S_{NG} = \frac{R^2}{2\pi} \int dy_1 dy_2 \frac{\sqrt{1 + (\partial_i r)^2 - (\partial_i y_0)^2 - (\partial_1 r \partial_2 y_0 - \partial_2 r \partial_1 y_0)^2}}{r^2}$$

- By scale invariance, edges of the square at $y_1, y_2 = \pm 1$

Boundary conditions

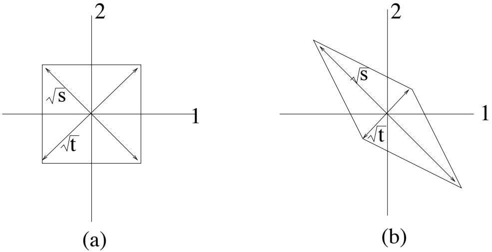
$$r(\pm 1, y_2) = r(y_1, \pm 1) = 0, \quad y_0(\pm 1, y_2) = \pm y_2, \quad y_0(y_1, \pm 1) = \pm y_1$$

- We know the solution near the cusps. We can make some guess

$$y_0(y_1, y_2) = y_1 y_2, \quad r(y_1, y_2) = \sqrt{(1-y_1^2)(1-y_2^2)}$$

- Easily seen to satisfy all the conditions and actually solves the eoms!
- However, $s = t$ is somehow a boring case...

- We would like to capture the kinematical dependence of the amplitude. We need to consider $s \neq t$.
- The square will be deformed to a rhombus



Embedding coordinates

$$-Y_{-1}^2 - Y_0^2 + Y_1^2 + Y_2^2 + Y_3^2 + Y_4^2 = -1$$

$$Y^\mu = \frac{y^\mu}{r}, \quad \mu = 0, ..., 3$$

$$Y_{-1} + Y_4 = \frac{1}{r}, \quad Y_{-1} - Y_4 = \frac{r^2 + y_\mu y^\mu}{r}$$

Embedding coordinates surface

$$Y_0 Y_{-1} = Y_1 Y_2 \qquad Y_3 = Y_4 = 0$$

- We can perform $SO(2,4)$ transformations and get new solutions. This is a "dual" conformal symmetry.

How do we change the distance between opposite vertices?

- *e.g.* a boost in the $0-4$ direction gives a new solution with $s \neq t$.

After the boost

$$Y_4 = 0, \quad Y_0 Y_{-1} = Y_1 Y_2 \rightarrow Y_4 - vY_0 = 0, \quad \sqrt{1-v^2}\, Y_0 Y_{-1} = Y_1 Y_2$$

Two parameters solution

- Size of the square we started with.
- Boost parameter v.

Conformal gauge action

$$iS = -\frac{R^2}{2\pi}\int du_1 du_2 \frac{1}{2}\frac{\partial r \partial r + \partial y_\mu \partial y^\mu}{r^2}$$

Solution for the rhombus

$$r = \frac{a}{\cosh u_1 \cosh u_2 + b \sinh u_1 \sinh u_2},$$

$$y_0 = r\sqrt{1+b^2}\sinh u_1 \sinh u_2$$

$$y_1 = r \sinh u_1 \cosh u_2, \qquad y_2 = r\cosh u_1 \sinh u_2$$

- The parameters a and b encode the kinematical information.

$$-s(2\pi)^2 = \frac{8a^2}{(1-b)^2}, \qquad -t(2\pi)^2 = \frac{8a^2}{(1+b)^2}$$

Let's compute the area...

