Modelling sources of GWs: non-vacuum spacetimes

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Plan of the talk

- A brief Introduction to Relativistic Hydrodynamics/MHD
- Numerical solutions: basic dos and don’ts
  - flux conservative formulations
  - high-resolution shock capturing schemes
- Non-vacuum sources at the AEI
- One representative example:
  - stellar collapse to a rotating black hole

A look at the equations

Let's recall the equations we are dealing with:

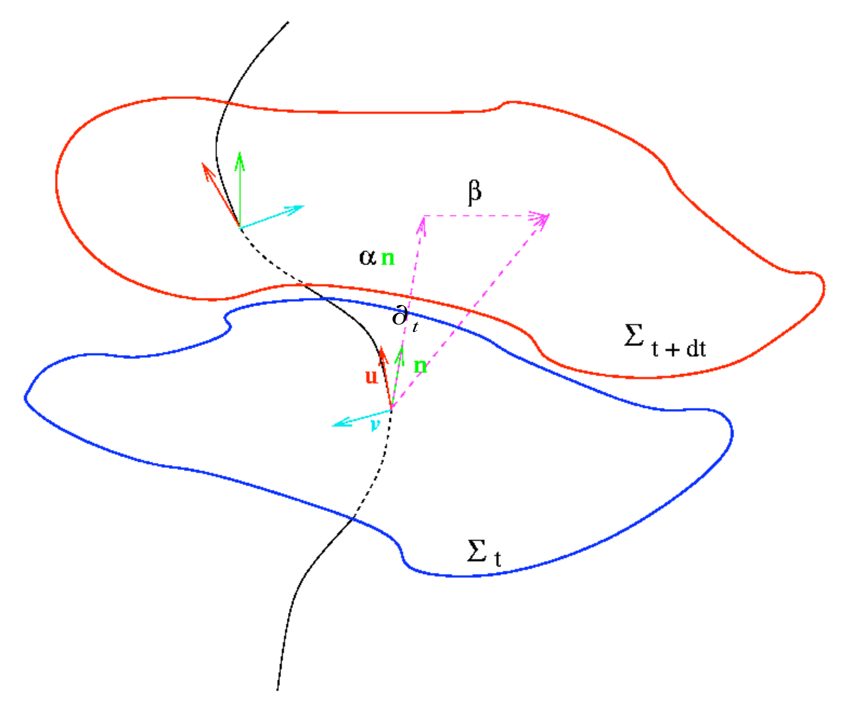
$$G_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu} \qquad \text{(field equations)}$$

$$\nabla_\mu T^{\mu\nu} = 0 \ , \qquad \text{(conservation of energy – momentum)}$$

$$\nabla_\mu (\rho u^\mu) = 0 \ , \qquad \text{(conservation of baryon number)}$$

$$p = p(\rho, \epsilon, \ldots) \ . \qquad \text{(equation of state)}$$

Handling the matter content of the spacetime



Let: $\boldsymbol{u}$ the fluid's 4-velocity, p the isotropic pressure, ρ the rest-mass density, ϵ the specific internal energy density and $e = \rho(1+\epsilon)$, the total energy density

Let also $\boldsymbol{v}$ be the fluid 3-velocity measured by the observers $\boldsymbol{n}$

$$\boldsymbol{v} \equiv -\frac{\boldsymbol{u}\cdot\partial_i}{\boldsymbol{n}\cdot\boldsymbol{u}} \longrightarrow v^i = \frac{1}{\alpha}\left(\frac{u^i}{u^t} + \beta^i\right)$$

From these quantities we can construct the ideal-fluid *stress-energy tensor*

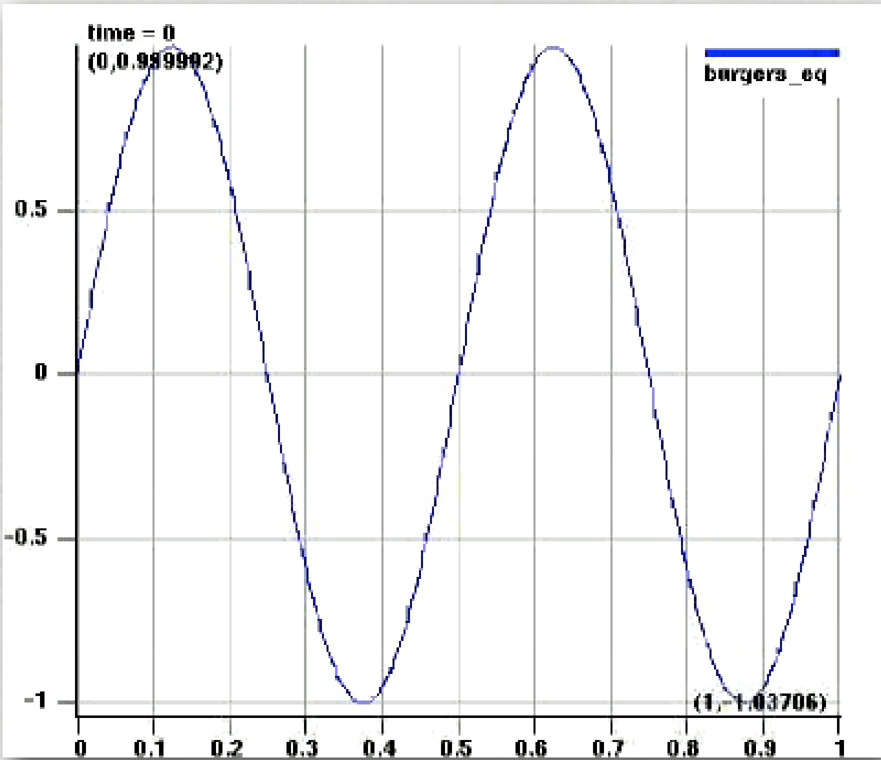
$$T^{\mu\nu} = T^{\mu\nu}_{\rm fluid} + T^{\mu\nu}_{\rm em} + \ldots$$

$$T^{\mu\nu} = (\rho + \rho\epsilon + p + b^2)u^\mu u^\nu + (p + \frac{b^2}{2})g^{\mu\nu} - b^\mu b^\nu + \ldots$$

A representative example: Burgers' equation

Before looking at the solution of the hydrodynamical equations there are some fundamental aspects of their nonlinear properties which must be clarified. For this it is easier to consider the simplest nonlinear hyperbolic equation: inviscid Burgers equation

$$\partial_t u(x,t) + u(x,t)\partial_x u(x,t) = 0$$

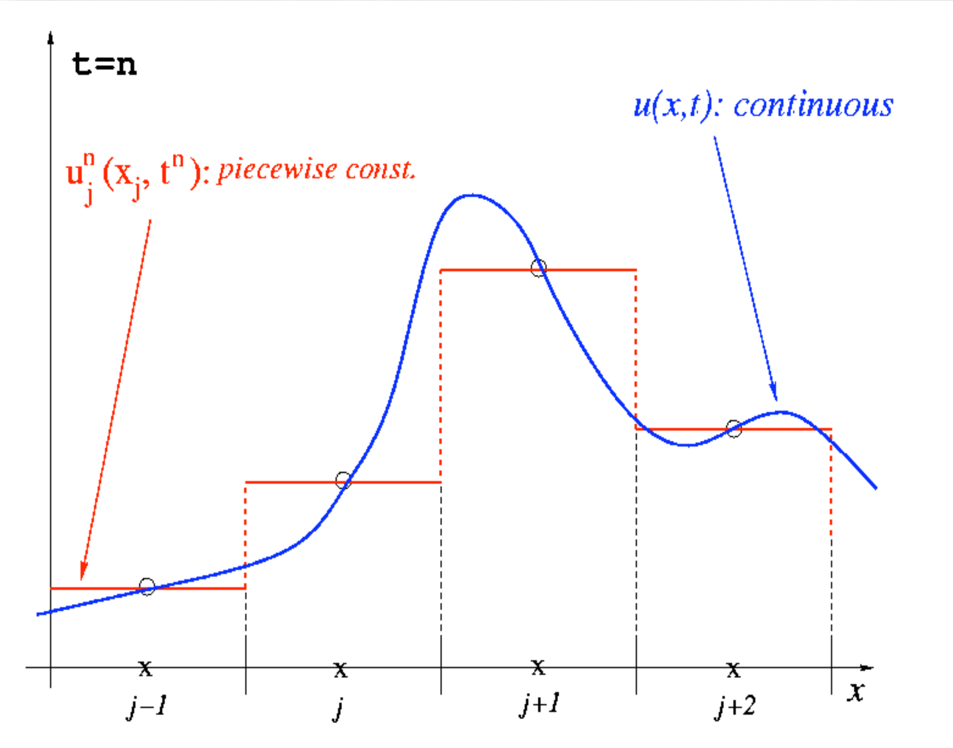


This is a well-know phenomenon which is usually referred to as "shock steepening". Mathematically this is due to the fact that the characteristics of the equation are space and time dependent and induce a focussing: ie a shock.

Note that this happens already with continuous initial data!

The problem of discretization…

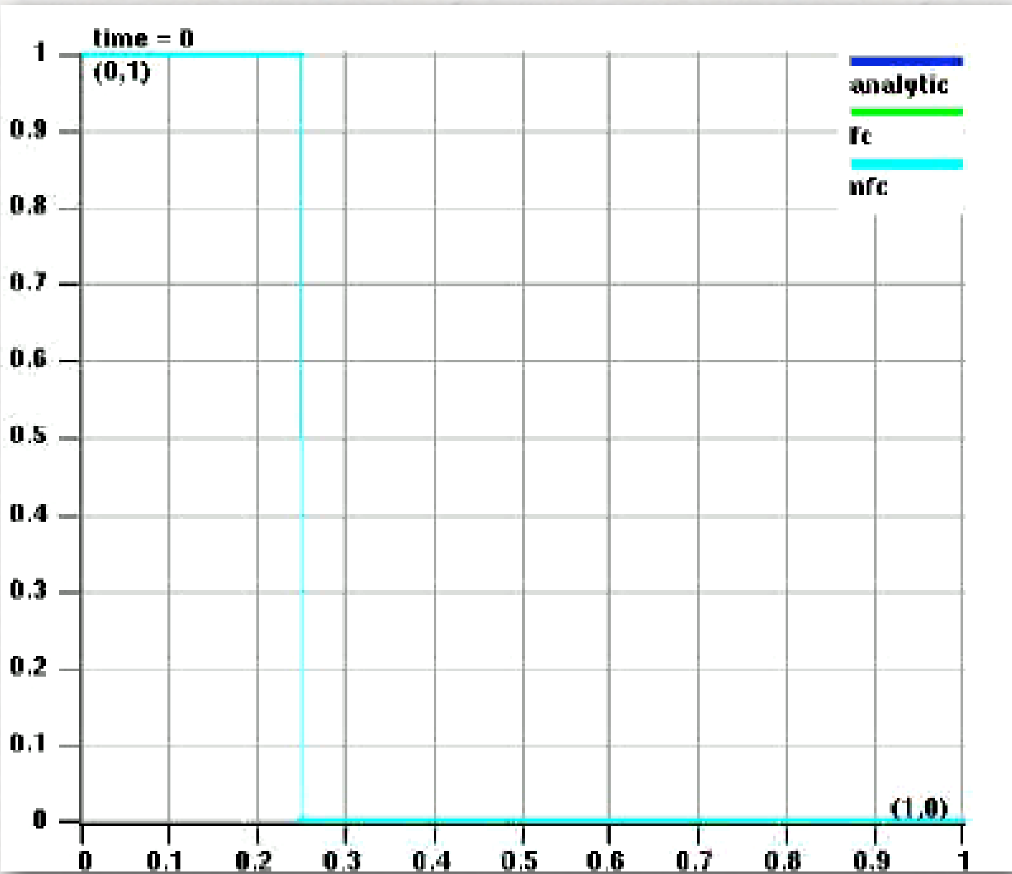
A generic problem arises when a Cauchy problem described by a set of *continuous* PDEs is solved in a *discretized form:* the numerical solution is, at best, *piecewise constant.*



