Modelling sources of GWs: vacuum spacetimes

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Plan of the talk

- Numerical relativity: Why?
- Numerical relativity: Why so hard?
- Numerical relativity: How (vacuum)?
- Numerical relativity at the AEI
- One example:
  - Inspiral and merger of binary black holes
- I will assume basic knowledge of general relativity

Numerical Relativity: why?

Among other things, numerical relativity aims at:

- solve Einstein equations without approximations(!)…
- solve the binary problem(s)…
- investigate the complex physics of gravitational collapse
- investigate the formation and dynamics of horizons
- investigate structure and stability of NSs
- modelling sources of gravitational waves

Modelling source of GWs

A simple, back-of-the-envelope calculation in the Newtonian quadrupole approximation shows that the luminosity in gravitational waves (energy emitted in gws per unit time) is

$$L_{\rm gw} = \left(\frac{G}{c^5}\right)\left(\frac{M\langle v^2\rangle}{\tau}\right)^2 \simeq \left(\frac{G}{c^5}\right)\left(\frac{M}{R}\right)^5$$

i.e. intense sources are compact, massive and move at relativistic speeds: general relativity is indispensable.

What makes gw-astronomy challenging is

$$\left(\frac{G}{c^5}\right) \simeq 3.8 \times 10^{-60} {\rm erg\ s}^{-1}$$

i.e. even the gws from the most intense sources will statistically reach us as very weak

Not just an academic exercise

The calculation of the waveforms is not just an academic achievement. Several millions €s and thousands man-hours are dedicated to one of the most challenging physical experiments.

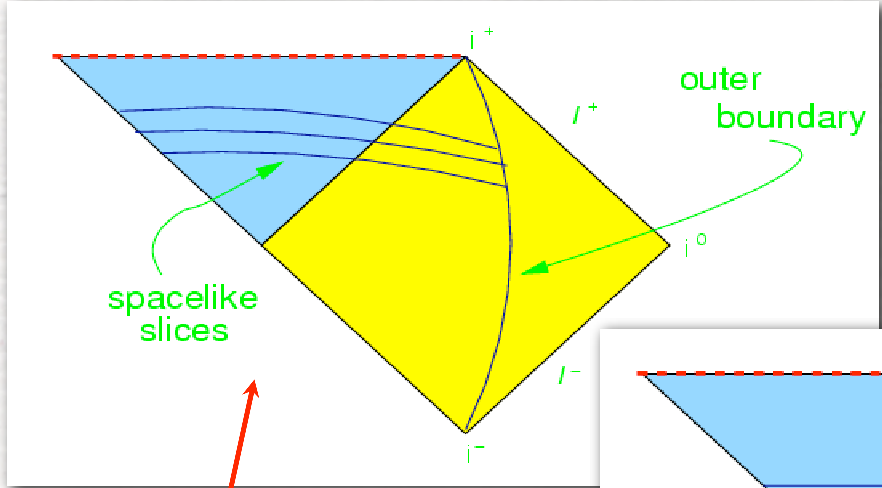
Knowledge of the waveforms can compensate for the very small S/N (matched-filtering). ⇒

Enhance detection and allow for source-characterization possible.

Numerical Relativity: why so hard?...

- ✳ No obviously "better" formulation of the Einstein equations
  - ADM, conformal traceless decomposition, first-order hyperbolic, harmonic, ...???
- ✳ Coordinates (spatial and time) do not have a specific meaning
  - this gauge freedom needs to be **handled with care!**

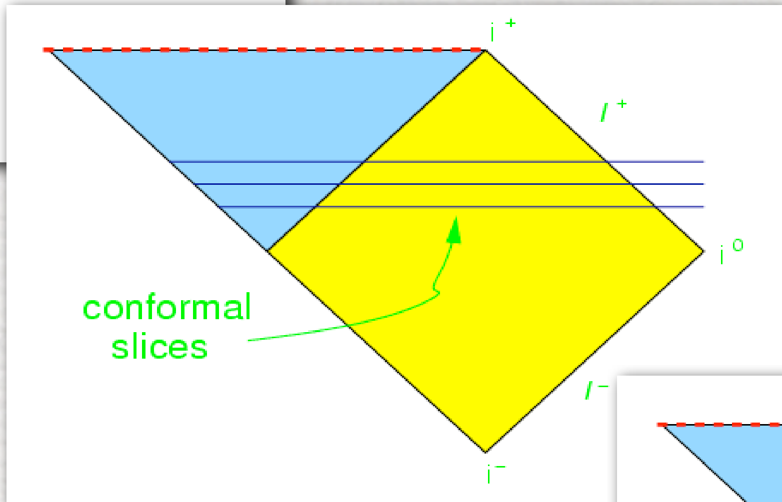
Which part of the spacetime to cover?...



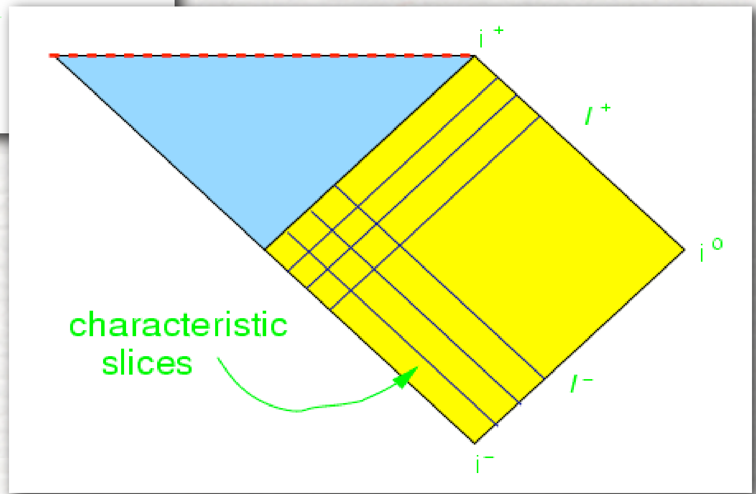
The spatial slices have finite extent (outer boundary in strong field region)

Images by D. Pollney

The standard choice for most groups



The spatial slices have infinite extent (infinity on the grid)



The slices are on in-outgoing light cones (ideal for radiation, less for intuition…)

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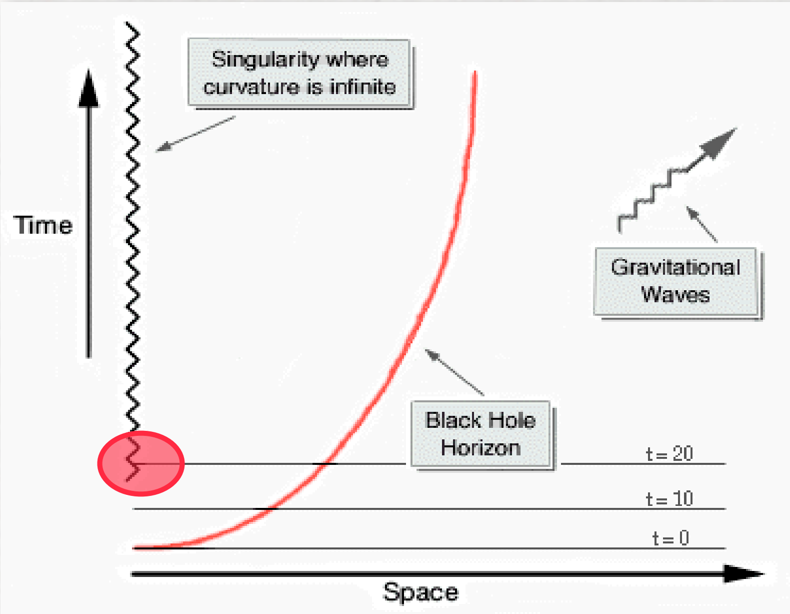
  - gauge conditions must *avoid singularities*
  - gauge conditions must counteract *grid stretching*