- Small problem: The area diverges!
- Dimensional reduction scheme: Theory in $D = 4 - 2\epsilon$ dimensions but with 16 supercharges.
- For integer D this is exactly the low energy theory living on $Dp-$branes $(p = D - 1)$

Gravity dual

$$ds^2 = h^{-1/2} dx_D^2 + h^{1/2}\left(dr^2 + r^2 d\Omega_{9-D}^2\right), \qquad h = \frac{c_D \lambda_D}{r^{8-D}}$$

$$\lambda_D = \frac{\lambda \mu^{2\epsilon}}{(4\pi e^{-\gamma})^{\epsilon}} \qquad c_D = 2^{4\epsilon}\pi^{3\epsilon}\Gamma(2+\epsilon)$$

T-dual coordinates

$$ds^2 = \sqrt{\lambda_D c_D}\left(\frac{dy_D^2 + dr^2}{r^{2+\epsilon}}\right) \rightarrow S_\epsilon = \frac{\sqrt{\lambda_D c_D}}{2\pi}\int \frac{\mathcal{L}_{\epsilon=0}}{r^\epsilon}$$

- Presence of ϵ will make the integrals convergent.
- The eoms will depend on ϵ but if we plug the original solution into the new action, the answer is accurate enough.
- plugging everything into the action...

$$iS = -\frac{\sqrt{\lambda_D c_D}}{2\pi a^\epsilon}\left(\frac{\pi\Gamma\left[-\frac{\epsilon}{2}\right]^2}{\Gamma\left[\frac{1-\epsilon}{2}\right]}\,{}_2F_1\left(\frac{1}{2},-\frac{\epsilon}{2},\frac{1-\epsilon}{2};b^2\right)+1/2\right)+\mathcal{O}(\epsilon)$$

- Just expand in powers of ϵ...

Final answer

$$\mathcal{A} = e^{iS} = \exp\left[iS_{div} + \frac{\sqrt{\lambda}}{8\pi}\left(\log\frac{s}{t}\right)^2 + \tilde{C}\right]$$

$$S_{div} = 2S_{div,s} + 2S_{div,t}$$

$$S_{div,s} = -\frac{1}{\epsilon^2}\frac{1}{2\pi}\sqrt{\frac{\lambda\mu^{2\epsilon}}{(-s)^\epsilon}} - \frac{1}{\epsilon}\frac{1}{4\pi}(1-\log 2)\sqrt{\frac{\lambda\mu^{2\epsilon}}{(-s)^\epsilon}}$$

- Should be compared to the field theory answer

$$\mathcal{A} \sim (\mathcal{A}_{div,s})^2 (\mathcal{A}_{div,t})^2 \exp\left\{\frac{f(\lambda)}{8}(\ln s/t)^2 + const\right\}$$

$$\mathcal{A}_{div,s} = \exp\left\{-\frac{1}{8\epsilon^2} f^{(-2)}\left(\frac{\lambda\mu^{2\epsilon}}{s^\epsilon}\right) - \frac{1}{4\epsilon} g^{(-1)}\left(\frac{\lambda\mu^{2\epsilon}}{s^\epsilon}\right)\right\}$$

- The general structure of the solution is perfect agreement with the BDS conjecture.
- $SO(2,4)$ transformations fixed somehow the kinematical dependence of the finite piece.
- This dual conformal symmetry constrains the form of the amplitude

$$\mathcal{A} \sim f\left(\left(\log\frac{r_{cut}^2}{s}\right)^2 + \left(\log\frac{r_{cut}^2}{t}\right)^2\right) + \tilde{g}\left(\log\frac{r_{cut}^2}{s} + \log\frac{r_{cut}^2}{t}\right) + \\ + f\left(\log\frac{s}{t}\right)^2 + const + \mathcal{O}(r_{cut})$$

- Up to a term independent on the kinematics, we get the same finite piece!

- Imagine the cusps are located at x_i and the cut-off depends on the point at the boundary, $r_{cut}(x)$: $r_{cut}(x_i) \to r_i$

$$\mathcal{A} \sim f \sum_i \left(\log \frac{r_i^2}{x_{i-1,i+1}^2} \right)^2 + g \sum_i \left(\log \frac{r_i^2}{x_{i-1,i+1}^2} \right) + Fin(x_i)$$

- The dual $SO(2,4)$ symmetries will move the points x_i, r_i and the area should be invariant.

$$a.K = \sum_i x_i^2 \; a.\partial_{x_i} - 2a.x_i \left(x_i.\partial_{x_i} + r_i \partial_{r_i} \right)$$

- $KA = 0$ fixes the finite piece for $n = 4$ and $n = 5$!