This is particularly problematic when discretizing hydrodynamical eqs in compressible fluids, whose nonlinear properties *generically* produce, in a finite time, *nonlinear waves* with discontinuities (ie shocks, rarefaction waves, etc) even from smooth initial data!

A representative example: Burgers' equation

$$\partial_t u(x,t) + u(x,t)\partial_x u(x,t) = 0$$



This is a well-know example of the importance of a proper writing of the equation. In particular, in the inviscid case it can be written a non-flux conservative but also in a conservative form as

$$\partial_t u + u\partial_x u = 0 \;, \qquad \text{(nfc)}$$

$$\partial_t u + \tfrac{1}{2}\partial_x u^2 = 0 \;, \qquad \text{(fc)}$$

Conservative form of the equations

The homogeneous partial differential equation

$$\partial_t u(x,t) + a[u(x,t)]\partial_x u(x,t) = 0$$

is written in said to be in flux-conservative (fc) form if written as

$$\partial_t u(x,t) + \partial_x F[u(x,t)] = 0$$

- In conservative systems, knowledge of the state vector u at one point in spacetime allows to determine the flux f (and so the evolution) for each state variable.
- Theorems (Lax, Wendroff; Hou, LeFloch)
    - fc formulation converges to the weak solution of the problem (ie a solution of the integral form of the fc form)
    - nfc converges to the wrong weak solution of the problem

Consider for simplicity an non-magnetized ideal fluid

$$\begin{aligned}
h^i_\nu T^{\mu\nu}{}_{;\mu} &= 0\ , && \text{(Euler eqs.)}\\
u_\nu T^{\mu\nu}{}_{;\mu} &= 0\ , && \text{(energy eq.)}\\
\rho_{,\mu} u^\mu + \rho u^\mu{}_{;\mu} &= 0\ , && \text{(continuity eq.)}\\
p &= p(\rho, \epsilon)\ , && \text{(EOS)}
\end{aligned}$$

The first step in rewriting the above equations in a fc form requires the identification of suitable "conserved" quantities in place of the "primitive" variables (ρ, ϵ, v^j). A little algebra shows that these are:

$$\begin{aligned}
D &= \rho W\ ,\\
S_j &= \rho h W^2 v_j\ ,\\
\tau &= \rho h W^2 - \rho W - p
\end{aligned}$$

where $h = 1 + \epsilon + p/\rho$ is the specific enthalpy and the Lorentz factor is defined as $W = (1 - \gamma_{ij} v^i v^j)^{-1/2} = \alpha u^0$

In this way one obtains the "Valencia" formulation (Banyuls et al. 97) of the relativistic hydrodynamics equations

$$\frac{1}{\sqrt{-g}}\left\{\partial_t\left[\sqrt{\gamma}\mathbf{F}^0(\mathbf{U})\right]+\partial_i\left[\sqrt{\gamma}\mathbf{F}^i(\mathbf{U})\right]\right\}=\mathbf{s}(\mathbf{U})\ ,$$

where

$$\mathbf{F}^0(\mathbf{U})=(D,S_j,\tau)^T\ ,$$

$$\mathbf{F}^i(\mathbf{U})=[D(\alpha v^i-\beta^i),S_j(\alpha v^i-\beta^i)+p\delta^i_j,\tau(\alpha v^i-\beta^i)+pv^i]^T$$

$$\mathbf{s}(\mathbf{U})=\left[0,T^{\mu\nu}\left(\partial_\mu g_{\nu j}+\Gamma^\delta_{\mu\nu}g_{\delta j}\right),\alpha\left(T^{\mu 0}\partial_\mu\ln\alpha-T^{\mu\nu}\Gamma^0_{\nu\mu}\right)\right]\ .$$

Note that the source terms do not contain derivatives of the hydrodynamical quantities (leaving intact the principal part) and vanish in a flat spacetime

Discretising the problem…

Let’s restrict to a simpler but instructive problem: a homogeneous, flux- conservative differential equation for the scalar $u=u(x,t)$ in one dimension

$$\partial_t u(x,t) + \partial_x F[u(x,t)] = 0$$

Its generic, finite-difference form is (1st-order in time, 2nd order in space)

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{\Delta x}\left(\hat{F}_{j+1/2} - \hat{F}_{j-1/2}\right)$$

where $u_{j+1/2}^n \equiv u(x_{j+1/2}, t^n)$ and

$$\hat{F}_{j+1/2} \equiv \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} F[u(x_{j+1/2},t)]\mathrm{d}t$$

“some approximation to the average flux at j+1/2”

Any finite-difference form of (1) must represent $\hat{F}_{j\pm 1/2}$ in the most accurate way. Different forms of calculating $\hat{F}_{j\pm 1/2}$ lead to different evolution schemes (Forward-Time-Centred-Space, Lax, Runge-Kutta, etc…, see www.aei.mpg.de/~rezzolla)

Possible solutions to the discontinuities problem:

★ **1st order accurate schemes**

- generally fine, but very inaccurate across discontinuities (eccessive diffusion, e.g. Lax method)

★ **2nd order accurate schemes**

- generally introduce oscillations across discontinuities

★ **2nd order accurate schemes with artificial viscosity**

- mimic Nature but not good in relativistic regimes

★ **Godunov Methods**

- discontinuities are not eliminated, rather they are exploited!

High Resolution Shock Capturing (Godunov) Methods

Based on a simple, yet brilliant idea by Godunov ('59). An example of how basic physics can boost research in computational physics.

Core idea: a piecewise constant description of hydrodynamical quantities will produce a collection of local Riemann problems whose solution can be found exactly.

$$\hat{F}_{j\pm1/2} \equiv \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} F[\tilde{u}(x_{j+1/2},t)]\mathrm{d}t$$

where $\tilde{u}(x_{j+1/2},t), \quad t \in [t^n, t^{n+1}]$

is the exact solution of the Riemann problem with initial data

$$\tilde{u}(x_{j\pm1/2},t^n) = \begin{cases} u_L(x,t^n) & \text{for } x < x_{j\pm1/2} \\ u_R(x,t^n) & \text{for } x > x_{j\pm1/2} \end{cases}$$

What is exactly a Riemann problem?...

It's the evolution of a fluid initially composed of two states with different and constant values of velocity, pressure and density.

If the problem is linear, it can be handled analytically after rewriting the flux conservative equation

$$\partial_t \bar{u} + \partial_x F(\bar{u}) = 0$$

as

$$\partial_t \bar{u} + A(\bar{u}) \partial_x \bar{u} = 0$$

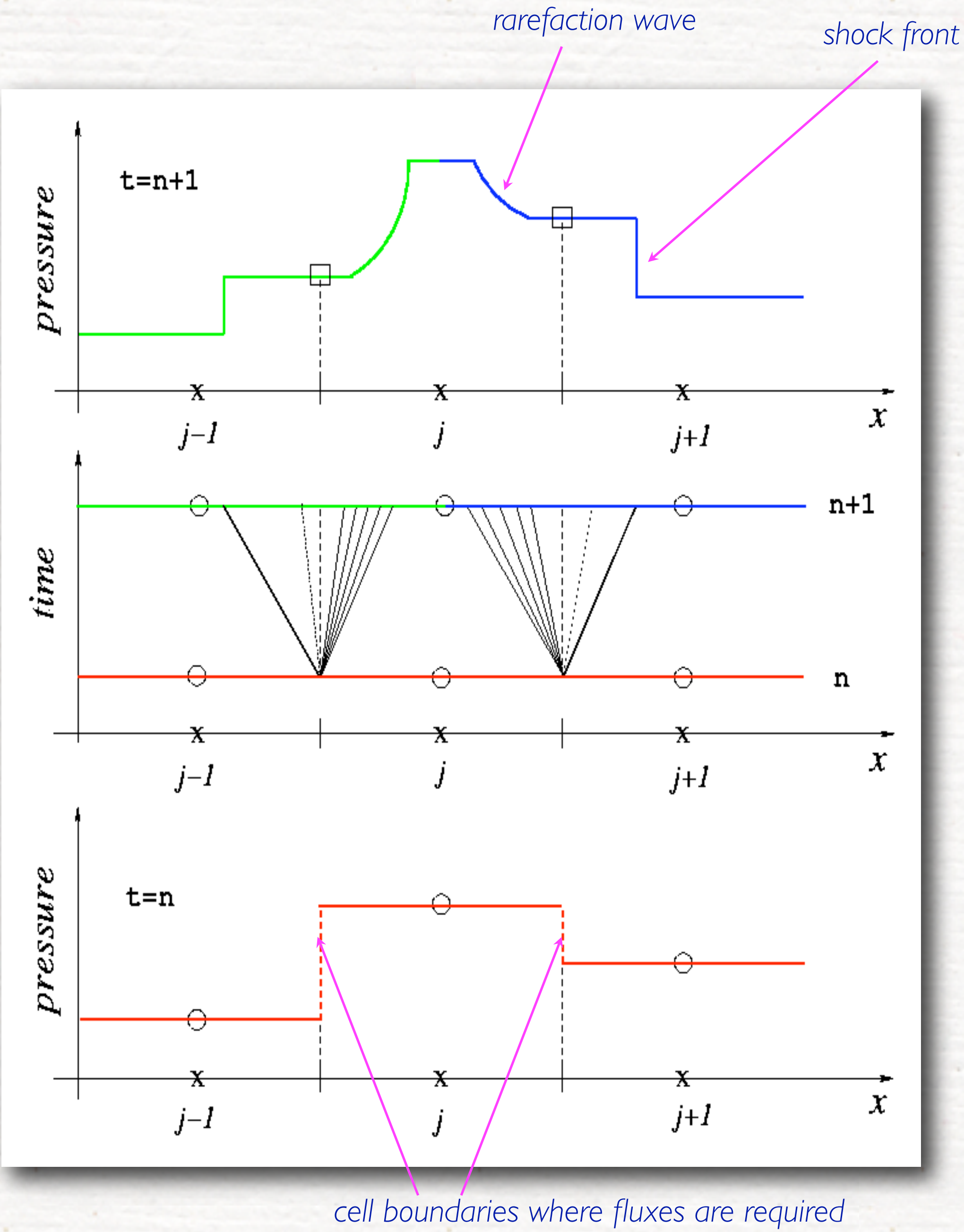
where *A(u)* is the Jacobian matrix of const. coefficients. In this way, (2) is written as a set of *i* linear equations for the characteristic variables $\bar{w}^i$

