# *Wilson loop / gluon amplitude duality in $\mathcal{N} = 4$ SYM*

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Based on work in collaboration with

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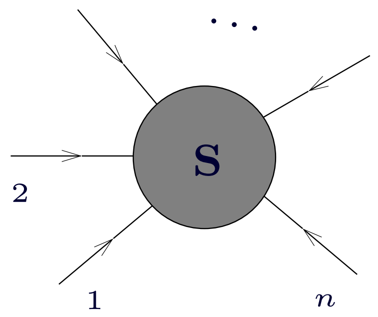
# Outline

- ✔ Perturbative gluon scattering amplitudes in $\mathcal{N} = 4$ SYM $\rightarrow$ Radu Roiban's lectures
  - ✗ Bern-Dixon-Smirnov (BDS) conjecture
  - ✗ hidden conformal symmetry of planar gluon amplitudes
- ✔ Alday-Maldacena proposal for gluon scattering at strong coupling using AdS/CFT
  $\longrightarrow$ Fernando Alday's lectures
- ✔ Wilson loops at weak coupling - a duality?
  $\longrightarrow$ Yuri Makeenko's and Gregory Korchemsky's lectures
- ✔ Conformal Ward identities for Wilson loops
- ✔ Hexagonal Wilson loop and BDS ansatz

# Gluon scattering amplitudes

✔ On-shell gluon scattering amplitudes

$\mathcal{A}_n =$ 


✗ on-shell gluons characterised by momentum $p_i^\mu$, $p_i^2 = 0$, helicity $\pm 1$, and colour

✗ amplitudes require infrared (IR) regularisation

✔ Colour-ordered planar partial amplitudes

$$\mathcal{A}_n = \mathrm{tr}\left[T^{a_1}T^{a_2}\dots T^{a_n}\right] A_n^{h_1,h_2,\dots,h_n}(p_1,p_2,\dots,p_n) + \text{[Bose symmetry]}$$

✔ Helicity structure

- ✗ Supersymmetry relations $\quad A^{++\dots+} = A^{-+\dots+} = 0$
- ✗ Maximally helicity violating (MHV) amplitudes $\quad A^{(\mathrm{MHV})} = A^{--+\dots+}, \quad \dots$
- ✗ NMHV amplitudes $\quad A^{(\mathrm{next-MHV})} = A^{---+\dots+}, \quad \dots$

✔ for MHV amplitudes:

$$A_n^{h_1,h_2,\dots,h_n}(p_1,p_2,\dots,p_n) = A_{\mathrm{tree}}^{h_1,h_2,\dots,h_n} \times A_{\mathrm{loops}}(p_1,p_2,\dots,p_n)$$

# Bern-Dixon-Smirnov conjecture for MHV amplitudes

- ✔ the structure of the IR divergences is known

$$\ln\left[A_n/A_n^{\text{tree}}\right] = \text{div} + F_n(a, p_i \cdot p_j) + O(\epsilon_{\text{IR}})$$

- ✔ the ABDK/BDS conjecture is a statement about the finite part:

[Anastasiou, Bern, Dixon, Kosower '03],[Bern, Dixon, Smirnov '05]

$$F_n = \frac{1}{2}\Gamma_{\text{cusp}}(a) F_n^{(1)}$$

- ✔ For example, for four and five points

$$F_4 = \frac{1}{4}\Gamma_{\text{cusp}}(a)\left[\ln^2 \frac{s}{t} + \text{const}\right]$$

$$F_5 = \frac{1}{4}\Gamma_{\text{cusp}}(a)\left[\sum_{i=1}^{5} \ln\frac{s_{i,i+1}}{s_{i+1,i+2}} \ln\frac{s_{i+2,i+3}}{s_{i+3,i+4}} + \text{const}\right]$$

- ✔ the BDS conjecture has been confirmed so far
  - ✗ up to three loops for $F_4$ [Bern, Dixon, Smirnov '05]
  - ✗ up to two loops for $F_5$ [Cachazo, Spradlin, Volovich '06],[Bern, Czakon, Korower, Roiban, Smirnov '06]