Choosing the right temporal gauge



Suppose you want to follow the gravitational collapse to a bh and assume a simplistic gauge choice:

$$\alpha = 1, \beta = 0 \quad \text{(geodesic slicing)}$$

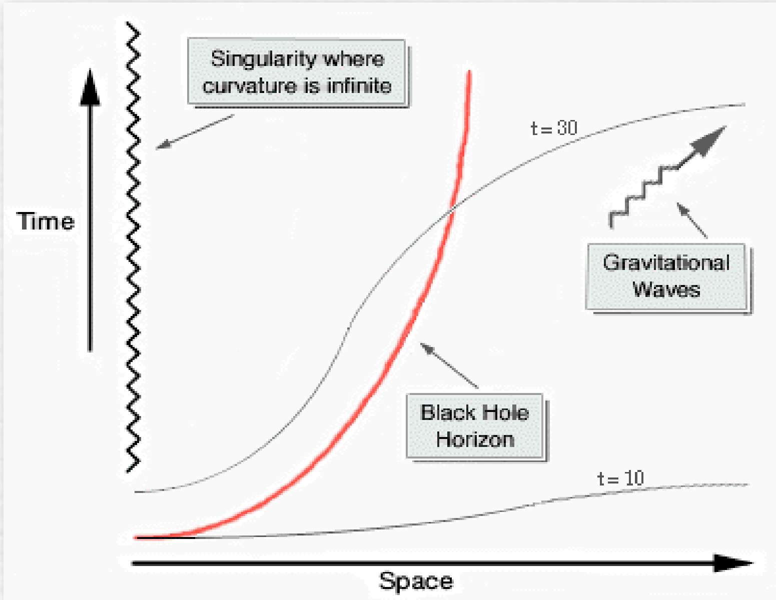
That would lead rapidly to a code crash! No chance of ever measuring gws!...

Need to use smarter gauges!

You want time to advance at different rates at different positions in the grid: *"singularity avoiding slicing"*

$\alpha = \alpha(t,x^i)$, $\beta = \beta(t,x^i)$ (e.g. maximal slicing)

Some chance of measuring gws!...



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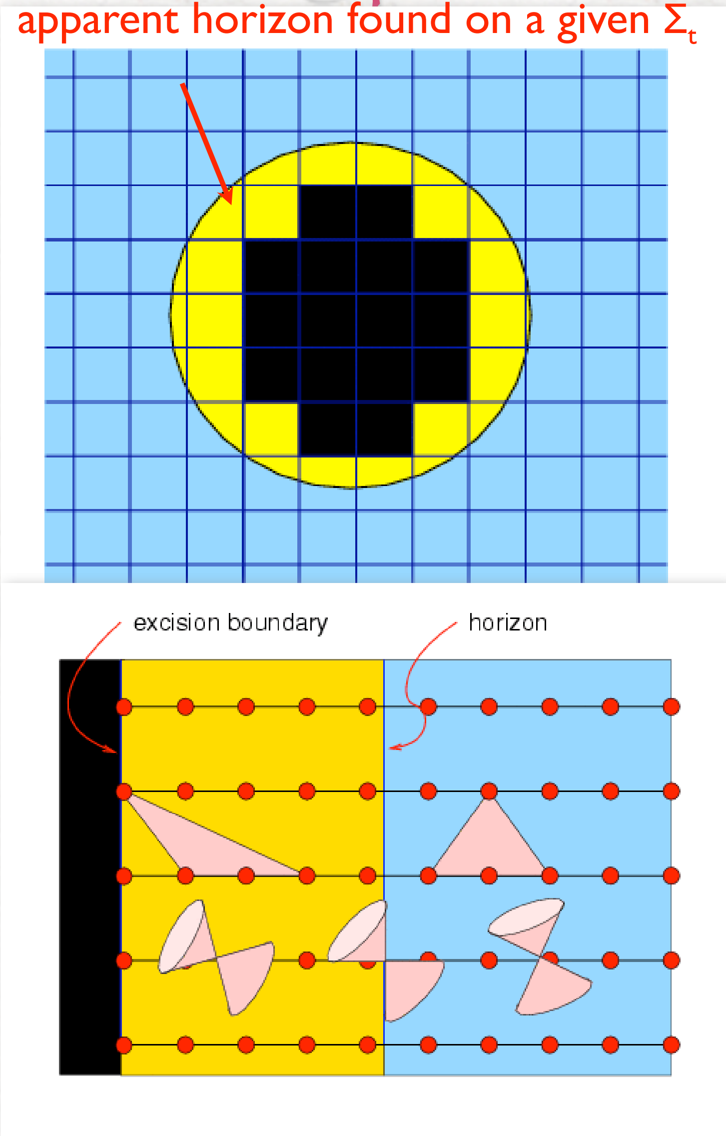
- **Einstein field equations are highly nonlinear**
  - essentially unknown in these regimes (well-posedeness not enough!...)
- **Physical singularities are the "butter-and-bread" of NR**
  - delicate techniques are needed to "excise" the troublesome region

excising parts of the spacetime with singularities…



Images by D. Pollney

In principle, the yellow region is causally disconnected from the blue one (ligth cones are "tilted in"); no boundary conditions would be needed at the apparent horizon.

In practice, the actual excision region (*"legosphere"*: black region) carved well inside the horizon.

NOTE:

o the Einstein equations are highly nonlinear in the yellow region! All sorts of numerical problems...

o the (apparent) horizon must be found; this is an expensive operation…

o the excised region has to move on the grid…

Numerical Relativity: why so hard?...

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  - delicate techniques are needed to "excise" the troublesome region
- ✴ **Simply more equations to solve: stretching supercomputers resources!**
  - large turn-around times make experimentation difficult (2-3 weeks/simulation)
  - implementations of AMR techniques is extremely problematic

Numerical Relativity: how?...