$$\partial_t \bar{w}_i + \Lambda \partial_x \bar{w}_i = 0 \longrightarrow \bar{w}_i(x,t) = \bar{w}_i(x - \lambda_i t, 0)$$

with $\mathbf{\Lambda}$ the diagonal matrix of the eigenvalues λ_i . The solution is

$$u(x,t) = \sum \bar{w}_i(x - \lambda_i t, 0) \bar{R}^i$$

and $\bar{R}^i$ are the right eigenvectors of *A*



Solution at the time n+1 of the two Riemann problems at the cell boundaries $x_{j+1/2}$ and $x_{j-1/2}$

Spacetime evolution of the two Riemann problems at the cell boundaries $x_{j+1/2}$ and $x_{j-1/2}$. Each problem leads to a shock wave and a rarefaction wave moving in opposite directions

Initial data at the time n for the two Riemann problems at the cell boundaries $x_{j+1/2}$ and $x_{j-1/2}$

Non-vacuum sources of gws at the AEI

- Neutron star oscillations: linear/nonlinear; magnetized/not

  Baiotti, Giacomazzo, LR

- Dynamical (barmode) instability

  Baiotti, De Pietri, Manca, LR

- Binary neutron stars

  Baiotti, Giacomazzo, LR

- Mixed Binary systems

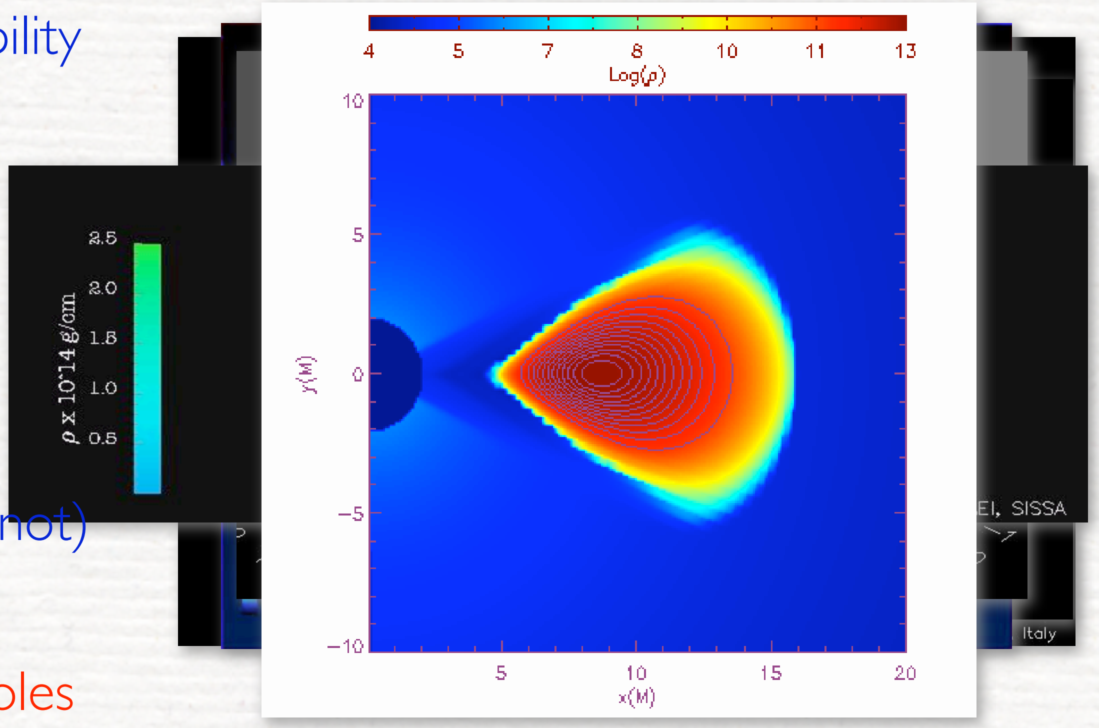
  Ansorg, Loeffler, LR

- Accretion torii (magnetized/not)

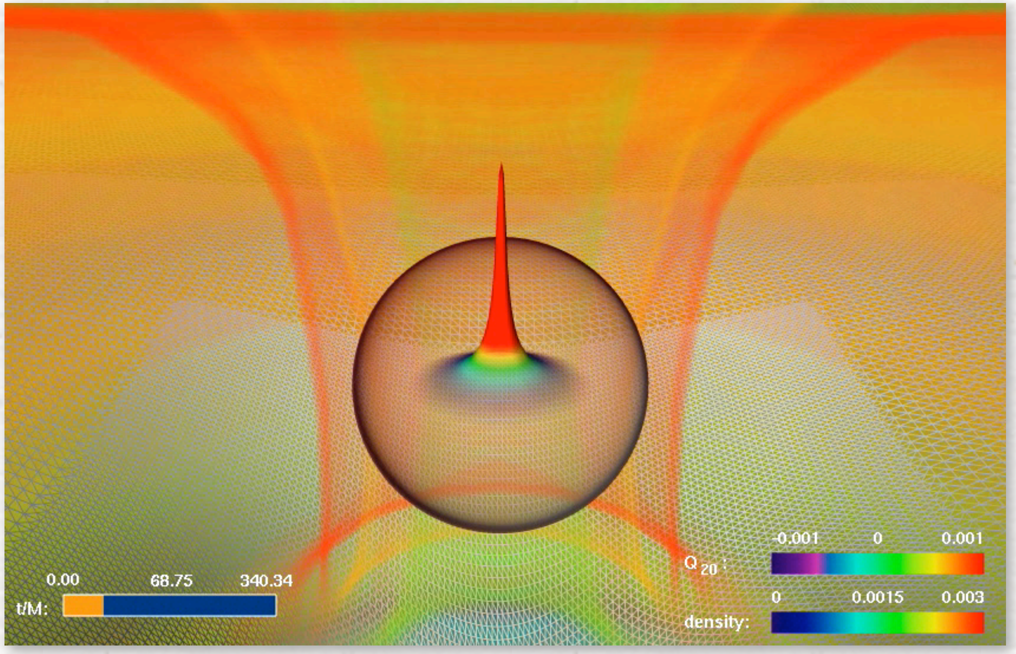
  Font, Montero, LR, Zanotti

- Rotating collapse to black holes

  Baiotti, Giacomazzo, Hawke, LR, Schnetter

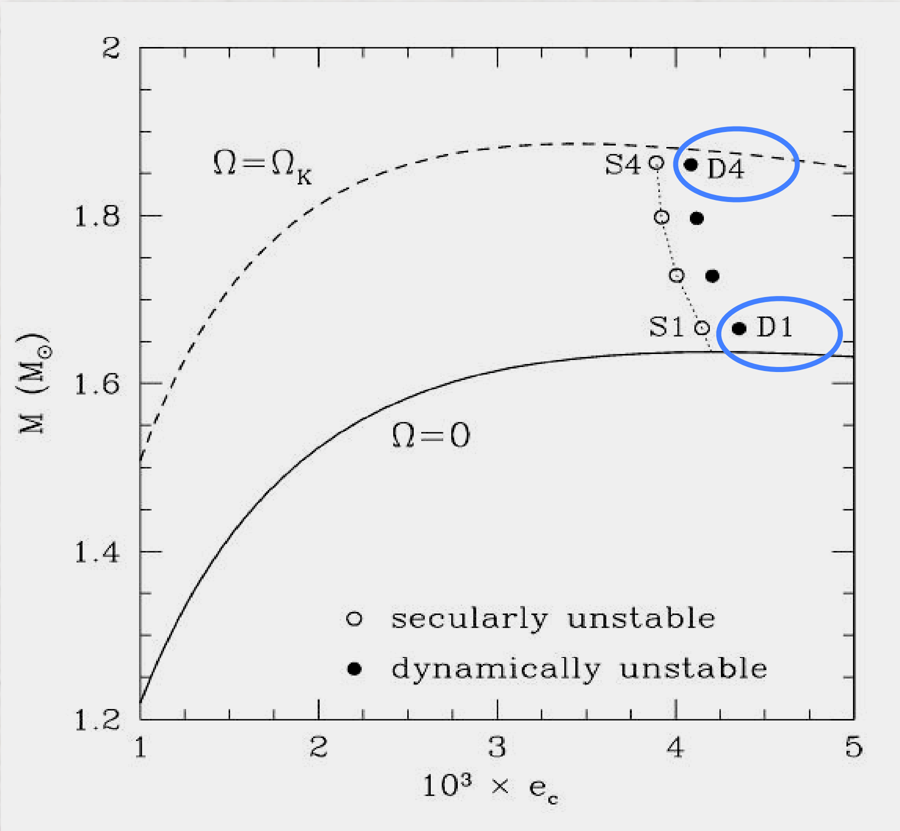


Stellar collapse to a rotating black hole



Baiotti, Hawke, Giacomazzo, LR, Schnetter, Stergioulas (05-07)

Initial Data: uniformly rotating polytropes



We have built sequences of *uniformly* rotating polytropes with constant value of angular momentum, up to the mass-shedding limit.

