Field Theory @ Finite Temperature & Phase Transitions

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Introduction

Motivation

The formalism used in conventional quantum field theory is suitable to describe observables (*e.g.* cross-sections) measured in empty space-time, as particle interactions in an accelerator. However, in the early stages of the universe/heavy ions collisions, at high temperature/density, the environment had a non-negligible matter and radiation density, making the hypotheses of conventional field theories impracticable. For that reason, under those circumstances, the methods of conventional field theories are no longer in use, and should be replaced by others, closer to thermodynamics, where the background state is a thermal bath. This field has been called field theory at finite temperature/density and it is extremely useful to study all phenomena which happened in the early universe: phase transitions, inflationary cosmology, ..

We shall give some definitions borrowed from thermodynamics and statistical mechanics.

- The microcanonical ensemble is used to describe an isolated system with fixed energy E, particle number N and volume V.
- The canonical ensemble describes a system in contact with a heat reservoir at temperature T: the energy can be exchanged between them and T, N and V are fixed.
- Finally, in the grand canonical ensemble the system can exchange energy and particles with the reservoir: T, V and the chemical potentials are fixed.

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Generating functionals

Consider now a dynamical system characterized by a hamiltonian [1] H and a set of conserved (mutually commuting) charges Q_A. The equilibrium state of the system at rest in the large volume V is described by the grand-canonical density operator ρ.

$$\rho = \exp(-\Phi)\exp\left\{-\sum_A \alpha_A Q_A - \beta H\right\}$$

where the Massieu function is defined as

$$\Phi = \log Tr \exp\left\{-\sum_A \alpha_A Q_A - \beta H\right\} \tag{1}$$

where α_A and β are Lagrange multipliers given by $\beta = T^{-1}$, $\alpha_A = -\beta\mu_A$, and μ_A are the chemical potentials.

[1] All operators will be considered in the Heisenberg picture.

One defines the grand canonical average of an arbitrary operator $\mathcal{O}$, as

$$\langle \mathcal{O} \rangle \equiv Tr(\mathcal{O}\rho) \tag{2}$$

satisfying the property $\langle \mathbf{1} \rangle = 1$

Some definitions

$$q_A = \frac{1}{V}\langle Q_A \rangle = -\frac{1}{V}\frac{\partial \Phi}{\partial \alpha_A} \tag{3}$$

$$E = \frac{1}{V}\langle H \rangle = -\frac{1}{V}\frac{\partial \Phi}{\partial \beta} \tag{4}$$

$$F = -P = -\frac{1}{\beta V}\Phi \tag{5}$$

$$S = -\frac{1}{V}\langle \log \rho \rangle = \beta\left[E - F - \sum_A \mu_A q_A\right] \tag{6}$$

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We will start considering the case of a real scalar field $\phi(x)$, carrying no charges ($\mu_A = 0$), with hamiltonian H, *i.e.*

$$\phi(x) = e^{itH}\phi(0, \vec{x})e^{-itH}$$

where the time $x^0 = t$ is analytically continued to the complex plane.

We define the thermal Green function as the grand canonical average of the ordered product of the n field operators

$$G^{(C)}(x_1, \ldots, x_n) \equiv \langle T_C \phi(x_1), \ldots, \phi(x_n)\rangle$$

where the T_C ordering means that fields should be ordered along the path C in the complex t-plane. For instance the product of two fields is defined as,

$$T_C\phi(x)\phi(y) = \theta_C(x^0 - y^0)\phi(x)\phi(y) + \theta_C(y^0 - x^0)\phi(y)\phi(x)$$

If we parameterize C as $t = z(\tau)$, where τ is a real parameter, T_C ordering means standard ordering along τ. Therefore the step and delta functions can be given as $\theta_C(t) = \theta(\tau)$, $\delta_C(t) = (\partial z/\partial\tau)^{-1}\,\delta(\tau)$.

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The rules of the functional formalism can be applied as usual, with the prescription $\delta j(y)/\delta j(x) = \delta_C(x^0 - y^0)\delta^{(3)}(\vec{x} - \vec{y})$

The generating functional $Z^\beta[j]$ for the full Green functions,

$$\begin{aligned} Z^\beta[j] &= \sum_{n=0}^{\infty} \frac{i^n}{n!} \int_C d^4x_1 \dots d^4x_n j(x_1) \dots j(x_n) G^{(C)}(\vec{x}) \\ &= \left\langle T_C \exp\left\{ i \int_C d^4x j(x)\phi(x) \right\} \right\rangle \end{aligned}$$

which is normalized to $Z^\beta[0] = \langle \mathbf{1} \rangle = 1$.

The generating functional for connected Green functions $W^\beta[j]$ is defined as

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The generating functional for 1PI Green functions $\Gamma^\beta[\overline{\phi}]$ is the Legendre transformation,

$$\Gamma^\beta[\overline{\phi}] = W^\beta[j] - \int_C d^4x \frac{\delta W^\beta[j]}{\delta j(x)} j(x)$$

where the current $j(x)$ is eliminated in favor of the classical field $\overline{\phi}(x)$ as

$$\overline{\phi}(x) = \delta W^\beta[j]/\delta j(x)$$

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It follows that $\delta\Gamma^\beta[\overline{\phi}]/\delta\overline{\phi}(x) = -j(x)$, and $\overline{\phi}(x) = \langle\phi(x)\rangle$ is the grand canonical average of the field $\phi(x)$.
As in field theory at zero temperature, in a translationally invariant theory $\overline{\phi}(x) = \phi_c$ is a constant. In this case we can define the effective potential at finite temperature as,

Effective Potential

$$\Gamma^\beta[\phi_c] = -\int d^4x V^\beta_{\rm eff}(\phi_c)$$

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Scalar fields

Not all the contours are allowed if we require Green functions to be analytic with respect to t.

$$G^{(C)}(x-y) = \theta_C(x^0-y^0)G_+(x-y) + \theta_C(y^0-x^0)G_-(x-y)$$

where

$$G_+(x-y) = \langle \phi(x)\phi(y) \rangle, \quad G_-(x-y) = G_+(y-x)$$

Now, take the complete set of states $|n\rangle$ with eigenvalues E_n: $H|n\rangle = E_n|n\rangle$. One can readily compute $G_+(x^0-y^0)$ at the point $\vec{x} = \vec{y} = 0$

$$e^{-\Phi} \sum_{m,n} |\langle m|\phi(0)|n\rangle|^2 \, e^{-iE_n(x^0-y^0)} e^{iE_m(x^0-y^0+i\beta)}$$

so that the convergence of the sum implies that $-\beta \leq Im(x^0-y^0) \leq 0$ which requires $\theta_C(x^0-y^0) = 0$ for $Im(x^0-y^0) > 0$.