Let's recall the equations we are dealing with:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi T_{\mu\nu} \qquad \text{(field equations)}$$

$$\nabla_\mu T^{\mu\nu} = 0 \, , \qquad \text{(conservation of energy – momentum)}$$

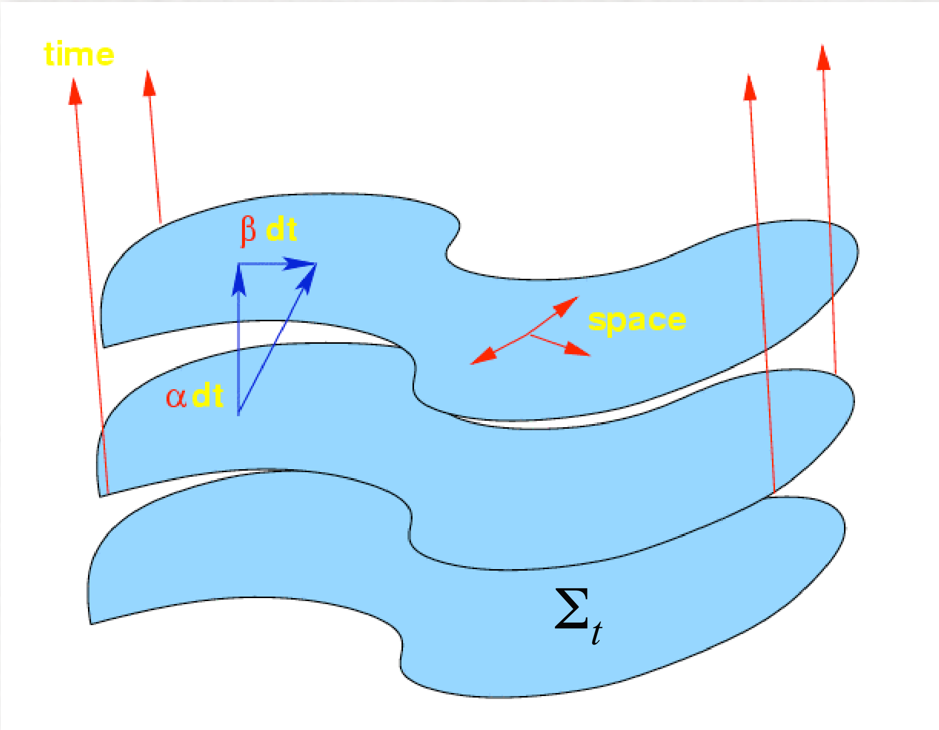
$$\nabla_\mu (\rho u^\mu) = 0 \, , \qquad \text{(conservation of baryon number)}$$

$$p = p(\rho, \epsilon, \ldots) \, . \qquad \text{(equation of state)}$$

In this first lecture we will consider $T_{\mu\nu} = 0$ and deal only with the Einstein field equations in vacuum: 6, highly-nonlinear, 2nd order partial differential equations

Disregarding the intrinsinc equality between spatial and time coordinates, numerical relativity follows the traditional approach in the solution of time-dependent problems and foliates the 4D spacetime in a series of *t=const* spatial hypersurfaces Σ_t , ie a stack of 3D spacetimes

$$\mathrm{d}s^2 = -(\alpha^2 - \beta_i\beta^i)\mathrm{d}t^2 + 2\beta_i \mathrm{d}x^i \mathrm{d}t + \gamma_{ij}\mathrm{d}x^i \mathrm{d}x^j$$



α :: *lapse function*

Measures the "clocks ticking rates" on two Σ_t

β_i :: *shift vector*

Measures the "stretching" of coordinates

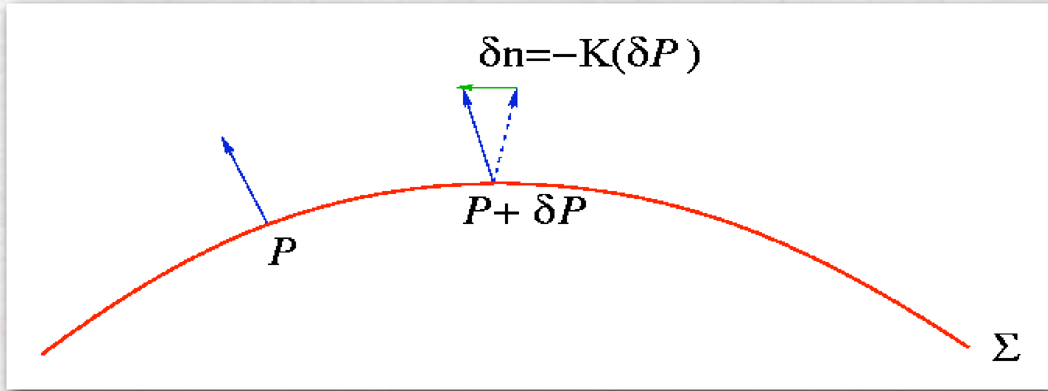
γ_{ij} :: 3 – *metric tensor*

Measures distances among points on a Σ_t

$$\mathrm{n} = \frac{1}{\alpha}\left(\partial_t - \beta^j \partial_j\right)$$

unit timelike 4-vector normal to Σ_t

Given a space-like slice Σ_t, while the three-metric γ_{ij} measures spatial (!) distances among points, the extrinsic curvature K_{ij} measures the curvature of the spatial hypersurface relative to the embedding 4D spacetime (i.e., it "bending")



Consider a vector at one position P and then parallel-transport it to a new location $P + \delta P$

The difference in the two vectors is proportional to the extrinsic curvature and this can either be positive or negative

The conformal factor ϕ, the conformal 3-metric $\tilde{\gamma}_{ij}$, the trace of the extrinsic curvature K, the trace-free conformal extrinsic curvature tensor $\tilde{A}_{ij}$, and the "Gammas" $\tilde{\Gamma}^i$ represent our evolution variables

$$\phi = \frac{1}{12}\ln(\det(g_{ij})) = \frac{1}{12}\ln(\gamma),$$

$$\tilde{\gamma}_{ij} = e^{-4\phi} g_{ij},$$

$$K = g^{ij} K_{ij},$$

$$\tilde{A}_{ij} = e^{-4\phi}(K_{ij} - \frac{1}{3} g_{ij} K),$$

$$\Gamma^i = \gamma^{jk}\Gamma^i_{jk}$$

$$\tilde{\Gamma}^i = \tilde{\gamma}^{jk}\tilde{\Gamma}^i_{jk}$$

$$\mathcal{D}_t \tilde{\gamma}_{ij} = -2\alpha \tilde{A}_{ij} \ ,$$

$$\mathcal{D}_t \phi = -\frac{1}{6}\alpha K \ ,$$

$$\mathcal{D}_t \tilde{A}_{ij} = e^{-4\phi}\left[-\nabla_i\nabla_j\alpha + \alpha\left(R_{ij} - S_{ij}\right)\right]^{\mathrm{TF}} + \alpha\left(K\tilde{A}_{ij} - 2\tilde{A}_{il}\tilde{A}^l_j\right) \ ,$$

$$\mathcal{D}_t K = -\gamma^{ij}\nabla_i\nabla_j\alpha + \alpha\left[\tilde{A}_{ij}\tilde{A}^{ij} + \frac{1}{3}K^2 + \frac{1}{2}\left(\rho + S\right)\right] \ ,$$

$$\begin{aligned}\mathcal{D}_t \tilde{\Gamma}^i = {} & -2\tilde{A}^{ij}\partial_j\alpha + 2\alpha\left(\tilde{\Gamma}^i_{jk}\tilde{A}^{kj} - \frac{2}{3}\tilde{\gamma}^{ij}\partial_j K - \tilde{\gamma}^{ij}S_j + 6\tilde{A}^{ij}\partial_j\phi\right) \\ & -\partial_j\left(\beta^l\partial_l\tilde{\gamma}^{ij} - 2\tilde{\gamma}^{m(j}\partial_m\beta^{i)} + \frac{2}{3}\tilde{\gamma}^{ij}\partial_l\beta^l\right) \ .\end{aligned}$$

where $\mathcal{D}_t \equiv \partial_t - \mathcal{L}_\beta$ and $\mathcal{L}_\beta$ is the Lie derivative along the shift

The evolution equations to be solved from one time slice to the next are therefore: 6+6+3+1+1=17 and it is important to underline that the above set of equations is hyperbolic.