For simplicity we consider polytropes

$$p = \rho\varepsilon(\Gamma-1)$$

with

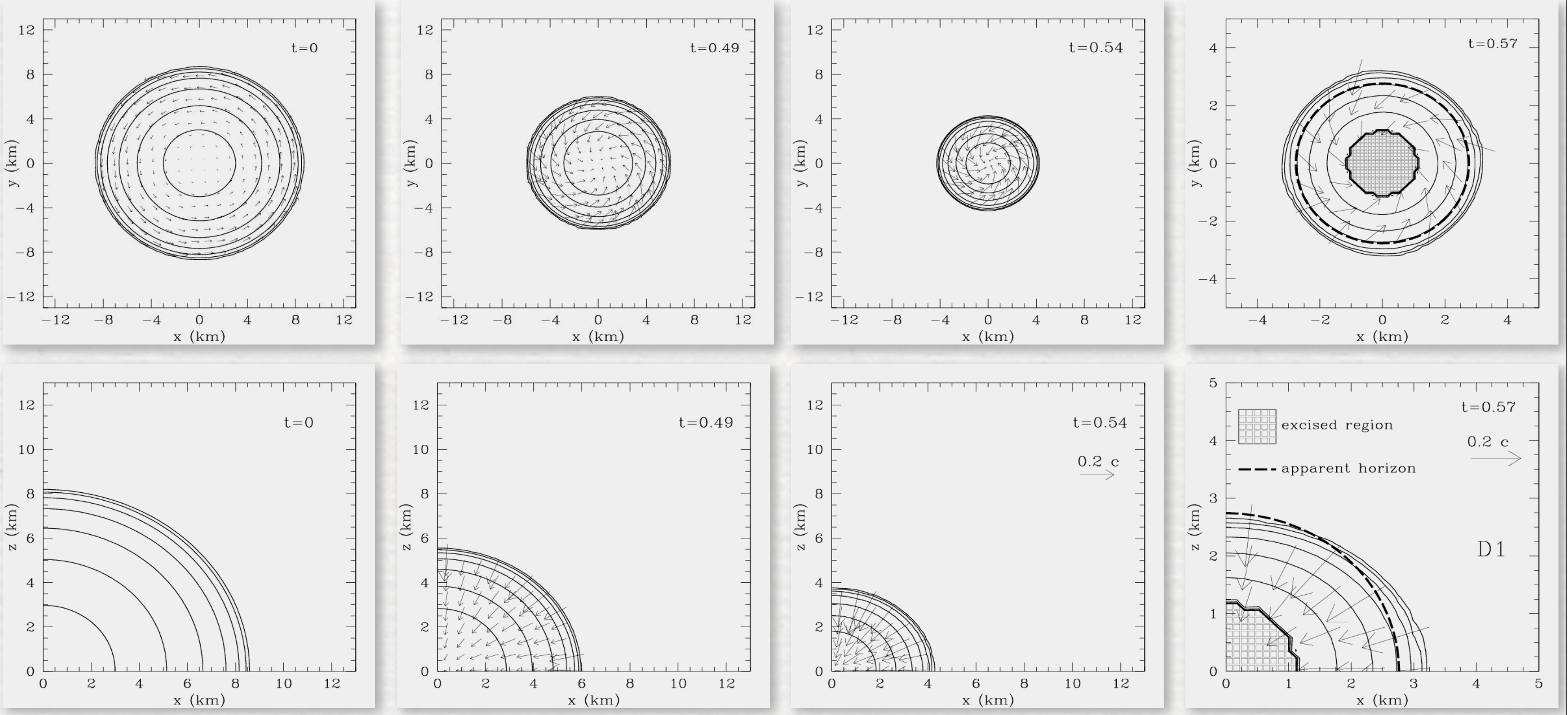
$$\Gamma=1+1/N=2$$

Baiotti, et al., PRD (2005)

Collapse of D1: slowly rotating model

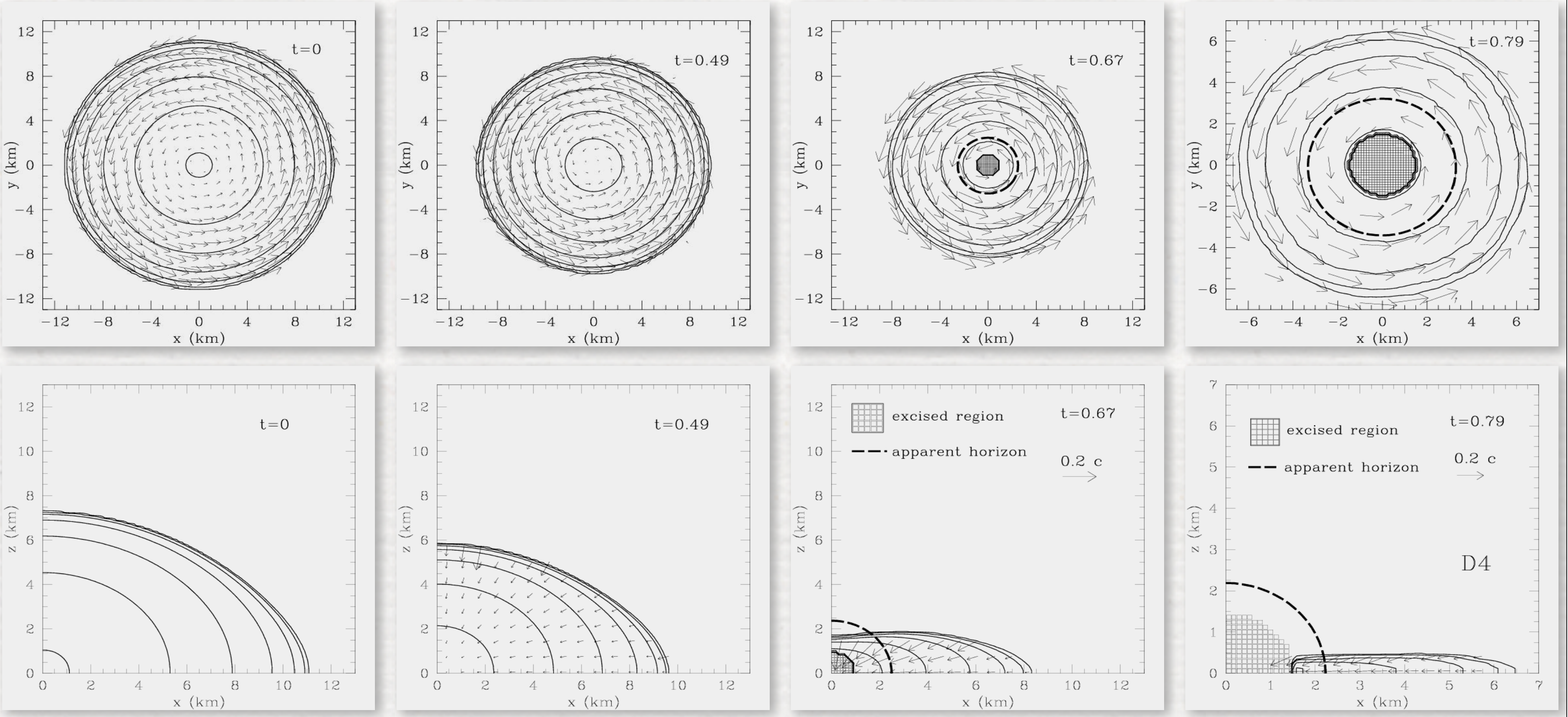
Baiotti, et al., PRD (2005)

The star is only slowly rotating and hence almost spherical. This holds essentially all the time till the formation of an apparent horizon

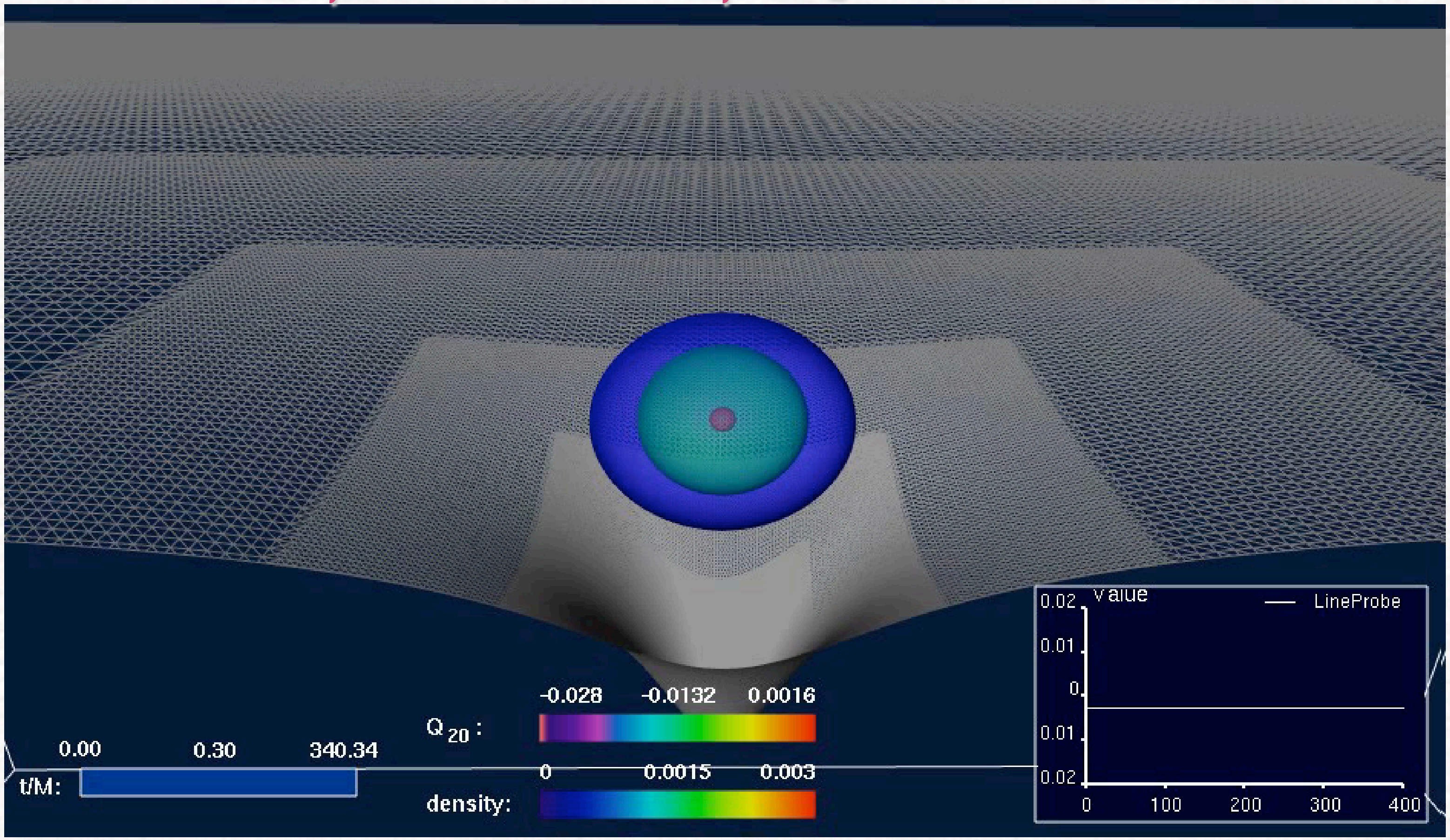


Collapse with excision: rapidly rotating model

Note that in this case the collapse is no longer homologous: the star first flattens, the collapse stalls and a disc is formed.