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The similar property for the convergence of $G_-(x^0 - y^0)$ is that $0 \leq Im(x^0 - y^0) \leq \beta$, which requires $\theta_C(y^0 - x^0) = 0$ for $Im(x^0 - y^0) < 0$.

The final condition for the convergence of the complete Green function on the strip

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is that we define the function $\theta_C(t)$ such that $\theta_C(t) = 0 \;\text{ for }\; Im(t) > 0.$

C must be such that *a point moving along it has a monotonically decreasing or constant imaginary part*

A very important periodicity relation affecting Green functions can be easily deduced from the very definition of $G_+(x)$ and $G_-(x)$

Kubo-Martin-Schwinger (KMS) relation

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We can now compute the two-point Green function for a free scalar field $[\omega_p = \sqrt{\vec{p}^{\,2} + m^2}]$

$$\phi(x) = \int \frac{d^3p}{(2\pi)^{3/2}(2\omega_p)^{1/2}} \left[a(p)e^{-ipx} + a^\dagger(p)e^{ipx}\right]$$

which satisfies $\left[\partial^\mu\partial_\mu + m^2\right] G^{(C)}(x-y) = -i\delta_C(x-y)$ and the equal time commutation relation,

$$\left[\phi(t,\vec{x}), \dot{\phi}(t,\vec{y})\right] = i\delta^{(3)}(\vec{x}-\vec{y})$$

One easily obtains the commutation relation for creation and annihilation operators,

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Defining the Hamiltonian of the field as

$$H = \int \frac{d^3p}{(2\pi)^3} \omega_p a^\dagger(p) a(p)$$

one can obtain the thermodynamical averages,

$$\langle a^\dagger(p) a(k) \rangle = n_B(\omega_p) \delta^{(3)}(\vec{p} - \vec{k})$$
$$\langle a(p) a^\dagger(k) \rangle = [1 + n_B(\omega_p)] \delta^{(3)}(\vec{p} - \vec{k})$$

Bose distribution function

$$n_B(\omega) = \frac{1}{e^{\beta\omega} - 1}$$

Two-point Green function

$$G^{(C)}(x) = \int \frac{d^4p}{(2\pi)^4}\rho(p)e^{-ipx}\left[\theta_C(x^0) + n_B(p^0)\right]$$

where the function $\rho(p)$ is defined by

Spectral function

$\rho(p) = 2\pi[\theta(p^0) - \theta(-p^0)]\delta(p^2 - m^2).$

Now the particular value of the Green function depends on the chosen contour C. We will show later on two particular contours giving rise to the so-called imaginary and real time formalisms. Before coming to them we will describe how the previous formulae apply to the case of fermion fields.

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Fermion fields

For fermion fields the T_C decomposition

$$\begin{aligned} T_C\psi_\alpha(x)\bar{\psi}_\beta(y) &= \theta_C(x^0-y^0\psi_\alpha(x)\bar{\psi}_\beta(y)) \\ &- \theta_C(y^0-x^0)\bar{\psi}_\beta(y)\psi_\alpha(x) \end{aligned}$$

and the Green function

$$S^{(C)}_{\alpha\beta}(x-y)=\theta_C(x^0-y^0)S^+_{\alpha\beta}-\theta_C(y^0-x^0)S^-_{\alpha\beta}$$

which satisfy the

Kubo-Martin-Schwinger relation

$$S^+_{\alpha\beta}(t-i\beta,\vec{x})=-S^-_{\alpha\beta}(t,\vec{x})$$

Relations for the Green function of fermions

It satisfies the Dirac equation

$$(i\gamma\cdot\partial - m)_{\alpha\sigma}\, S^{(C)}_{\sigma\beta}(x-y) = i\delta_C(x-y)\delta_{\alpha\beta}$$

One can define a Green function $S^{(C)}$ as

$$S^{(C)}_{\alpha\beta}(x-y) \equiv (i\gamma\cdot\partial + m)_{\alpha\beta} S^{(C)}(x-y)$$

where $S^{(C)}(x-y)$ satisfies the Klein-Gordon propagator equation. One can obtain for $S^{(C)}$

$$S^{(C)}(x-y) = \int \frac{d^4p}{(2\pi)^4}\rho(p)e^{-ip(x-y)}\left[\theta_C(x^0-y^0) - n_F(p^0)\right]$$

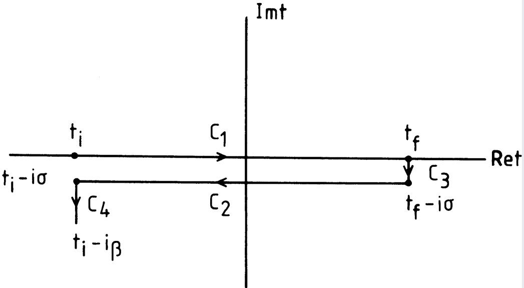
$n_F(\omega)$ is the Fermi distribution function

$$n_F(\omega) = \frac{1}{e^{\beta\omega}+1}.$$

Imaginary time formalism

The calculation of the propagators in the previous sections depends on the chosen path C going from an initial arbitrary time t to $t - i\beta$, provided by the Kubo-Martin-Schwinger periodicity properties of Green functions. The simplest path is to take a straight line along the imaginary axis $t = -i\tau$. It is called Matsubara contour, since Matsubara was the first to set up a perturbation theory based upon this contour. In that case $\delta_C(t) = i\delta(\tau)$.