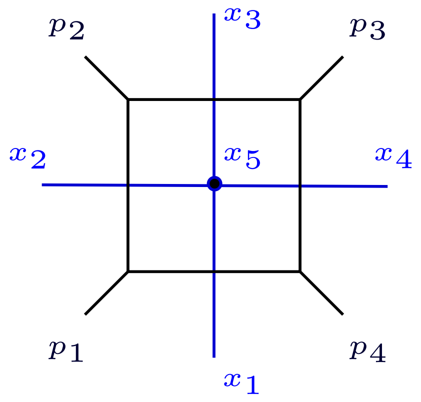
☞ *Where does the simplicity of the finite part come from?*

# Dual conformal symmetry

One-loop: 'scalar box' integral

✔ Change variables to go to a *dual 'coordinate space'* picture (not a Fourier transform!)

$$p_1 = x_1 - x_2 \equiv x_{12}\,, \quad p_2 = x_{23}\,, \quad p_3 = x_{34}\,, \quad p_4 = x_{41}\,, \quad k = x_{15}$$




$$= \int \frac{d^D k}{k^2 (k-p_1)^2 (k-p_1-p_2)^2 (k+p_4)^2} = \int \frac{d^D x_5}{x_{15}^2 x_{25}^2 x_{35}^2 x_{45}^2}$$

*Conformal inversion:* $x_i^\mu \to x_i^\mu / x_i^2\,, x_{ij}^2 \to \frac{x_{ij}^2}{x_i^2 x_j^2}$

[Broadhurst '93],[Drummond,J.H.,Smirnov,Sokatchev '06]

✔ Consider the integral off-shell and for $D = 4$

- ✗ The integral is conformal in the dual space
- ✗ The symmetry *is not related* to the conformal symmetry of $\mathcal{N} = 4$ SYM

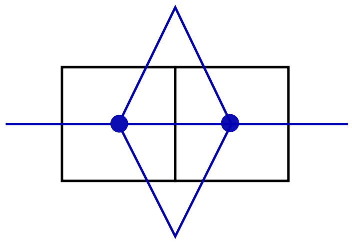
✔ The dual conformal symmetry is *broken* by the infrared regulator, $D = 4 - 2\epsilon$.

☞ *Is this broken symmetry present at higher loops?*

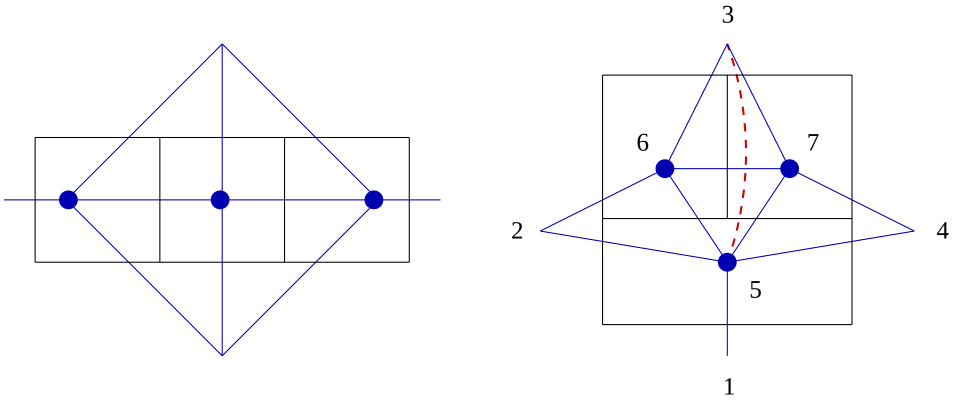
# Dual conformal symmetry

The dual conformal structure continues to higher loops [Drummond,J.H.,Smirnov,Sokatchev '06]

- ✔ Two loops [Bern, Rozowski, Yan '97], [Anastasiou, Bern, Dixon, Kosower '03]



- ✔ Three loops [Bern, Dixon, Smirnov '05]




- ✔ Continues to four loops [Bern, Czakon, Dixon, Kosower, Smirnov '06]
- ✔ Even five? [Bern, Carrasco, Johannson, Kosower '07]