Other formulations have different properties and the ADM one (used for many years) is only weakly hyperbolic.

In addition, we also compute 3+1=4 elliptic equations: the "constraints".

$$\mathcal{H} \equiv {}^{(3)}R + K^2 - K_{ij}K^{ij} = 0 \,, \quad \text{(Hamiltonian constraint)}$$

$$\mathcal{M}^i \equiv D_j(K^{ij} - g^{ij}K) = 0 \,, \quad \text{(momentum constraints)}$$

Note we don't actually search for a solution but just monitor how large the violation is, i.e. $\mathcal{H}$ and $\mathcal{M}^i$

Wave-extraction techniques

Computing the waveforms is the ultimate goal of most numerical relativity and there are several ways of extracting gws from numerical relativity codes:

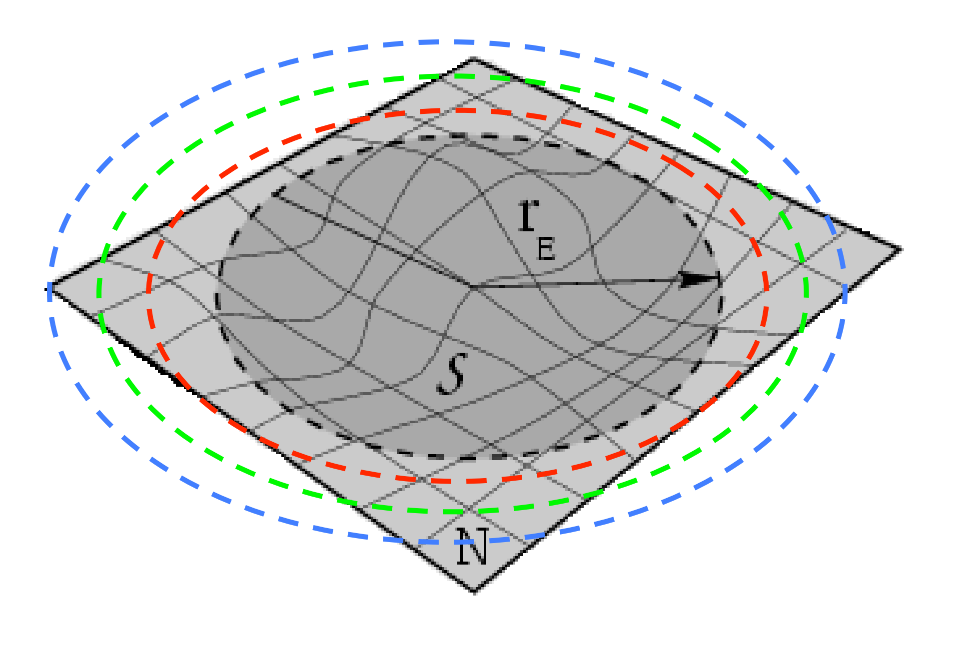
- asymptotic measurements
  - null slicing
  - conformal compactification
- non-asymptotic measurements (finite-size extraction worldtube)
  - Weyl scalars
  - perturbative matching to a Schwarzschild background

All have different degrees of success and this depends on the efficiency of the process which is very different for different sources

$$\frac{\Delta M}{M} \sim 10^{-2} \text{ (binary bhs)} - 10^{-7} \text{ (collapse to bh)}$$

Wave-extraction techniques

In both approaches, "*observers*" are placed on nested 2-spheres and calculate there either the Weyl scalars or decompose the metric into tensor spherical-harmonics to calculate the gauge-invariant perturbations of a Schwarschild black hole



Once the waveforms are calculated, all the related quantities: energy, momentum and angular momentum radiated can be derived simply.

Weyl scalars

At a sufficiently large distance from the source and in a Newman-Penrose tetrad frame the gws l in the two polarizations $h_\times, h_+$ can be written as

$$h_+ - \mathrm{i}h_\times = \lim_{r\to\infty} \int_0^t \mathrm{d}t' \int_0^{t'} \mathrm{d}t'' \Psi_4$$

where, according to the Peeling theorem, ψ_4 is the scalar with the smallest fall-off O(1/r).

It is then possible, for instance, to compute the projection of the momentum flux on the equatorial plane as

$$\mathcal{F}_i = \frac{\mathrm{d}P_i}{\mathrm{d}t} = \lim_{r\to\infty} \left\{ \frac{r^2}{16\pi} \int d\Omega \, n_i \left| \int_{-\infty}^t dt \Psi_4 \right|^2 \right\} .$$

This quantity will be used later to calculate the recoil velocity.

Gauge-invariant pertubations

$$h_+ - \mathrm{i}h_\times = \frac{1}{\sqrt{2}r}\sum_{l,m}\left(Q^+_{\ell m} - \mathrm{i}\int_{-\infty}^{t} Q^\times_{\ell m}(t')\mathrm{d}t'\right) Y^{-2}_{\ell m} + \mathcal{O}\left(\frac{1}{r^2}\right)$$

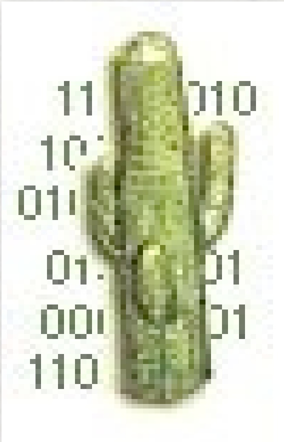
where $Q^\times_{\ell\, m}$, $Q^+_{\ell m}$ are the odd and even-parity *gauge-invariant* perturbations of a Schwarschild black hole. Similarly, it is possible to compute the projection of the momentum flux in on the equatorial plane as

$$\mathcal{F}^{\ell m}_x + \mathrm{i}\mathcal{F}^{\ell m}_y \equiv \frac{(-1)^m}{16\pi\ell(\ell+1)}\Bigg\{ -2\mathrm{i}\left[a_{\ell m}\dot{Q}^+_{\ell -m} Q^\times_{\ell\, m-1} + b_{\ell m}\dot{Q}^+_{\ell m} Q^\times_{\ell\, -(m+1)}\right]$$

$$+\sqrt{\frac{\ell^2(\ell-1)(\ell+3)}{(2\ell+1)(2\ell+3)}}\Bigg[c_{\ell m}\left(\dot{Q}^+_{\ell\, -m}\dot{Q}^+_{\ell+1\, m-1} + Q^\times_{\ell\, -m}\dot{Q}^\times_{\ell+1\, m-1}\right)$$

$$+d_{\ell m}\left(\dot{Q}^+_{\ell m}\dot{Q}^+_{\ell+1\, -(m+1)} + Q^\times_{\ell m}\dot{Q}^\times_{\ell+1\, -(m+1)}\right)\Bigg]\Bigg\},$$