Contours



The two-point Green functions for scalar and fermion fields

$$G(\tau,\vec{x})_{B,F} = \int \frac{d^4p}{(2\pi)^4}\rho(p)e^{i\vec{p}\vec{x}}e^{-\tau p^0}\left[\theta(\tau)\pm n_{B,F}(p^0)\right]$$

The Green function can be decomposed as
$G(\tau,\vec{x}) = G_+(\tau,\vec{x})\theta(\tau) \pm G_-(\tau,\vec{x})\theta(-\tau)$

KMS relation $\Rightarrow$ (Anti)Periodicity condition

$$G(\tau+\beta) = \pm G(\tau) \text{ for } -\beta \le \tau \le 0,$$
$$G(\tau-\beta) = \pm G(\tau) \text{ for } 0 \le \tau \le \beta$$

It follows that the Fourier transform

$$\widetilde{G}(\omega_n,\vec{p}) = \int_{\alpha-\beta}^{\alpha} d\tau \int d^3x e^{i\omega_n\tau - i\vec{x}\vec{p}}G(\tau,\vec{x})$$

(where $0 \le \alpha \le \beta$) is independent of α and the discrete frequencies satisfy the relation $e^{i\omega_n\beta} = \pm 1$, *i.e.* $\omega_n = 2n\pi\beta^{-1}$ for bosons, and $\omega_n = (2n+1)\pi\beta^{-1}$ for fermions.

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$$G(\tau,\vec{x})_{B,F}=\int\frac{d^4p}{(2\pi)^4}\rho(p)e^{i\vec{p}\vec{x}}e^{-\tau p^0}\left[\theta(\tau)\pm n_{B,F}(p^0)\right]$$

The Green function can be decomposed as
$G(\tau,\vec{x})=G_+(\tau,\vec{x})\theta(\tau)\pm G_-(\tau,\vec{x})\theta(-\tau)$

KMS relation $\Rightarrow$ (Anti)Periodicity condition

$$G(\tau+\beta)=\pm G(\tau)\ \text{for}\ -\beta\le\tau\le 0,$$
$$G(\tau-\beta)=\pm G(\tau)\ \text{for}\ \ 0\le\tau\le\beta$$

It follows that the Fourier transform

$$\widetilde{G}(\omega_n,\vec{p})=\int_{\alpha-\beta}^{\alpha}d\tau\int d^3x e^{i\omega_n\tau-i\vec{x}\vec{p}}G(\tau,\vec{x})$$

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\begin{aligned}
\text{Boson propagator} &: \quad \frac{i}{p^2 - m^2}; \ \ p^\mu = [2ni\pi\beta^{-1}, \vec{p}\ ] \\
\text{Fermion propagator} &: \quad \frac{i}{\gamma \cdot p - m}; \ \ p^\mu = [(2n+1)i\pi\beta^{-1}, \vec{p}\ ] \\
\text{Loop integral} &: \quad \frac{i}{\beta}\sum_{n=-\infty}^{\infty}\int \frac{d^3p}{(2\pi)^3} \\
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There is a standard trick to perform infinite summations. For the case of bosons [fermions] we can have frequency sums as,

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\frac{1}{\beta}\sum_{n=-\infty}^{\infty} f(p^0 = i\omega_n)
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where the contour γ encircles anticlockwise all the previous poles of the imaginary axis. The contour γ can be deformed to a new contour consisting in two straight lines: the first one starting at $-i\infty+\epsilon$ and going to $i\infty+\epsilon$, and the second one starting at $i\infty-\epsilon$ and ending at $-i\infty-\epsilon$.

Rearranging the exponentials in the hyperbolic cotangent one can write the previous expression as

$$\begin{aligned}&\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} dz\frac{1}{2}[f(z)+f(-z)]\\ +\ &\frac{1}{2\pi i}\int_{-i\infty+\epsilon}^{i\infty+\epsilon} dz[f(z)+f(-z)]\frac{1}{e^{\beta z}-1}\end{aligned}$$

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The contour of the second integral can be deformed to a contour C which encircles clockwise all singularities of the functions $f(z)$ and $f(-z)$ in the right half plane. Similar manipulations for the case of fermions lead to the general identity

Infinity sum identity

$$\frac{1}{\beta}\sum_{n=-\infty}^{\infty} f(p^0 = i\omega_n) = \int_{-i\infty}^{i\infty}\frac{dz}{4\pi i}[f(z)+f(-z)]$$
$$\pm\int_C \frac{dz}{2\pi i} n_{B,F}(z)[f(z)+f(-z)]$$

The frequency sum naturally separates into a T independent piece, which should coincide with the similar quantity computed in field theory at zero temperature, and a T dependent piece which vanishes in the limit $T \to 0$, *i.e.* $\beta \to \infty$.

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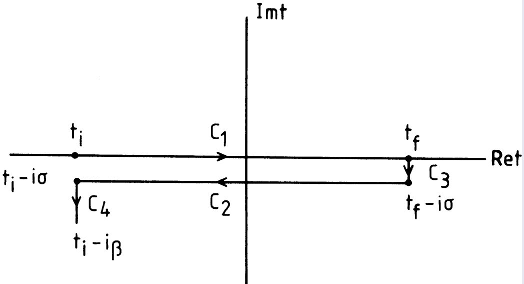
Real time formalism

The obvious disadvantage of the imaginary time formalism is to compute Green functions along imaginary time, so that going to the real time has to be done through a process of analytic continuation. The family of real time contours is depicted in the figure where the contour C is $C = C_1 \bigcup C_2 \bigcup C_3 \bigcup C_4$ and C_1 goes from the initial time t_i to the final time t_f, C_3 from t_f to $t_f - i\sigma$, with $0 \leq \sigma \leq \beta$, C_2 from $t_f - i\sigma$ to $t_i - i\sigma$, and C_4 from $t_i - i\sigma$ to $t_i - i\beta$.

Contours



Different choices of σ lead to an equivalence class of quantum field theories at finite temperature. For instance the choice $\sigma = 0$ leads to the Keldysh perturbation expansion, while the choice $\sigma = \beta/2$ is the preferred one to compute Green functions.

One can prove that the contribution from the contours C_3 and C_4 can be neglected. Therefore, for the propagator between x^0 and y^0 there are four possibilities depending on whether they are on C_1 or C_2. Correspondingly, there are four propagators which are labeled by (11), (12), (21) and (22).