Collapse of D4: rapidly rotating star

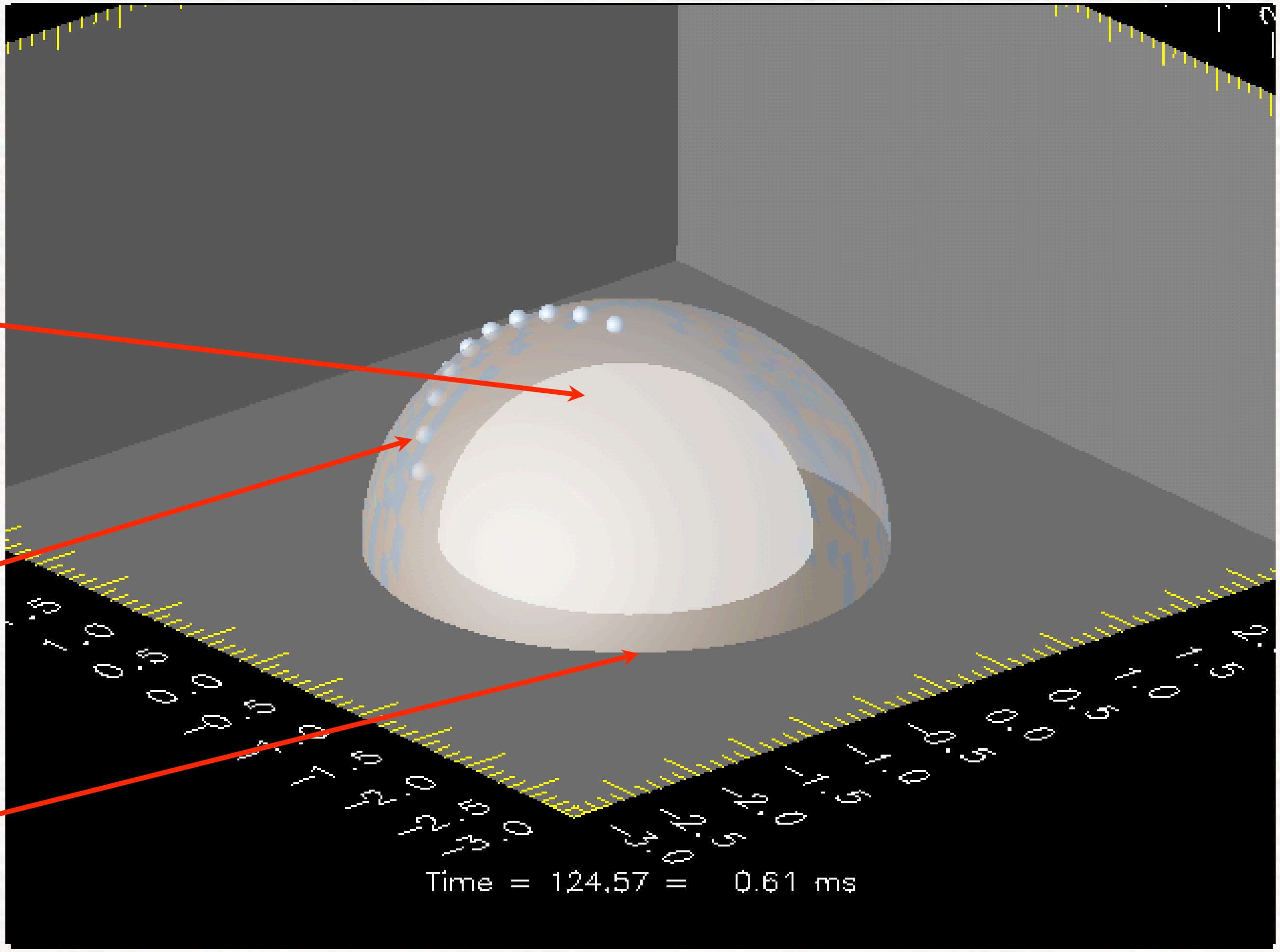


Animation by R. Kaehler

No excision is used. A small amount of numerical dissipation and suitable gauge-conditions provide stability

Dynamics of trapped surfaces

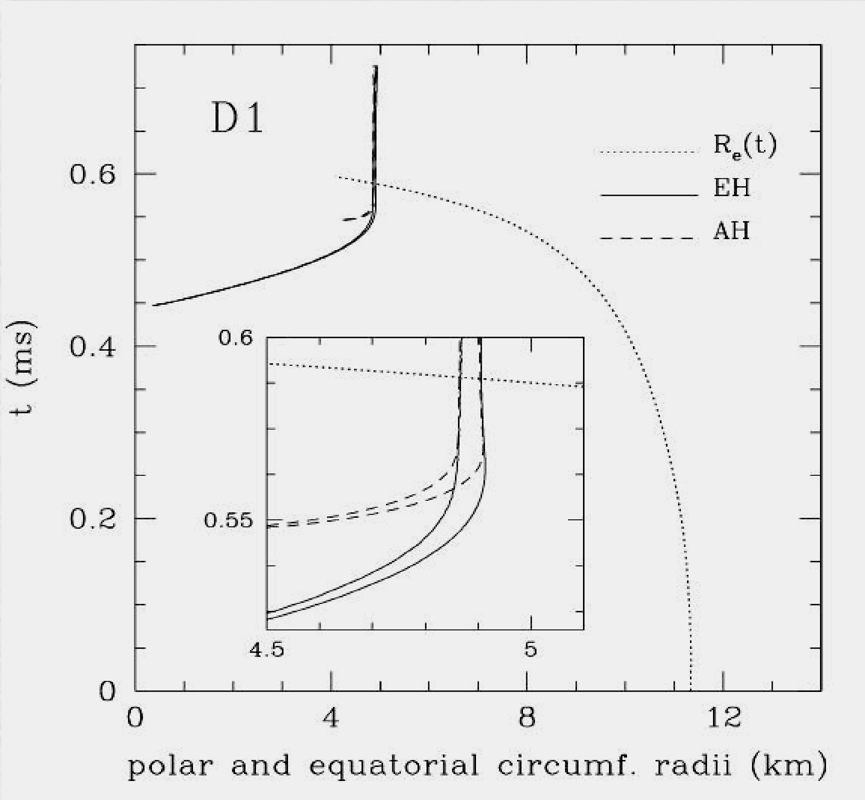
- White surface: *apparent horizon*
- Filled circles: *event horizon generators*
- Grey surface: *event horizon*