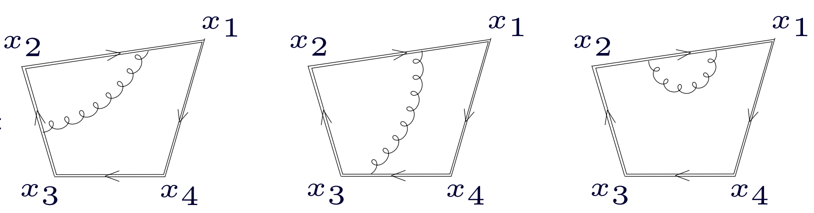
☞ *Where does the dual conformal symmetry come from?*

# Light-like Wilson loops

Expectation value of light-like Wilson loop in $\mathcal{N}=4$ SYM

$$W(C_4) = \frac{1}{N_c} \langle 0 | \mathrm{Tr}\, \mathrm{P} \exp \left( ig \oint_{C_4} dx^\mu A_\mu(x) \right) | 0 \rangle$$

✔ One-loop Feynman diagrams

$\ln W(C_4) =$ 


✔ The light-like Wilson loop is IR finite but has UV divergences due to cusps on contour $C_4$

✔ One-loop result [Drummond,Korchemsky,Sokatchev '07]

$$\ln W(C_4) = \frac{g^2}{4\pi^2} C_F \left\{ -\frac{1}{{\epsilon_{\rm UV}}^2} \left[ \left(-x_{13}^2 \mu^2\right)^{\epsilon_{\rm UV}} + \left(-x_{24}^2 \mu^2\right)^{\epsilon_{\rm UV}} \right] + \frac{1}{2} \ln^2 \left( \frac{x_{13}^2}{x_{24}^2} \right) + \text{const} \right\} + O(g^4)$$

✔ identification of dual variables with momenta $\longrightarrow$ $x_{13}^2 / x_{24}^2 = s/t$

# Gluon scattering amplitudes / Wilson loops duality

- Proposal: gluon amplitudes at weak coupling are dual to light-like Wilson loops

$$\ln\left[A_n/A_n^{\text{tree}}\right] = \ln W(C_n) + O(1/N_c) + O(\epsilon)$$

- motivated by computation of scattering amplitudes at strong coupling via AdS/CFT

  [Alday, Maldacena '07]

- At one loop,
  - proposed and checked for $n=4$ [Drummond,Korchemsky,Sokatchev '07]
  - later, extended to arbitrary $n$ [Brandhuber, Heslop, Travaglini '07]

- What about higher loops?
  - Two loop calculation for $n=4,5$ [Drummond,J.H.,Korchemsky,Sokatchev '07]

# Conformal Ward identity

- ✔ Cusp divergences break conformal invariance $\to$ anomalous conformal Ward identity

[Drummond,J.H.,Korchemsky,Sokatchev '07]

$$\mathbb{K}^{\mu}\,F_n \equiv \sum_{i=1}^{n}\left[2x_i^{\mu}(x_i\cdot\partial_{x_i}) - x_i^2\partial_{x_i}^{\mu}\right]F_n = \frac{1}{2}\Gamma_{\rm cusp}(a)\sum_{i=1}^{n} x_{i,i+1}^{\mu}\ln\left(\frac{x_{i,i+2}^2}{x_{i-1,i+1}^2}\right)$$

- ✔ Four and five points: the Ward identity has a *unique all-loop solution* (up to an additive constant)

$$F_4 = \frac{1}{4}\Gamma_{\rm cusp}(a)\ln^2\left(\frac{x_{13}^2}{x_{24}^2}\right) + \text{ const },$$

$$F_5 = -\frac{1}{8}\Gamma_{\rm cusp}(a)\sum_{i=1}^{5}\ln\left(\frac{x_{i,i+2}^2}{x_{i,i+3}^2}\right)\ln\left(\frac{x_{i+1,i+3}^2}{x_{i+2,i+4}^2}\right) + \text{ const}$$

*Exactly the functional form of the BDS ansatz for the 4- and 5-point gluon amplitudes!*

- ✔ Starting from six points there are conformal invariants in the form of cross-ratios

$$u_1 = \frac{x_{13}^2x_{46}^2}{x_{14}^2x_{36}^2}, \qquad u_2 = \frac{x_{24}^2x_{15}^2}{x_{25}^2x_{14}^2}, \qquad u_3 = \frac{x_{35}^2x_{26}^2}{x_{36}^2x_{25}^2}$$

- ✔ For *arbitrary* $n$ the BDS ansatz is still a solution of the conformal Ward identity, but the solution is no longer unique.