Making the choice $\sigma = \beta/2$, for scalar fields

$$G(p) \equiv \begin{pmatrix} G^{(11)}(p) & G^{(12)}(p) \\ G^{(21)}(p) & G^{(22)}(p) \end{pmatrix} \tag{7}$$

$$\begin{aligned} G^{(11)}(p) &= \Delta(p) + 2\pi n_B(\omega_p)\delta(p^2 - m^2) \\ G^{(12)} &= 2\pi e^{\beta\omega_p/2} n_B(\omega_p)\delta(p^2 - m^2) \\ G^{(22)}(p) &= G^{(11)*}, \quad G^{(21)} = G^{(12)} \end{aligned}$$

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Real time rules

The main feature of the real time formalism is that the propagators come in two terms: one which is the same as in the zero temperature field theory, and a second one where all the temperature dependence is contained. However the propagators (12), (21) and (22) are unphysical since one of their time arguments has an imaginary component. The only physical propagator is the (11) component.

The Feynman rules in the real time formalism are very similar to those in the zero temperature field theory. In fact all diagrams have the same topology as in the zero temperature field theory and the same symmetry factors. However, associated to every field there are two possible vertices, 1 and 2, and four possible propagators, (11), (12), (21) and (22) connecting them. All of them have to be considered for the consistency of the theory. In the Feynman rules, type 2 vertices are hermitian conjugate with respect to type 1 vertices. The golden rule is that: *physical legs must always be attached to type 1 vertices*.

Effective potential

We will construct the (one-loop) effective potential at finite temperature, using all the tools provided in the previous sections.

As we will see the effective potential at finite temperature contains the effective potential at zero temperature that requires regularization.

The usefulness of this construction is addressed to the theory of phase transitions at finite temperature. The latter being essential for the understanding of phenomena as: inflation, baryon asymmetry generation, quark-gluon plasma transition in QCD,...

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Scalar fields

We will consider here the simplest model of one self-interacting scalar fields described by the lagrangian

$$\mathcal{L} = \frac{1}{2}\partial^\mu \phi \partial_\mu \phi - V_0(\phi)$$

$$V_0 = \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4$$

We will compute the diagrams in

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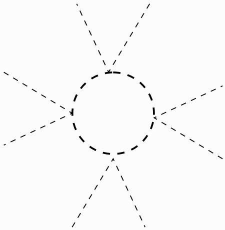

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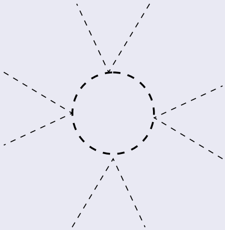

Technical

The n-th diagram has n propagators, n vertices and $2n$ external legs. The n propagators will contribute a factor of $i^n(p^2 - m^2 + i\epsilon)^{-n}$. The external lines contribute a factor of ϕ_c^{2n} and each vertex a factor of $-i\lambda/2$, where the factor $1/2$ comes from the fact that interchanging the 2 external lines of the vertex does not change the diagram. There is a global symmetry factor $\frac{1}{2n}$, where $\frac{1}{n}$ comes from the symmetry of the diagram under the discrete group of rotations $\mathbf{Z}_n$ and $\frac{1}{2}$ from the symmetry of the diagram under reflection. Finally there is an integration over the loop momentum and an extra global factor of i from the definition of the generating functional.

Using the Feynman rules the CW effective potential

$$V_1(\phi_c) = \frac{1}{2}\int \frac{d^4 p}{(2\pi)^4} \log\left[p^2 + m^2(\phi_c)\right]$$

$$m^2(\phi_c) = m^2 + \frac{1}{2}\lambda\phi_c^2 = \frac{d^2 V_0(\phi_c)}{d\phi_c^2}$$

translates into,

$$V_1^\beta(\phi_c) = \frac{1}{2\beta}\sum_{n=-\infty}^{\infty}\int \frac{d^3 p}{(2\pi)^3} \log(\omega_n^2 + \omega^2)$$

where ω_n are the bosonic Matsubara frequencies and

$$\omega^2 = \vec{p}^{\;2} + m^2(\phi_c)$$

The sum over n diverges, but the infinite part does not depend on ϕ_c. The finite part contains the ϕ_c dependence

$$V_1^\beta(\phi_c) = \int \frac{d^3 p}{(2\pi)^3}\left[\frac{\omega}{2} + \frac{1}{\beta}\log\left(1 - e^{-\beta\omega}\right)\right]$$

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Calculation of the infinite sum

Define,

$$v(\omega) = \sum_{n=-\infty}^{\infty} \log(\omega_n^2 + \omega^2)$$

then,

$$\frac{\partial v}{\partial \omega} = \sum_{n=-\infty}^{\infty} \frac{2\omega}{\omega_n^2 + \omega^2}$$

Using the identity,

$$\begin{aligned} f(y) = \sum_{n=1}^{\infty} \frac{y}{y^2 + n^2} &= -\frac{1}{2y} + \frac{1}{2}\pi \coth \pi y \\ &= -\frac{1}{2y} + \frac{\pi}{2} + \pi \frac{e^{-2\pi y}}{1 - e^{-2\pi y}} \end{aligned}$$

with $y = \beta\omega/2\pi$ we obtain,

$$\frac{\partial v}{\partial \omega} = 2\beta \left[\frac{1}{2} + \frac{e^{-\beta\omega}}{1 - e^{-\beta\omega}} \right] \tag{8}$$

One can prove that the first integral is the one-loop effective potential at zero temperature.

Effective potential at zero temperature

$$\frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\omega = \frac{1}{2}\int \frac{d^4p}{(2\pi)^4}\log[p^2+m^2(\phi_c)]$$

Thermal correction

$$\frac{1}{\beta}\int \frac{d^3p}{(2\pi)^3}\log\left(1-e^{-\beta\omega}\right) = \frac{1}{2\pi^2\beta^4}J_B[m^2(\phi_c)\beta^2]$$

where the thermal bosonic function J_B is defined as,

$$J_B[m^2\beta^2] = \int_0^\infty dx\; x^2 \log\left[1-e^{-\sqrt{x^2+\beta^2m^2}}\right]$$

There is a very simple way of computing the effective potential: it consists in *computing its derivative* **in the shifted theory** *and then integrating*! In fact the derivative of the effective potential is described diagrammatically by the tadpole diagram.