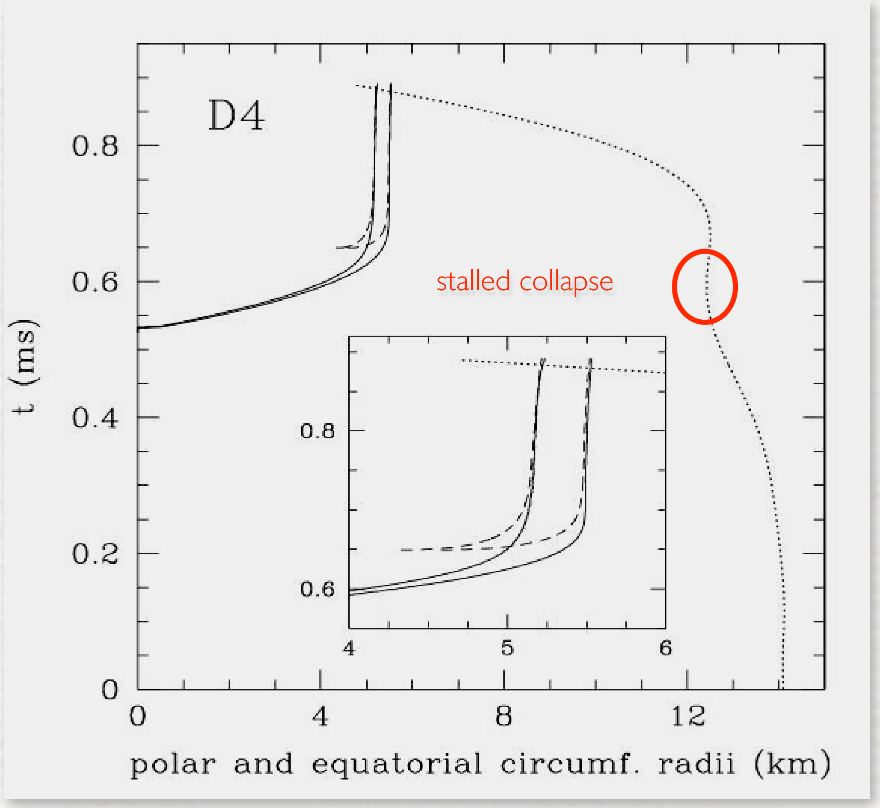
Animation by P. Diener

The collapse in a spacetime diagram

Slowly rotating star



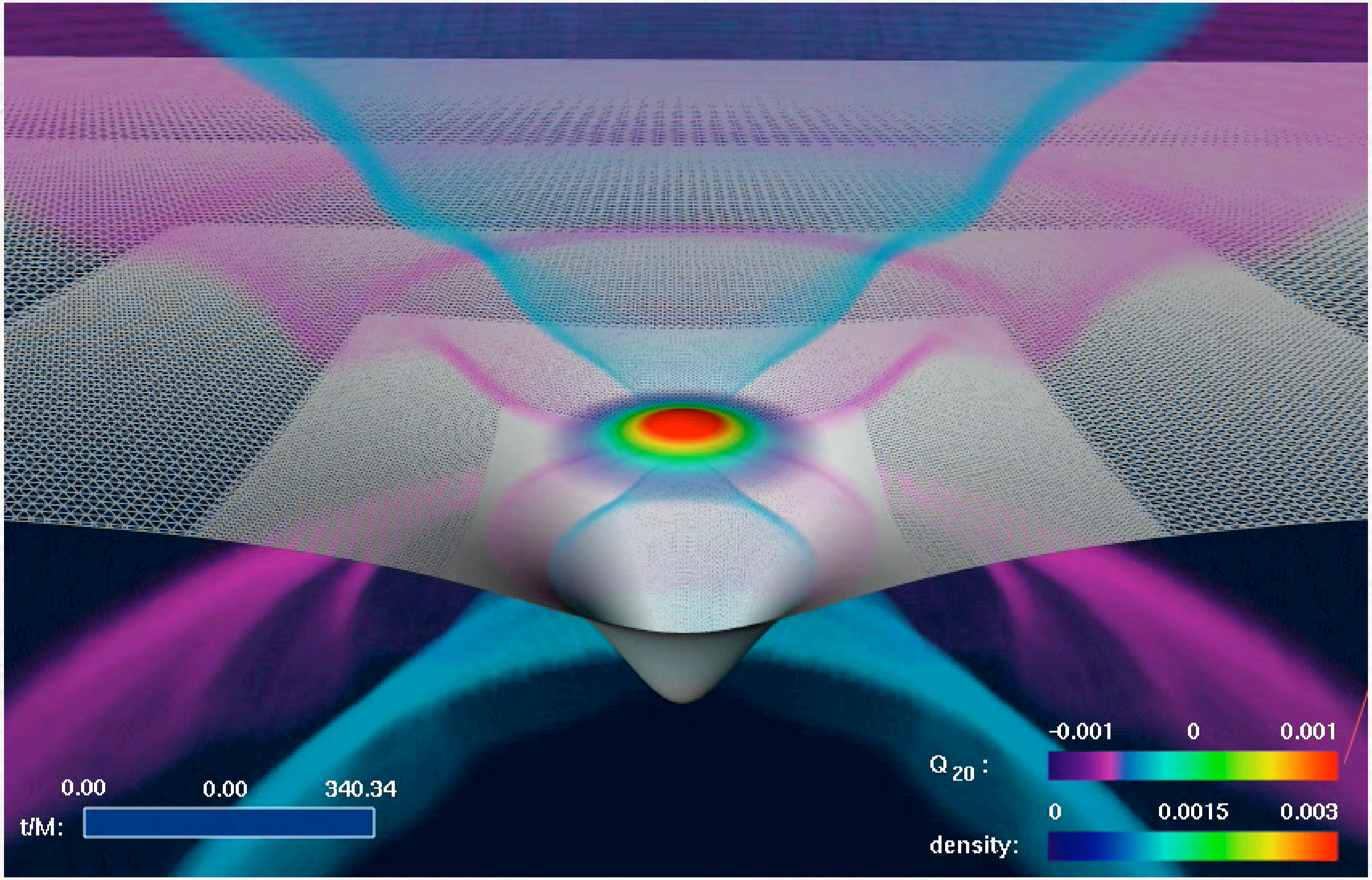
Rapidly rotating star



Each point on these diagrams summarizes $\sim 10^{11}$ fp operations!

The first waveforms form the birth of a black hole

Simulation for D1; similar behaviour for D4

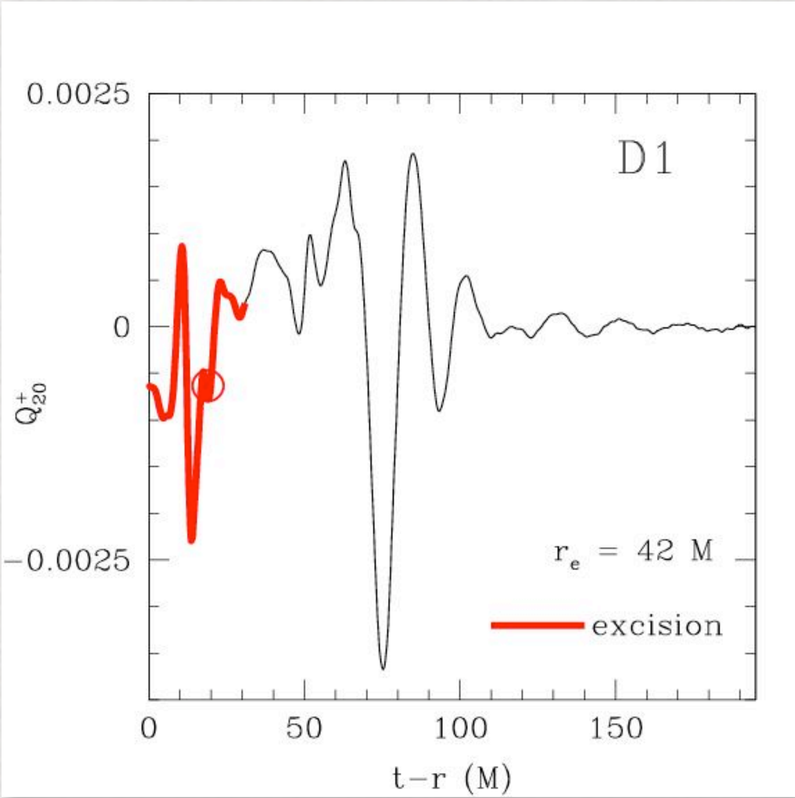


Animation by R. Kaehler

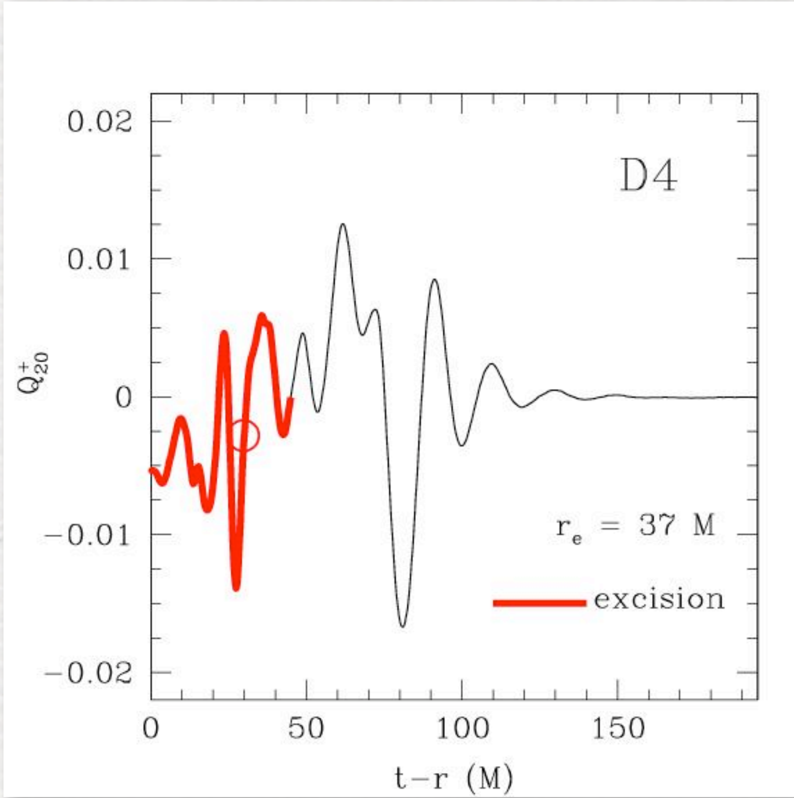
Waveforms!

$$h_{+} - \mathrm{i}h_{\times} = \frac{1}{2r}\sum_{\mathrm{l}m}\left(Q^{+}_{\mathrm{l}m} - \mathrm{i}\int_{-\infty}^{t} Q^{\times}_{\mathrm{l}m}(t')dt' \right)_{-2}Y^{\mathrm{l}m}$$

Slowly rotating model



Rapidly rotating model



Energy losses and detectability

The amplitudes can be used to calculate the energy lost to gws

$$\frac{dE}{dt} = \frac{1}{32\pi}\sum_{lm}\left(\left|\frac{dQ^{+}_{lm}}{dt}\right|^2 + \left|Q^{\times}_{lm}\right|^2\right) \longrightarrow \Delta E = \begin{cases} 3.3\times10^{-7}\, M/M_{\bullet} & (D1) \\ 3.1\times10^{-6}\, M/M_{\bullet} & (D4) \end{cases}$$

Smaller efficiency than calculated by Stark & Piran ($\Delta E \sim 1.5\text{x}10^{-4\div-5}\, M/M_{\odot}$), but consistent with estimates from core collapse

$$\frac{S}{N} \equiv \frac{h_c}{h_{rms}(f_c)} = \frac{h_c}{\sqrt{f_c S_h(f_c)}} =$$