Numerical Relativity at the AEI

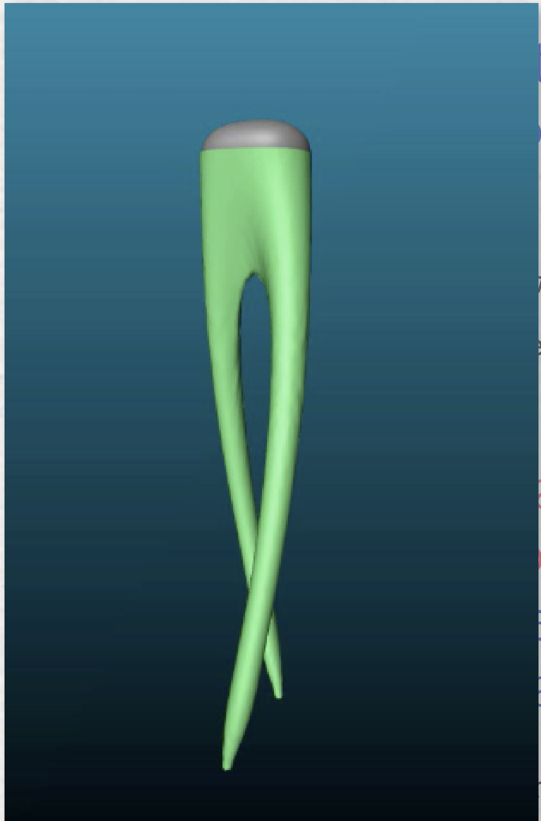


Cactus (www.cactuscode.org) is a computational "toolkit" developed at the AEI over the last 10 years the AEI and provides a general infrastructure for the solution in 3D and on parallel computers of PDEs in general and of the Einstein equations in particular.

Whisky (www.cactuscode.org) is a more recent code, developed at the AEI and SISSA, for the solution of the relativistic hydrodynamics and magnetohydrodynamics equations in arbitrary curved spacetimes.

Vacuum sources: binary black holes

- Binary BH evolutions for one or more orbits, calculating waveforms, energy, angular momentum, emitted, etc