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The thermal bosonic effective potential admits a high-temperature expansion which will be very useful for practical applications. It is given by

Finite temperature expansion

$$J_B(m^2/T^2) = -\frac{\pi^4}{45} + \frac{\pi^2}{12}\frac{m^2}{T^2} - \frac{\pi}{6}\left(\frac{m^2}{T^2}\right)^{3/2}$$
$$-\frac{1}{32}\frac{m^4}{T^4}\log\frac{m^2}{a_b T^2}$$
$$-2\pi^{7/2}\sum_{\ell=1}^{\infty}(-1)^{\ell}\frac{\zeta(2\ell+1)}{(\ell+1)!}\Gamma\left(\ell+\frac{1}{2}\right)\left(\frac{m^2}{4\pi^2 T^2}\right)^{\ell+2}$$

where $a_b = 16\pi^2 \exp(3/2 - 2\gamma_E)$ ($\log a_b = 5.4076$) and ζ is the Riemann ζ-function.

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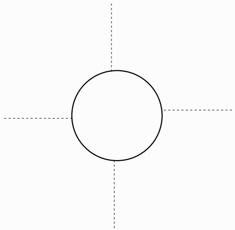

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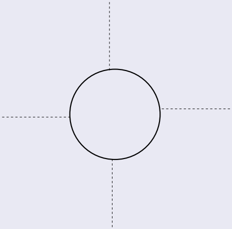

Technical

Diagrams with an odd number of vertices are zero because of the γ-matrices property: $tr(\gamma^{\mu_1} \cdots \gamma^{\mu_{2n+1}}) = 0$. The diagram with $2n$ vertices has $2n$ fermionic propagators. The propagators yield a factor

$$Tr_s[i^{2n}(\gamma \cdot p)^{2n}(p^2 + i\epsilon)^{-2n}]$$

where Tr_s refers to spinor indices. The vertices contribute as

$$Tr[-i^{2n} M_f(\phi_c)^{2n}]$$

where Tr runs over the different fermionic fields. There is also a combinatorial factor $\frac{1}{2n}$ (from the cyclic and anticyclic symmetry of diagrams) and an overall -1 coming from the fermions loop. The factor $Tr_s\mathbf{1}$ just counts the number of degrees of freedom of the fermions. It is equal to 4 if Dirac fermions are used, and 2 if Weyl fermions (and σ-matrices) are present. So we will write, $Tr_s\mathbf{1} = 2\lambda$ where $\lambda = 1$ ($\lambda = 2$) for Weyl (Dirac) fermions.

Using the Feynman rules the CW effective potential

$$V_1 = -2\lambda \frac{1}{2} Tr \int \frac{d^4 p}{(2\pi)^4} \log\left[p^2 + M_f^2(\phi_c)\right]$$

one gets

$$V_1^\beta(\phi_c) = -\frac{2\lambda}{2\beta} \sum_{n=-\infty}^{\infty} \int \frac{d^3 p}{(2\pi)^3} \log(\omega_n^2 + \omega^2)$$

where ω_n are the fermionic Matsubara frequencies and

$$\omega^2 = \vec{p}^{\ 2} + M_f^2 .$$

The infinite sum of over n gives

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Using the Feynman rules the CW effective potential

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Infinite summation: 1

Let $f(y)$ be given by

$$f(y) = \sum_{n=1}^{\infty} \frac{y}{y^2 + n^2}$$

then

$$\begin{aligned}
\sum_{m=2,4,\ldots} \frac{y}{y^2+m^2} &= \sum_{n=1}^{\infty} \frac{y}{y^2+4n^2} = \frac{1}{2} f\left(\frac{y}{2}\right) \\
\sum_{m=1,3,\ldots} \frac{y}{y^2+m^2} &= f(y) - \frac{1}{2} f\left(\frac{y}{2}\right)
\end{aligned}$$

and using the definition of $f(y)$ we get,

$$\sum_{m=1,3,\ldots} \frac{y}{y^2+m^2} = \frac{\pi}{4} - \frac{\pi}{2} \frac{1}{e^{\pi y}+1}$$

The function $v(\omega)$ in this case can be written as,

Infinite summation: 2

$$v(\omega) = 2 \sum_{n=1,3,\ldots} \log\left[\frac{\pi^2 n^2}{\beta^2} + \omega^2\right]$$

and its derivative,

$$\frac{\partial v}{\partial \omega} = \frac{4\beta}{\pi} \sum_{1,3,\ldots} \frac{y}{y^2 + n^2}$$

where $y = \beta\omega/\pi$. Then using the previous equations we get

$$\frac{\partial v}{\partial \omega} = 2\beta\left[\frac{1}{2} - \frac{1}{1 + e^{\beta\omega}}\right]$$

and, after integration with respect to ω,

$$v(\omega) = 2\beta\left[\frac{w}{2} + \frac{1}{\beta}\log\left(1 + e^{-\beta\omega}\right)\right] + \omega\mathrm{-independent\quad terms}$$

The first integral in

$$V_1^\beta(\phi_c) = -2\lambda \int \frac{d^3p}{(2\pi)^3}\left[\frac{\omega}{2} + \frac{1}{\beta}\log\left(1+e^{-\beta\omega}\right)\right]$$

leads to the one-loop effective potential at zero temperature

Effective potential at zero temperature

$$V_1 = -2\lambda\frac{1}{2}Tr\int\frac{d^4p}{(2\pi)^4}\log\left[p^2+M_f^2(\phi_c)\right]$$

Thermal correction

$$-2\lambda\frac{1}{\beta}\int\frac{d^3p}{(2\pi)^3}\log\left(1+e^{-\beta\omega}\right) = -2\lambda\frac{1}{2\pi^2\beta^4}J_F[M_f^2(\phi_c)\beta^2]$$

where the thermal fermionic function J_F is defined as,

$$J_F[m^2\beta^2] = \int_0^\infty dx\; x^2\log\left[1+e^{-\sqrt{x^2+\beta^2m^2}}\right]$$

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The thermal fermionic effective potential admits a high-temperature expansion which will be very useful for practical applications. It is given by

Finite temperature expansion

$$J_F(m^2/T^2) = \frac{7\pi^4}{360} - \frac{\pi^2}{24}\frac{m^2}{T^2} - \frac{1}{32}\frac{m^4}{T^4}\log\frac{m^2}{a_f T^2}$$
$$-\frac{\pi^{7/2}}{4}\sum_{\ell=1}^{\infty}(-1)^\ell \frac{\zeta(2\ell+1)}{(\ell+1)!}\left(1-2^{-2\ell-1}\right)\Gamma\left(\ell+\frac{1}{2}\right)\left(\frac{m^2}{\pi^2 T^2}\right)^{\ell+2}$$

where $a_f = \pi^2 \exp(3/2 - 2\gamma_E)$ ($\log a_f = 2.6351$) and ζ is the Riemann ζ-function.