Slow	Rapid	detector
0.2	2.1	Virgo/LIGO
1.1	11	Advanced
3.3	28	Dual

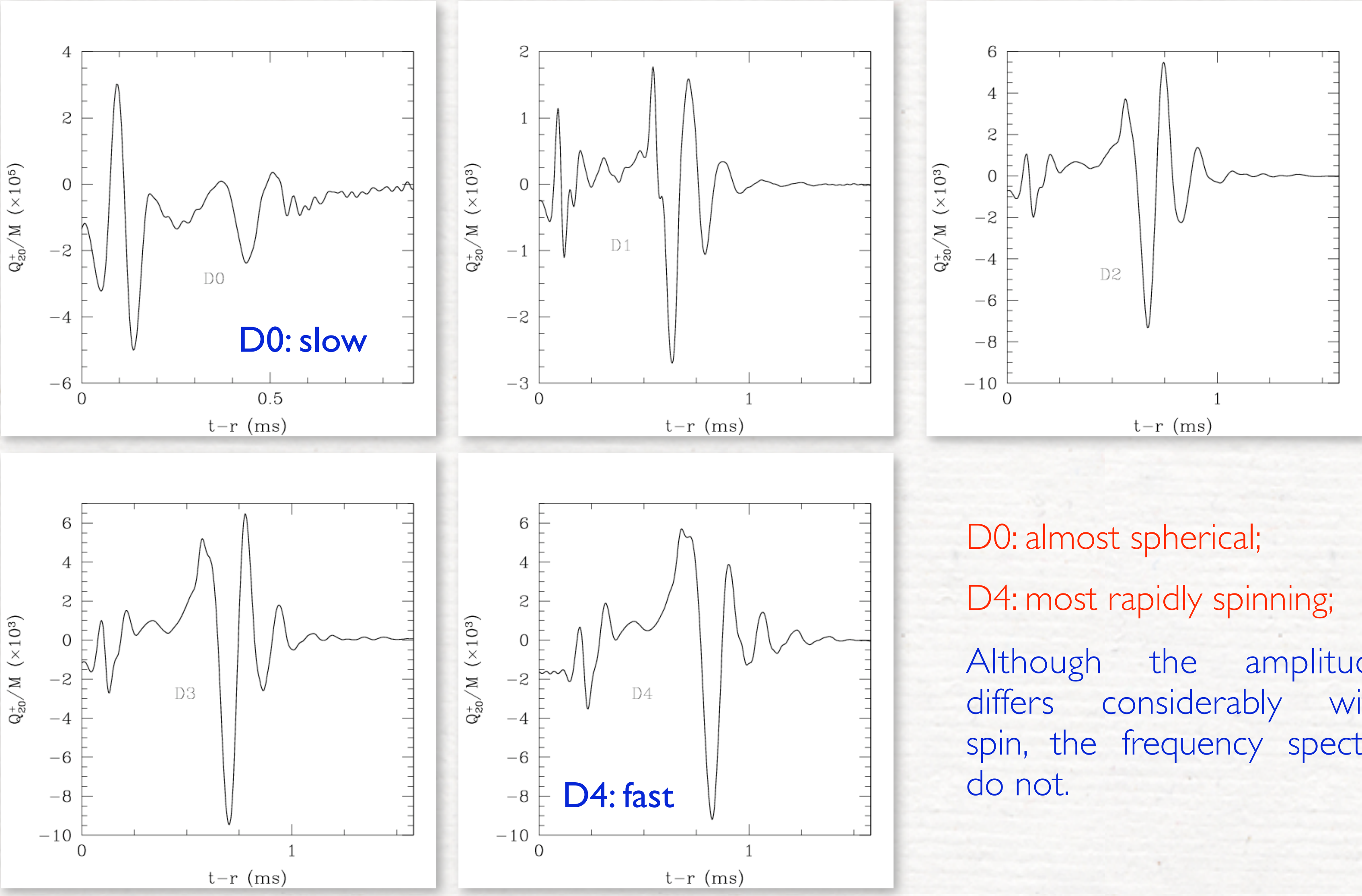
Building our understanding

With the basic picture clear, we are now looking at how the properties of waveforms depend (at times sensitively) on a number of factors:

- *perturbation type and amplitude*
- *rotation rate*
- *velocity distribution (ie differential rotation)*
- *EOS*

First steps towards source characterization and GW-astronomy

Influencing factors: rotation

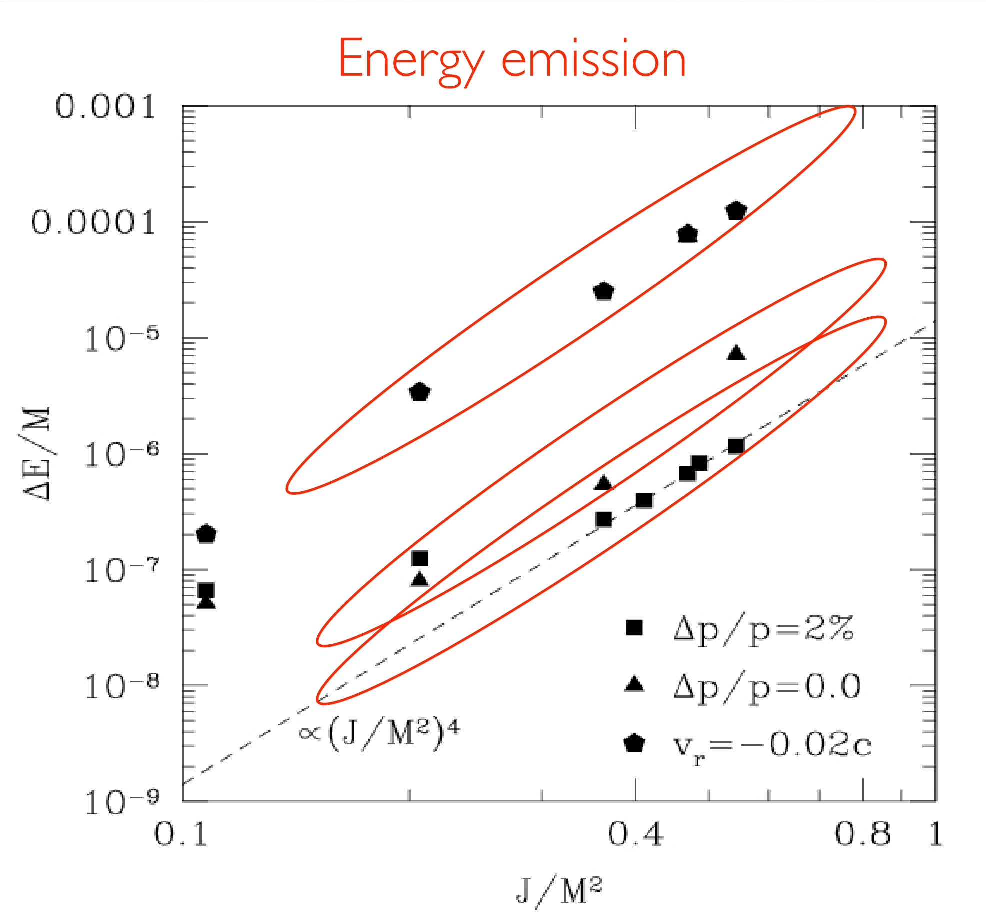


D0: almost spherical;

D4: most rapidly spinning;

Although the amplitude differs considerably with spin, the frequency spectra do not.

Energy emission from uniformly rotating stars



- The energy emitted is ~ $(J/M^2)^4$ for sufficiently large rotations

- Even small (eg. ~2%) pressure perturbations modify the collapse and reduce the energy emission

- Additional radial velocities (2%) can boost the collapse and enhance the emission

Overall, the gw emission from uniformly rotating stars is efficient only at very large rotation rates with *S/N>1* at 10 kpc.

Differentially rotating polytropes

Giacomazzo, LR, stergioulas (in progress)

- The efficiency in the gw emission can be increased if large deviations from axisymmetry develop via dynamical instabilities
- Differential rotation is expected in a star produced as a result of core-collapse or in the merger of a binary system of NSs
- Differential rotation can easily yield stars with $J/M^2<1$ (ie *sub-Kerr*) as well as stars with $J/M^2>1$ (ie *supra-Kerr*)
- A supra-Kerr star cannot collapse promptly to a Kerr bh; something must intervene to remove angular momentum, eg. non-axisymmetric instabilities

Collapse of sub-Kerr models

We have constructed differentially rotating sub-Kerr models close to the stability limit with

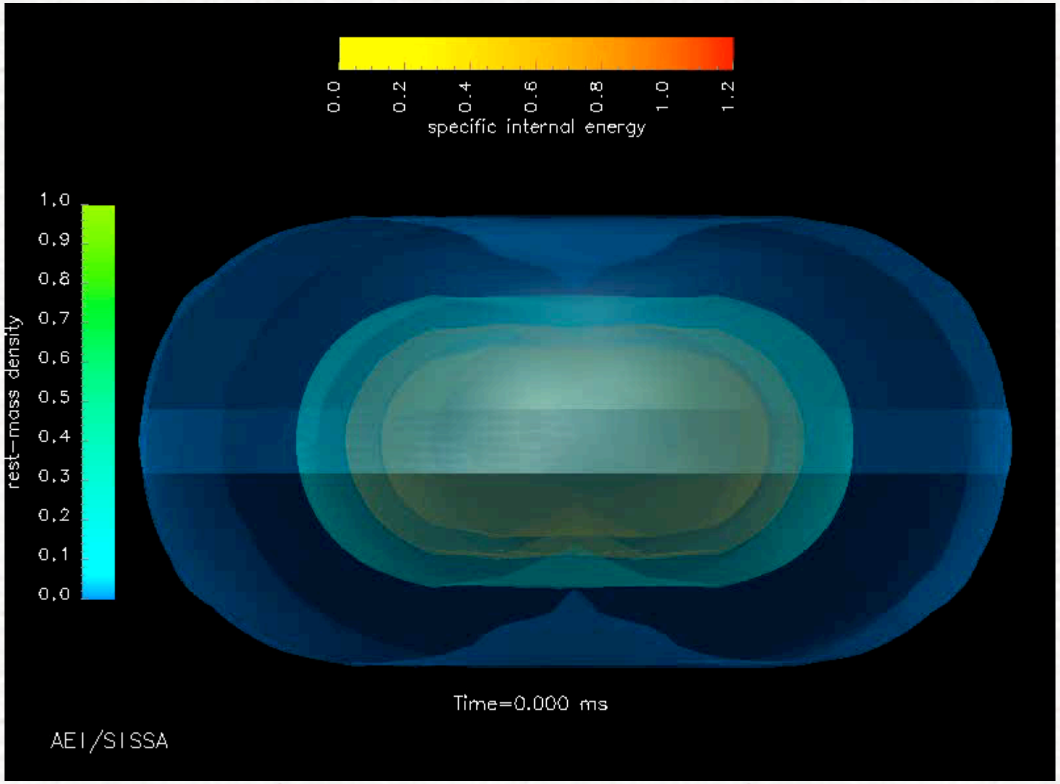
$$\Omega_c - \Omega = \left(\frac{r_e}{\hat{A}}\right)^2 \left[\frac{(\Omega-\omega)r^2\sin^2\vartheta e^{-2\rho}}{1-(\Omega-\omega)^2 r^2 \sin^2\vartheta e^{-2\rho}}\right]$$

As an example: **A10**

$M = 1.812\, M;\ J/M^2 = 0.477$

$T/|W| = 0.06;\ \Gamma = 2;\ A=1$

A ~**2%** pressure perturbation to trigger a collapse which is essentially axisymmetric but has nonzero contributions from higher multipoles



Animation by B. Giacomazzo

Collapse of a supra-Kerr model

Simple is also to build supra-Kerr models.