# Hexagonal Wilson loop at two loops

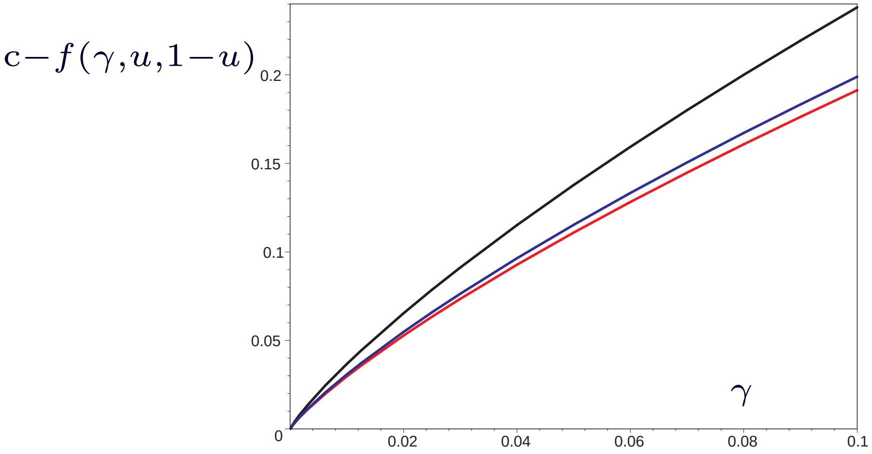
Wilson loop at six points and two loops was computed recently

[Drummond, J.H., Korchemsky, Sokatchev '08]

$$F_6 = F_6^{(BDS)} + f(u_1, u_2, u_3)$$

✔ The Wilson loop is not given by the BDS ansatz for the gluon amplitude

✔ $f(u_1, u_2, u_3)$ goes to a constant in the collinear limit
$\longrightarrow$ consistent with behaviour of gluon amplitudes




✔ Regge behaviour at two loops disagrees with BDS ansatz for six gluons [Bartels, Lipatov, Vera '08]

☞ Does the duality hold?

# Hexagon Wilson loop = six-gluon amplitude

[Bern, Dixon, Kosower, Roiban, Spradlin, Vergu, Volovich '08]

[Drummond, J.H., Korchemsky, Sokatchev '08]

✔ Numerical tests for different kinematical configurations $K^{(i)}$.

| Kinematical point | $(u_1, u_2, u_3)$ | $f_{\rm WL} - f_{\rm WL}^{(0)}$ | $f_{\rm A} - f_{\rm A}^{(0)}$ |
|---|---|---|---|
| $K^{(1)}$ | $(1/4, 1/4, 1/4)$ | $< 10^{-5}$ | $-0.018 \pm 0.023$ |
| $K^{(2)}$ | $(0.547253, 0.203822, 0.88127)$ | $-2.75533$ | $-2.753 \pm 0.015$ |
| $K^{(3)}$ | $(28/17, 16/5, 112/85)$ | $-4.74460$ | $-4.7445 \pm 0.0075$ |
| $K^{(4)}$ | $(1/9, 1/9, 1/9)$ | $4.09138$ | $4.12 \pm 0.10$ |
| $K^{(5)}$ | $(4/81, 4/81, 4/81)$ | $9.72553$ | $10.00 \pm 0.50$ |

☞ The BDS ansatz needs to be corrected at six gluons and two loops!

☞ The duality holds!

# Conclusions and outlook

- ✔ Evidence of a (broken) dual conformal symmetry of planar MHV amplitudes
- ✔ This symmetry becomes manifest within the gluon amplitude / Wilson loop duality
- ✔ The duality relation was verified in several nontrivial cases
- ✔ At two-loops and six gluons, the BDS ansatz fails, but the duality is preserved!
- ☞ Conjecture that the duality holds to all loops and for an arbitrary number of gluons

- ☞ Can the duality be extended beyond the MHV case? E.g. NMHV amplitudes appear starting from six points.
- ☞ What is the origin of the dual conformal symmetry of gluon amplitudes? ... Related to integrability of planar $\mathcal{N} = 4$ SYM?!