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The Standard Model

The spin-zero fields of the Standard Model are described by the $SU(2)$ doublet,

$$\Phi = \begin{pmatrix} \chi_1 + i\chi_2 \\ \dfrac{\phi_c + h + i\chi_3}{\sqrt{2}} \end{pmatrix}$$

where ϕ_c is the real constant background, h the Higgs field, and χ_a (a=1,2,3) are the three Goldstone bosons.

The tree level potential reads, in terms of the background field, as

$$V_0(\phi_c) = -\frac{m^2}{2}\phi_c^2 + \frac{\lambda}{4}\phi_c^4$$

with positive λ and m^2, and the tree level minimum corresponding to

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The spin-zero field dependent masses are

$$\begin{aligned} m_h^2(\phi_c) &= 3\lambda\phi_c^2 - m^2 \\ m_\chi^2(\phi_c) &= \lambda\phi_c^2 - m^2 \end{aligned}$$

so that $m_h^2(v) = 2\lambda v^2 = 2m^2$ and $m_\chi^2(v) = 0$.

The gauge bosons contributing to the one-loop effective potential are $W^\pm$ and Z, with tree level field dependent masses,

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Finally, the only fermion which can give a significant contribution to the one loop effective potential is the top quark, with a field-dependent mass

$$m_t^2(\phi_c) = \frac{h_t^2}{2}\phi_c^2$$

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In the $\overline{\mathrm{MS}}$ renormalization scheme one easily arrives to the finite effective potential provided by

Finite effective potential at zero temperature

$$V(\phi_c) = V_0(\phi_c) + \frac{1}{64\pi^2}\sum_i n_i m_i^4(\phi_c)\left[\log\frac{m_i^2(\phi_c)}{\mu^2} - C_i\right]$$

where C_i are constants given by,

$$\begin{aligned} C_W = C_Z &= \frac{5}{6} \\ C_h = C_\chi = C_t &= \frac{3}{2} \end{aligned}$$

and n_i are the degrees of freedom

$$n_W = 6,\ n_Z = 3,\ n_h = 1,\ n_\chi = 3,\ n_t = -12$$

One can easily see that the finite-temperature part of the one-loop effective potential can be written as,

Thermal correction

$$\Delta V^{(1)}(\phi_c, T) = \frac{T^4}{2\pi^2} \sum_{i=W,Z,\chi,h} n_i J_B[m_i^2(\phi_c)/T^2]$$
$$+ \quad \frac{T^4}{2\pi^2} n_t J_F[m_t^2(\phi_c)/T^2]$$

where the thermal integrals J_B and J_F were previously defined.

Using now the high temperature expansions and the one loop effective potential at zero temperature, one can write the total potential as,

Standard Model effective potential

$$V(\phi_c, T) = D(T^2 - T_o^2)\phi_c^2 - ET\phi_c^3 + \frac{\lambda(T)}{4}\phi_c^4$$

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Coefficients of the polynomial

$$D = \frac{2m_W^2 + m_Z^2 + 2m_t^2}{8v^2}$$

$$E = \frac{2m_W^3 + m_Z^3}{4\pi v^3}$$

$$T_o^2 = \frac{m_h^2 - 8Bv^2}{4D}$$

$$B = \frac{3}{64\pi^2 v^4}\left(2m_W^4 + m_Z^4 - 4m_t^4\right)$$

$$\lambda(T) = \lambda - \frac{3}{16\pi^2 v^4}\left(2m_W^4 \log\frac{m_W^2}{A_B T^2} + m_Z^4 \log\frac{m_Z^2}{A_B T^2} - 4m_t^4 \log\frac{m_t^2}{A_F T^2}\right)$$

where $\log A_B = \log a_b - 3/2$ and $\log A_F = \log a_F - 3/2$

Cosmological phase transitions

Many cosmological applications of field theories are based on the theory of phase transitions at finite temperature. The main point here is that at finite temperature, the equilibrium value of the scalar field ϕ, $\langle\phi(T)\rangle$, does not correspond to the minimum of the effective potential $V_{\rm eff}^{T=0}(\phi)$, but to the minimum of the finite temperature effective potential $V_{\rm eff}^{\beta}(\phi)$. Thus, even if the minimum of $V_{\rm eff}^{T=0}(\phi)$ occurs at $\langle\phi\rangle = \sigma \neq 0$, very often, for sufficiently large temperatures, the minimum of $V_{\rm eff}^{\beta}(\phi)$ occurs at $\langle\phi(T)\rangle = 0$: this phenomenon is known as **symmetry restoration** at high temperature, and gives rise to the phase transition from $\phi(T) = 0$ to $\phi = \sigma$. It was discovered by Kirzhnits in the context of the electroweak theory (symmetry breaking between weak and electromagnetic interactions occurs when the universe cools down to a critical temperature $T_c \sim 10^2\ GeV$) and subsequently confirmed and developed by other authors.

Cosmological scenario

The cosmological scenario can be drawn as follows: In the theory of the hot big bang, the universe is initially at very high temperature and, depending on the function $V_{\rm eff}^{\beta}(\phi)$, it can be in the **symmetric phase** $\langle\phi(T)\rangle = 0$, *i.e.* $\phi = 0$ can be the stable absolute minimum. At some critical temperature T_c the minimum at $\phi = 0$ becomes metastable and the phase transition may proceed. The phase transition may be **first** or **second** order. First-order phase transitions have supercooled (out of equilibrium) symmetric states when the temperature decreases and are of use for baryogenesis purposes. Second-order phase transitions are used in the so-called new inflationary models. We will illustrate these kinds of phase transitions with very simple examples.