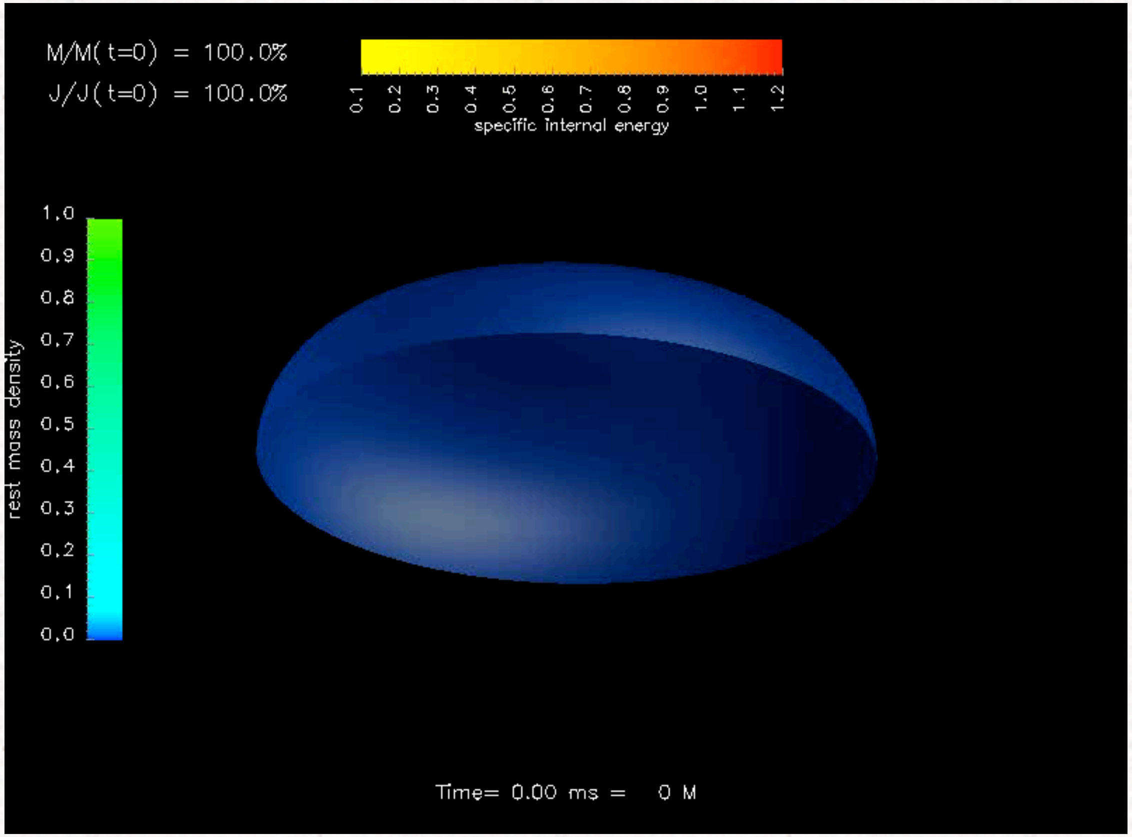
As an example **B1**:

$M=1.91M_{sun}$; $r_p/r_e=0.390$

$J/M^2=1.09$; $T/|W|=0.215$

$\Gamma = 2$

Note the collapse was triggered with a pressure depletion of **99%**



Animation by B. Giacomazzo

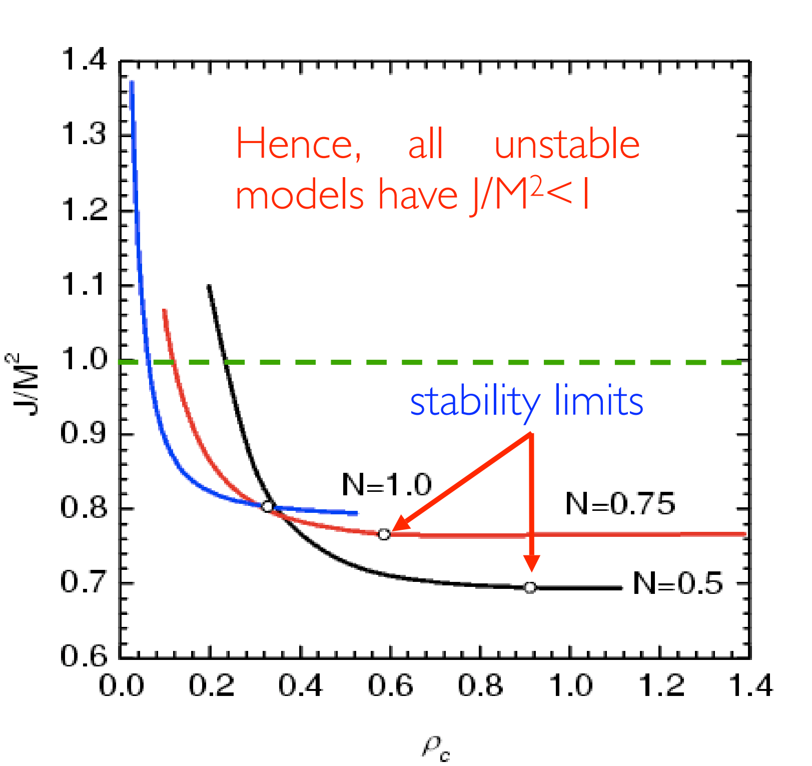
Modulo the Cartesian "imprints", this seems an excellent source of gravitational radiation. *Unfortunately, this is rather unrealistic…*

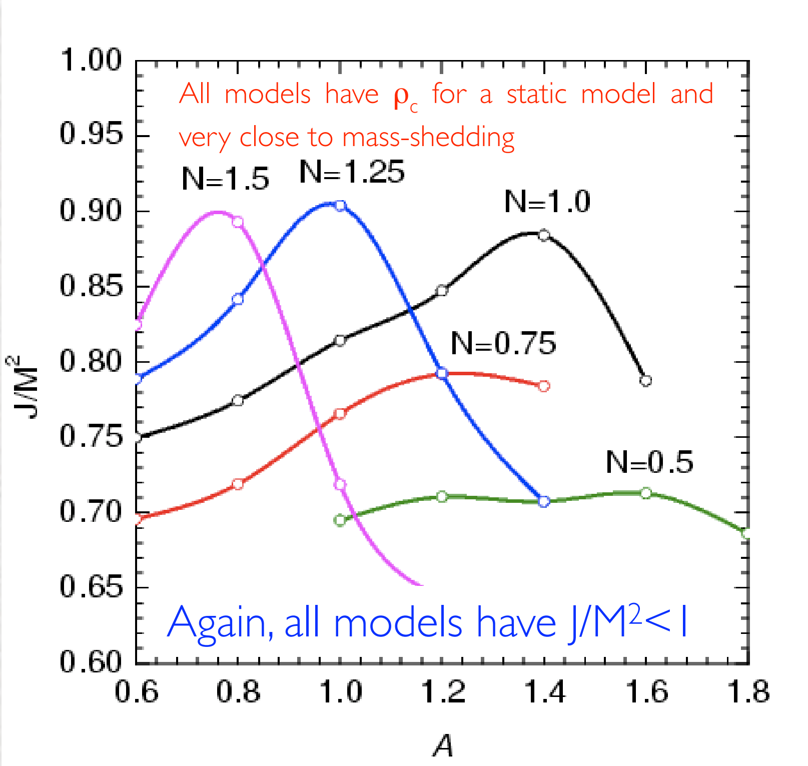
More on differentially rotating polytropes

No systematic investigation has been made on the equilibrium models and this has some surprises…

Sequences of typical stars at mass-shedding limit with A=1, r_p/r_e=0.35

Sequences of stars very close to the stability limit (the exact position tobe determined via simulations).

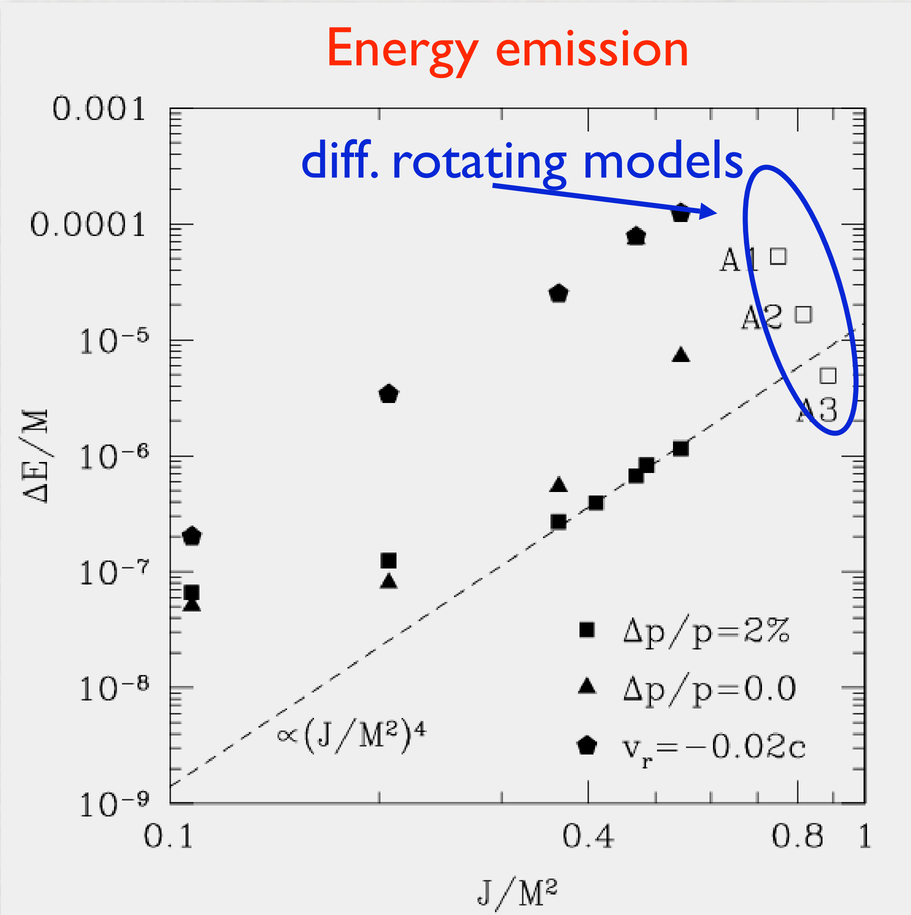




In other words:

- All differentially-rotating models that are dynamically unstable have $J/M^2 \leq 1$; collapse's dynamics/efficiency is similar to the one for uniformly rotating models
- All differentially-rotating models $J/M^2 > 1$ are stable; unlikely to produce non-axisymmetric instabilities without huge pressure reductions; not yet excluded (e.g. through phase transition for stars near the threshold).

Characterizing the energy emission



Note the increase in the energy efficiency as the differential rotation is increased:

$$\text{model A1} :: \quad \hat{A} = 0.6$$

$$\text{model A2} :: \quad \hat{A} = 1.0$$

$$\text{model A3} :: \quad \hat{A} = 1.4$$

Slow	Rapid	Diffrntl rotation	detector
0.2	2.1	4.5	Virgo/LIGO
1.1	11	27.6	Advanced
3.3	28	48.9	Dual

Summary

- Also for matter, the modelling of sources of gravitational waves in fully nonlinear regimes requires the numerical solution of the Einstein equations: ie numerical relativity techniques
- When dealing with the relativistic hydrodynamics or MHD equations, shocks are to be expected and need to be properly handled
- The best way of doing this is to write the equations in a flux-conservative form and to use HRSC methods
- Also for matter, there is all the evidence that numerical relativity is living its "renaissance", and our simulations have never been as accurate and stable.
- Much remains to be done towards a more realistic description of these sources. Introducing a magnetic field is a first step but also realistic equations of state, radiation transport, neutrino transport, etc...