- Also observed perturbatively (Drummond, Henn, Korchemsky, Sokatchev)

Wilson loops vs. Scattering amplitudes

- This computation shows a relation between Wilson loops and scattering amplitudes.
- This relation holds also at weak coupling! Drummond, Korchemsky, Sokatchev; Henn, Brandhuber...

Write BDS on a slightly different way

$$\log \mathcal{M}_n = Div_n + \frac{f(\lambda)}{4} a_1(k_1, k_2, ..., k_n) + h(\lambda) + nk(\lambda)$$

Scattering amplitudes vs. WL (Brandhuber, Heslop, Travaglini)

$$< W_{k_i} >= 1 + \lambda \left(Div + w_1(k_1, ..., k_n) + c + n\tilde{c} \right)$$

$$\Downarrow$$

$$w_1(k_1, ..., k_n) = a_1(k_1, ..., k_n)$$

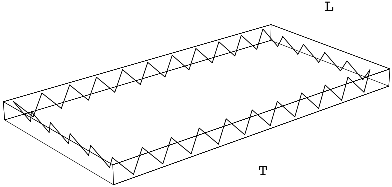
- BDS $\Rightarrow$ $a_{strong} = f^{strong} a_1(k_1, ..., k_n)$
- WL vs. Amplitudes at strong coupling $\Rightarrow a_{strong} = w_{strong}$
- WL vs. Amplitudes at weak coupling $\Rightarrow a_1 = w_1$

$$\Downarrow$$

$$w_{strong} = f^{strong} w_1(k_1, ..., k_n)$$

- For $n = 4$ and $n = 5$ that is the case! but fixed by symmetries.
- We need to take $n > 5$, what about $n = \infty$?

We choose a zig-zag configuration that approximates the rectangular Wilson loop.



- $\log < W_{rect}^{weak} >= \frac{\lambda}{8\pi} \frac{T}{L}$
- $\log < W_{rect}^{strong} >= \sqrt{\lambda} \frac{4\pi^2}{\Gamma(\frac{1}{4})^4} \frac{T}{L}$

- The BDS ansatz predicts $\log < W_{rect}^{strong} >= \sqrt{\lambda} \frac{1}{4} \frac{T}{L}$
- The strong coupling result is not what we would expect from the BDS ansatz, hence something needs to be revised...

At which order in perturbation theory and for how many gluons will BDS fail?

- BDS $\Rightarrow \quad a_\ell = f^{(\ell)} a_1(k_1, ..., k_n)$
- WL vs. Amplitudes at all orders $\Rightarrow a_\ell = w_\ell$

$$\Downarrow$$

$$w_2 = f^{(2)} w_1(k_1, ..., k_n)$$

- An explicit computation for the rectangular Wilson loop, shows that either the BDS conjecture or the relation between WL and amplitudes (or both!) fail at two loops for a large number of gluons.

- You will hear more about it!!

What have we done?

- A prescription for computing planar scattering amplitudes on $\mathcal{N} = 4$ SYM at strong coupling by using the AdS/CFT duality.
- We have done detailed computations for $n = 4$ but the prescription is valid for any number of gluons.
- Our results agree in all detail with the conjecture of Bern, Dixon and Smirnov for $n = 4$ and for $n = 5$ they should also agree, but the conjecture may need to be revised for large number of gluons.
- A small step towards understanding the iterative structures for gluon amplitudes from the string theory point of view.

What things need to be done?

- Try to make explicit computations for $n > 4$, *e.g.* $n = 6$ is a good one.
- We haven't assume/use at all the machinery of integrability.
- Subleading corrections in $1/\sqrt{\lambda}$? Information about helicity of the particles, etc.
- Gross and Mende computed higher genus amplitudes (in flat space) using similar ideas, can we do the same?
- Can we repeat the computation in other backgrounds?
- Deeper relation among Wilson loops and scattering amplitudes?
- Some powerful alternative to BDS?