Second order phase transitions

We will illustrate the difference between **first** and **second** order phase transitions by considering first the simple example of a potential [2] described by the function,

$$V(\phi, T) = D(T^2 - T_o^2)\phi^2 + \frac{\lambda(T)}{4}\phi^4$$

where D, T_o^2 and λ are constant terms.

Zero temperature

At zero temperature, the potential has a negative mass-squared term, which indicates that the state $\phi = 0$ is unstable, and the energetically favored state corresponds to the minimum at $\phi(0) = \pm\sqrt{\frac{2D}{\lambda}}T_o$, where the symmetry $\phi \leftrightarrow -\phi$ of the original theory is spontaneously broken.

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The curvature of the finite temperature potential is now T-dependent,

$$m^2(\phi, T) = 3\lambda\phi^2 + 2D(T^2 - T_o^2)$$

and its stationary points, solutions to $dV(\phi, T)/d\phi = 0$,

Stationary points at finite temperature

$$\begin{aligned} \phi(T) &= 0 \\ &\text{and} \\ \phi(T) &= \sqrt{\frac{2D(T_o^2 - T^2)}{\lambda(T)}} \end{aligned}$$

Therefore the critical temperature is given by T_o. This phase transition is called of **second order**, because there is no barrier between the symmetric and broken phases. The phase transition may be achieved by a thermal fluctuation for a field located at the origin. Actually, when the broken phase is formed, the origin (symmetric phase) becomes a maximum.

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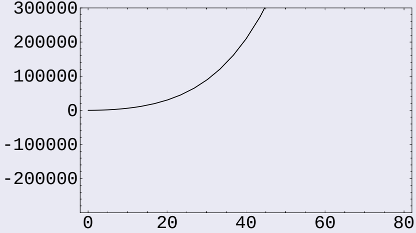
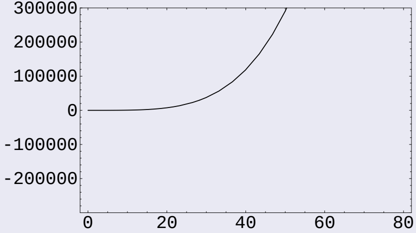
Cartoon of second order phase transition

- At $T > T_o$, $m^2(0, T) > 0$ and the origin $\phi = 0$ is a minimum



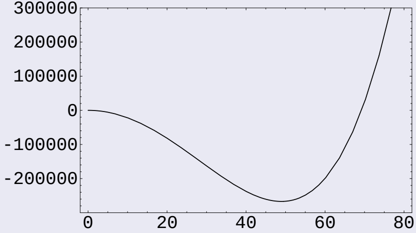
Cartoon of second order phase transition

- At $T = T_o$, $m^2(0, T_o) = 0$ and both solutions collapse at $\phi = 0$



Cartoon of second order phase transition

- At $T < T_o$, $m^2(0, T) < 0$ and the origin becomes a maximum. The solution $\phi(T) \neq 0$ does appear



First order phase transitions

In many interesting theories there is a **barrier** between the symmetric and broken phases. This is characteristic of **first order** phase transitions.

A typical example is provided by the potential

$$V(\phi, T) = D(T^2 - T_o^2)\phi^2 - ET\phi^3 + \frac{\lambda(T)}{4}\phi^4$$

where, as before, D, T_0 and E are T independent coefficients, and λ is a slowly varying T-dependent function. Notice the addition of the cubic term with coefficient E from bosonic fields

Critical temperatures

$$T_1^2 = \frac{8\lambda(T_1)DT_o^2}{8\lambda(T_1)D - 9E^2}$$

$$T_c^2 = \frac{\lambda(T_c)DT_o^2}{\lambda(T_c)D - E^2}$$

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The first order phase transition: 1

- At $T > T_1$ the only minimum is at $\phi = 0$

$$T_1^2 = \frac{8\lambda(T_1) D T_o^2}{8\lambda(T_1) D - 9E^2}$$

- At $T = T_1$ a local minimum at $\phi(T) \neq 0$ appears as an inflection point

$$\langle \phi(T_1) \rangle = \frac{3ET_1}{2\lambda(T_1)}$$

- A barrier develops between a maximum and a local minimum

$$\begin{aligned} \phi_{M,m}(T) &= \frac{3ET}{2\lambda(T)} \\ &\mp \frac{1}{2\lambda(T)}\sqrt{9E^2T^2 - 8\lambda(T)D(T^2 - T_o^2)} \end{aligned}$$

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The first order phase transition: 2

- $$T_c^2 = \frac{\lambda(T_c) D T_o^2}{\lambda(T_c) D - E^2}$$
  At $T = T_c$ the origin and the minimum become degenerate
  $$\phi_M(T_c) = \frac{E T_c}{\lambda(T_c)}$$
  and
  $$\phi_m(T_c) = \frac{2 E T_c}{\lambda(T_c)}$$
- For $T < T_c$ the minimum at $\phi = 0$ becomes metastable
- At $T = T_o$ the barrier disappears, the origin becomes a maximum and the second minimum becomes equal to
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  $$\phi_m(T_o) = \frac{3 E T_o}{\lambda(T_o)}$$

The first order phase transition: 2

- $$T_c^2 = \frac{\lambda(T_c) D T_o^2}{\lambda(T_c) D - E^2}$$

  At $T = T_c$ the origin and the minimum become degenerate

  $$\phi_M(T_c) = \frac{E T_c}{\lambda(T_c)}$$

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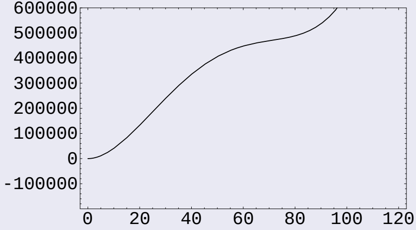
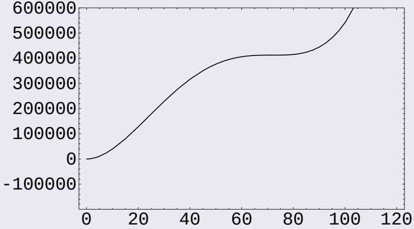
Cartoon of first order phase transition