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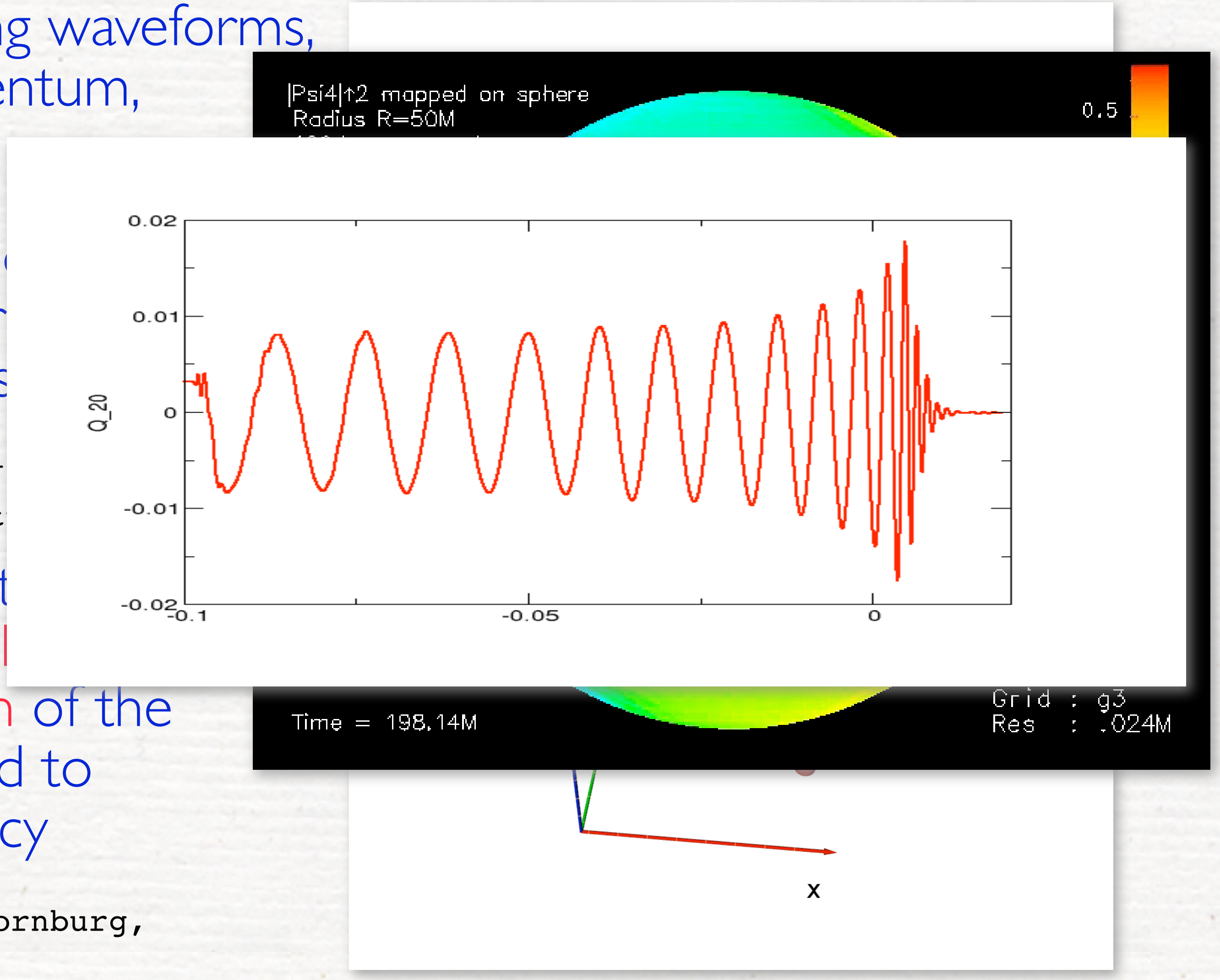
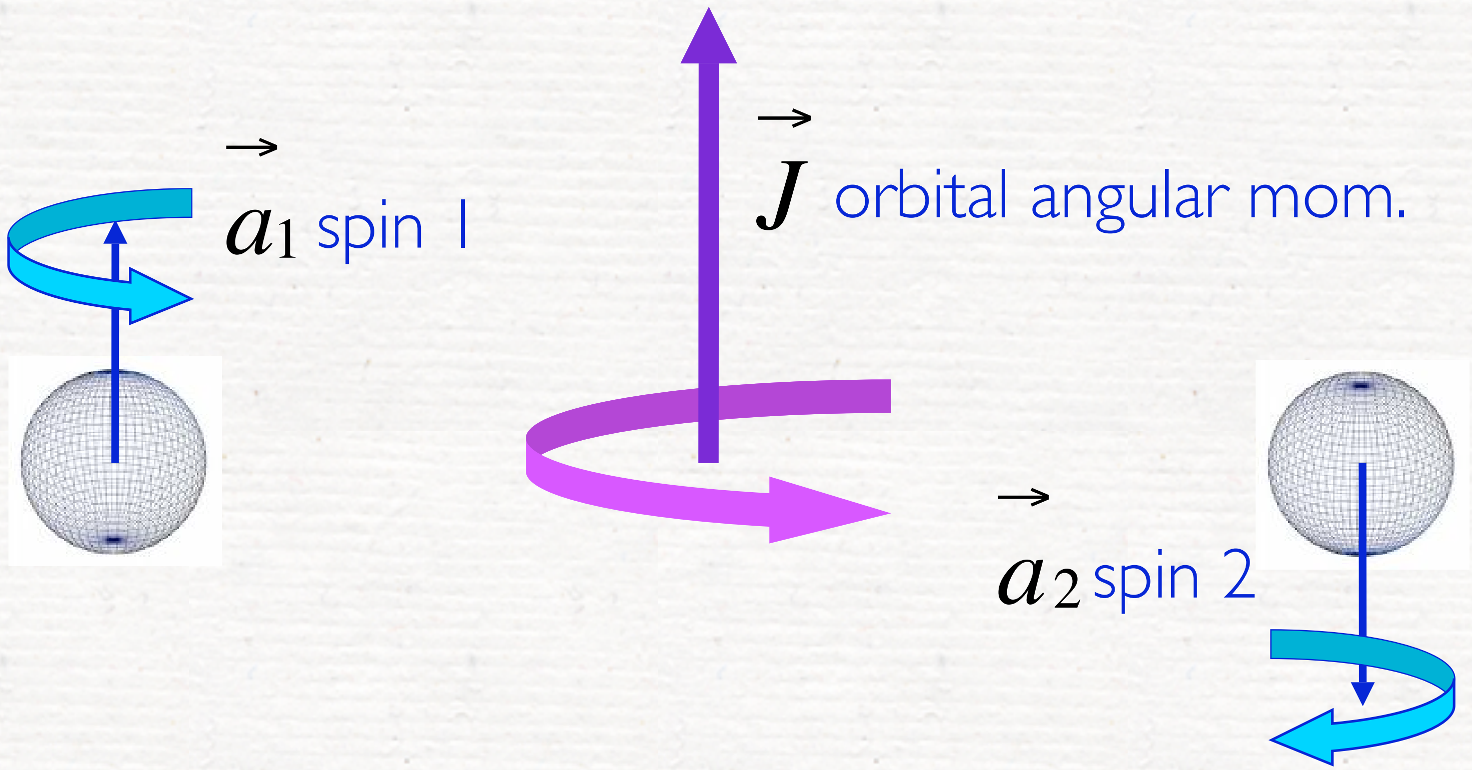
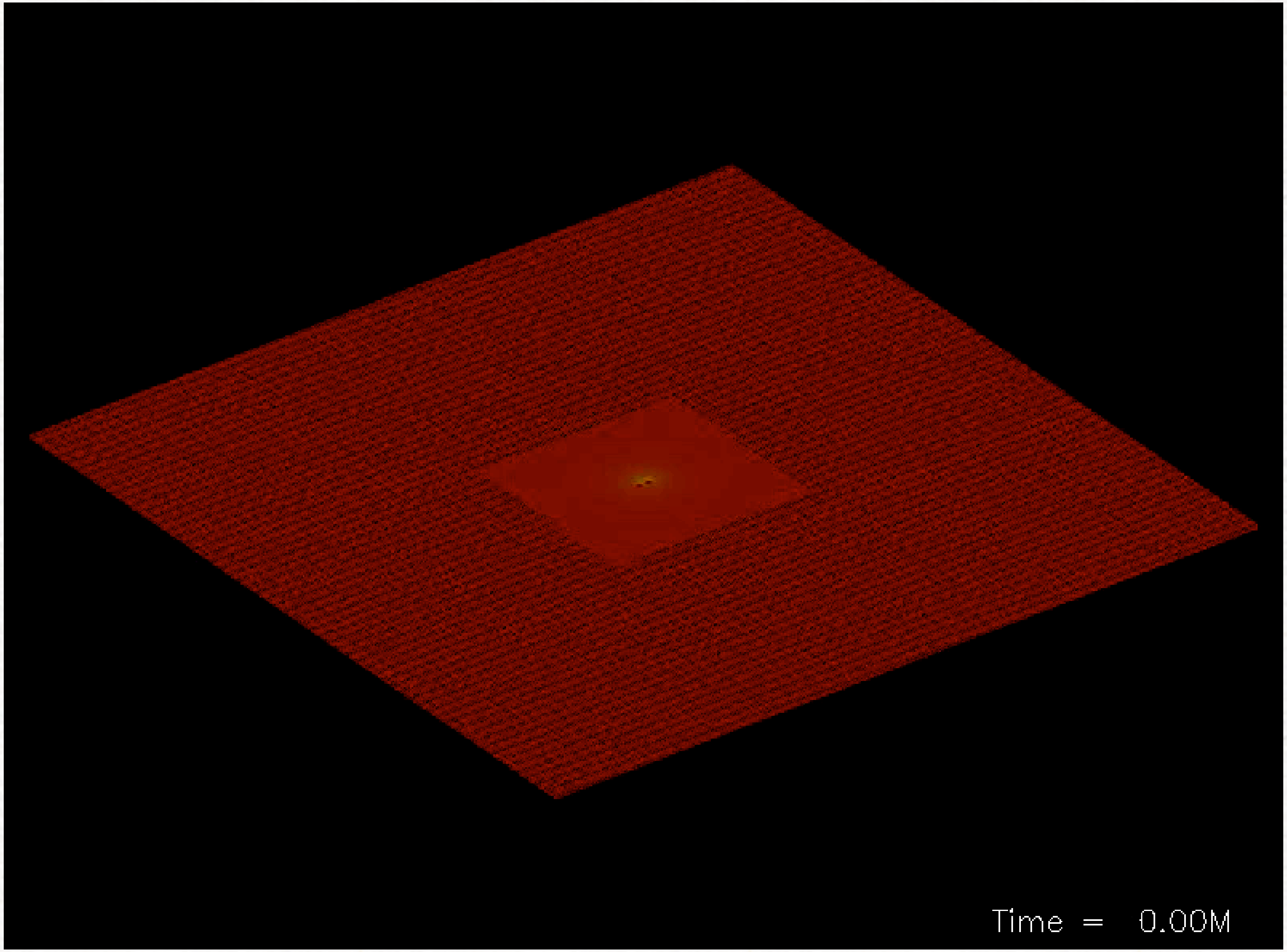


An interesting example



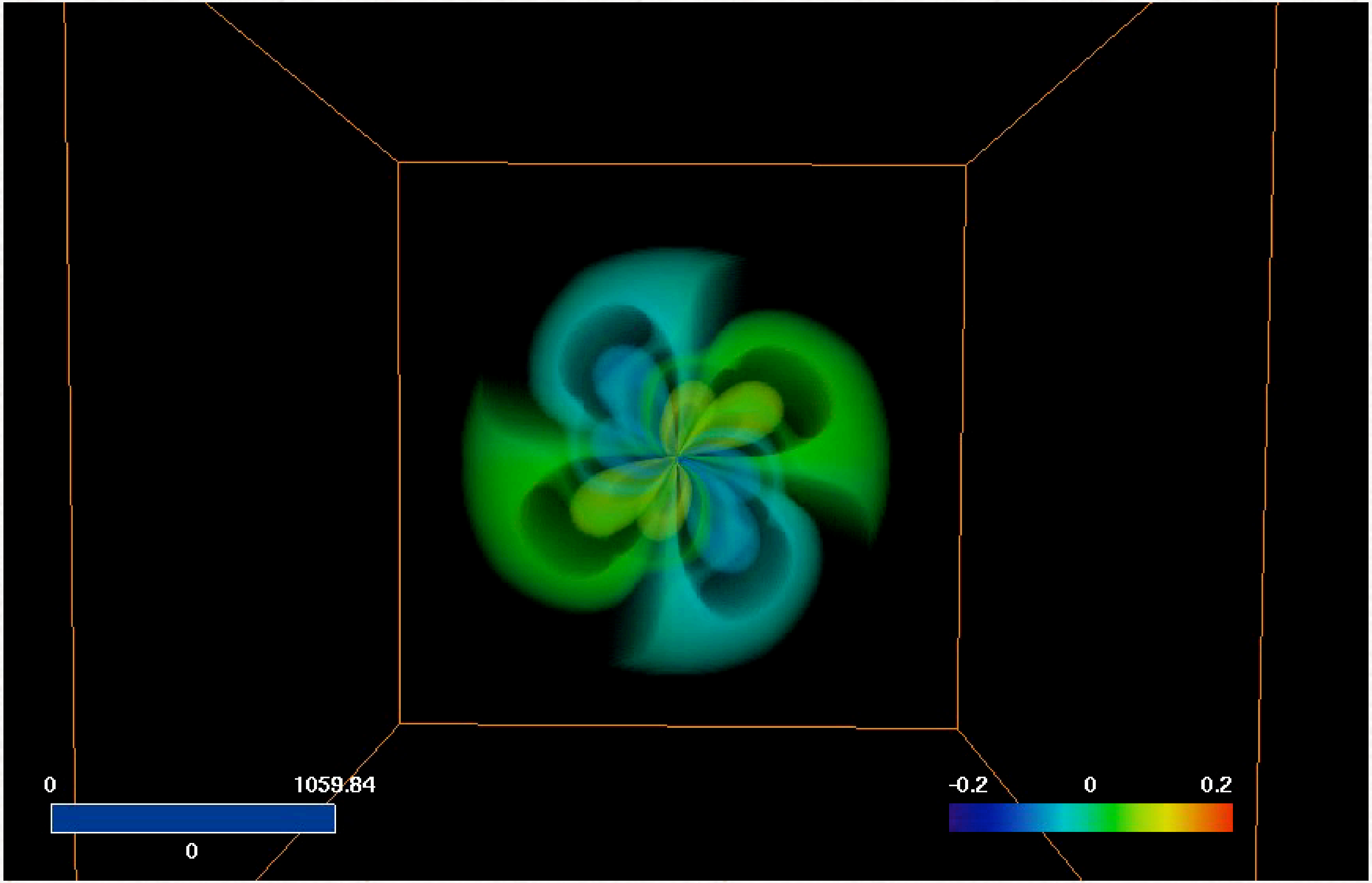
Hereafter I will concentrate on a binary system of black holes with equal masses but unequal spin (either in direction or modulus) set in quasi-circular orbit

Dynamics of the horizons



Animation by C. Riesswig

Waveforms: ψ_4



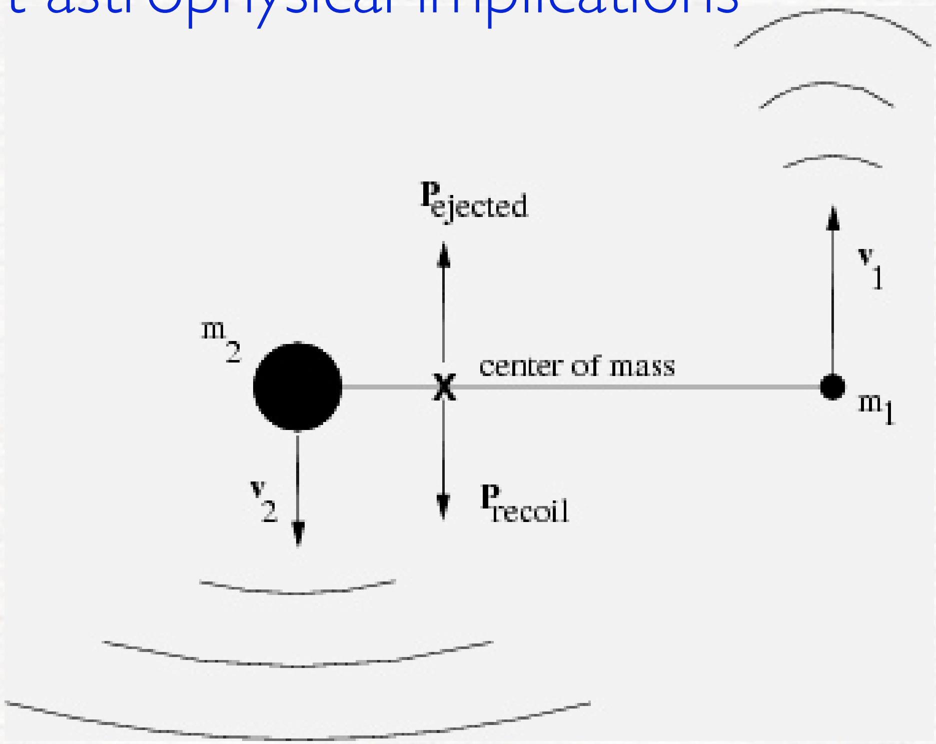
Animation by R. Kaehler

Astrophysical impact

Is there more physics and astrophysics that can be studied in this system besides the waveforms?

YES! The asymmetry in the system will lead to a nonzero recoil velocity ("bh kicks") with important astrophysical implications

The emission of gravitational waves is beamed and asymmetrical: momentum radiated at an angle will not be compensated by the momentum after 1/2 orbit ("garden hose")



We have studied the recoil velocities in a sequence of bhs with same mass but different spin-ratio

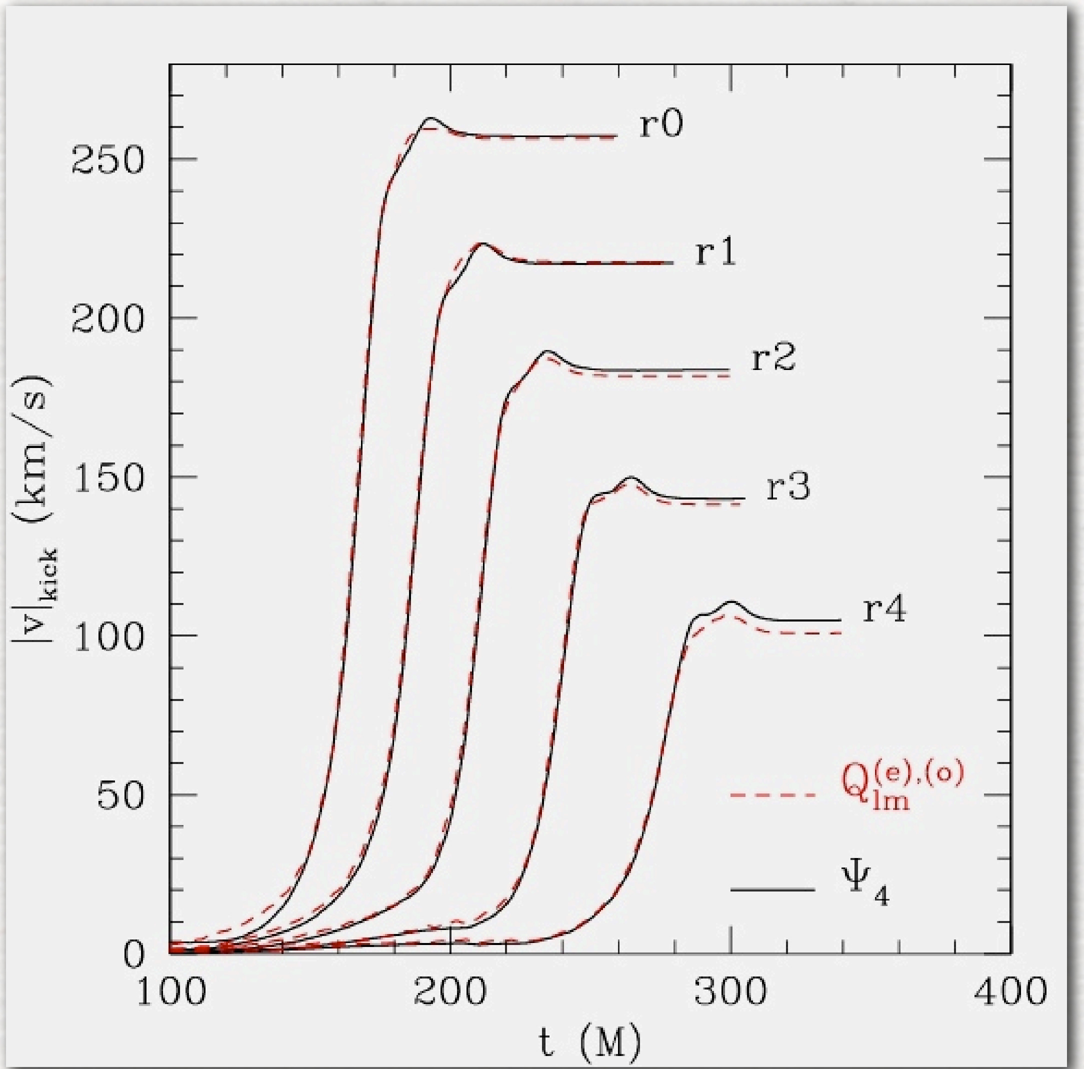
r0: ↑ ↓ (a_1/a_2=-4/4)

r1: ↑ ↓ (a_1/a_2=-3/4)

r2: ↑ ↓ (a_1/a_2=-2/4)

r3: ↑ ↓ (a_1/a_2=-1/4)

r4: ↑ . (a_1/a_2=-0/4)



The general behaviour

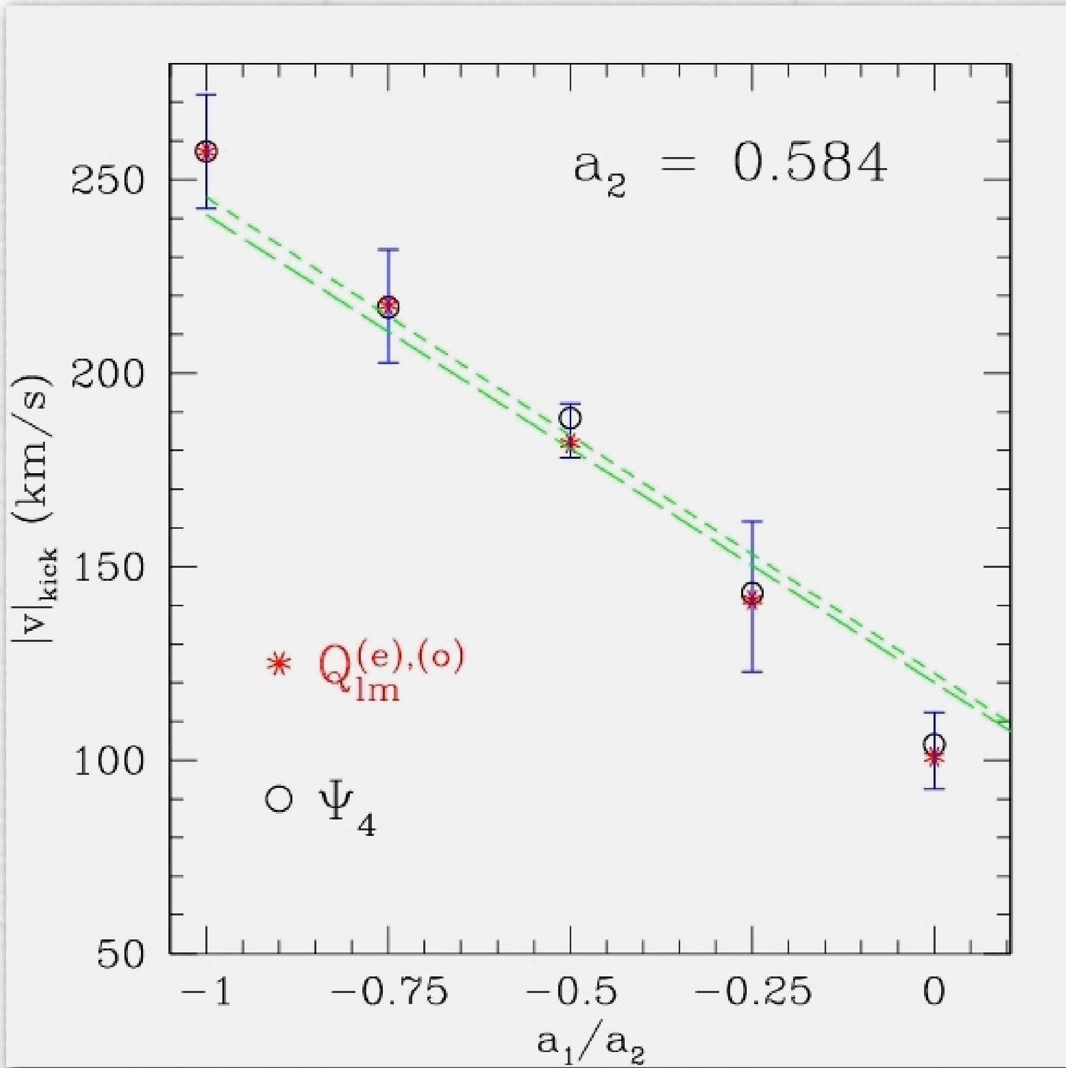
Post-Newtonian prediction

$$|v|_{kick} = c_1 \frac{q^2(1-q)}{(1+q)^5} + c_2 \frac{q^2 a_2(1-q a_1/a_2)}{(1+q)^5}$$

q: mass ratio ($q=1$ here)

The constants c_1, c_2 can be calculated only in full GR, but the PN prediction holds well.

We are working to extend this to a larger space of parameters



Summary

- The modelling of sources of gravitational waves in fully nonlinear regimes requires the numerical solution of the Einstein equations: ie numerical relativity techniques
- There is all the evidence that numerical relativity is living its "renaissance", and our simulations have never been as accurate and stable.
- Several groups in the world are now able to successfully compute the inspiral and merger of binary black holes in full generality (different masses and spins).
- Work on vacuum sources is already having an impact on related fields such as astrophysics (cf. recoil velocity calculations) or gw-data analysis (hybrid waveforms being injected in detectors pipelines).
- Much remains to be done and tomorrow will take a view of the progress when the right-hand-side is not zero: non-vacuum sources