- At $T > T_1$ the only minimum is at $\phi = 0$



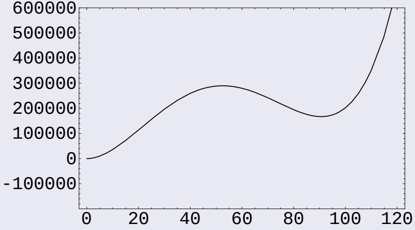
Cartoon of first order phase transition

- At $T = T_1$ an inflection point appears



Cartoon of first order phase transition

- At $T < T_1$ local minimum appears



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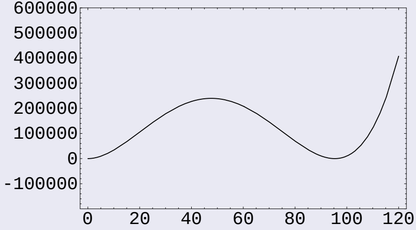
- At $T = T_c$ origin and minimum degenerate



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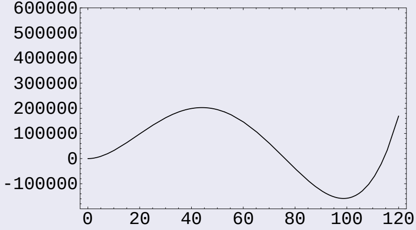

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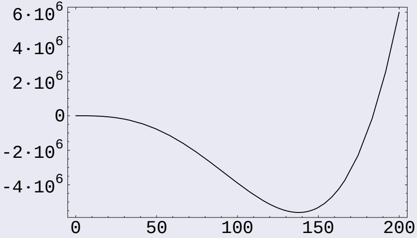
- For $T = T_o$ the origin becomes a maximum



Thermal tunneling

The transition from the false to the true vacuum proceeds via **thermal tunneling** at finite temperature.

It can be understood in terms of formation of bubbles of the broken phase in the sea of the symmetric phase. Once this has happened, the bubble spreads throughout the universe converting false vacuum into true one.

The tunneling rate is computed by using the rules of field theory at finite temperature.

We defined the critical temperature T_c as the temperature at which the two minima of the potential $V(\phi, T)$ have the same depth. However, tunneling with formation of bubbles of the field ϕ corresponding to the second minimum starts somewhat later, and goes sufficiently fast to fill the universe with bubbles of the new phase only at some lower temperature T_t when the corresponding euclidean action $S_E = S_3/T$ suppressing the tunneling becomes $\mathcal{O}(130 - 140)$, as we will see.

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We will use as prototype the Standard Model potential

$$V(\phi, T) = D(T^2 - T_o^2)\phi^2 - ET\phi^3 + \frac{\lambda(T)}{4}\phi^4$$

which can trigger, as we showed in this section, a first order phase transition. In this case the false minimum is $\phi = 0$, and the value of the potential at the origin is zero, $V(0, T) = 0$.

The tunneling probability per unit time per unit volume is given by

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The bounce

At very high temperature the **bounce** solution has $O(3)$ symmetry and the euclidean action is then simplified to,

$$S_3 = 4\pi \int_0^\infty r^2 dr \left[\frac{1}{2} \left(\frac{d\phi}{dr} \right)^2 + V(\phi(r), T) \right]$$

where $r^2 = \vec{x}^{\,2}$, and the euclidean equation of motion yields,

$$\frac{d^2\phi}{dr^2} + \frac{2}{r}\frac{d\phi}{dr} = V'(\phi, T)$$

with the boundary conditions

$$\begin{aligned} \lim_{r\to\infty} \phi(r) &= 0 \\ \left.\frac{d\phi}{dr}\right|_{r=0} &= 0 \end{aligned}$$

Let us take $\phi = 0$ outside a bubble. Then S_3, which is also the surplus free energy of a true vacuum bubble, can be written as

$$S_3 = 4\pi \int_0^R r^2 dr \left[\frac{1}{2} \left(\frac{d\phi}{dr} \right)^2 + V(\phi(r), T) \right]$$

where R is the bubble radius. There are two contributions to S_3: a surface term F_S, coming from the derivative term, and a volume term F_V, coming from the second term. They scale like,

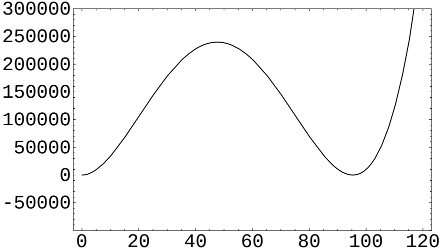
$$S_3 \sim 2\pi R^2 \left(\frac{\delta\phi}{\delta R} \right)^2 \delta R + \frac{4\pi R^3 \langle V \rangle}{3}$$

where δR is the thickness of the bubble wall, $\delta\phi = \phi_m$ and $\langle V \rangle$ is the average of the potential inside the bubble.

Thin wall bubbles

For temperatures just below T_c, the height of the barrier $V(\phi_M, T)$ is large compared to the depth of the potential at the minimum, $-V(\phi_m, T)$. In that case, the solution of minimal action corresponds to minimizing the contribution to F_V coming from the region $\phi = \phi_M$. This amounts to a very small bubble wall $\delta R/R \ll 1$ and so a very quick change of the field from $\phi = 0$ outside the bubble to $\phi = \phi_m$ inside the bubble.

Therefore, the first formed bubbles after T_c are **thin wall** bubbles.



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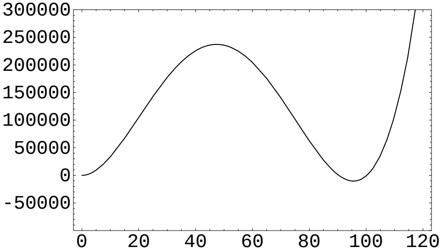

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In the limit of $\epsilon(T) \ll 1$

$$\epsilon(T) = \frac{T_c - T}{T_c - T_o}$$

the initial bounce is very close to ϕ_m for large r and the viscosity damping force can very soon be neglected. Then

$$\frac{d\phi}{dr} = \sqrt{2V(\phi, T)}$$

Thin bubbles

The critical radius is obtained by extremizing the action. For the Standard Model potential

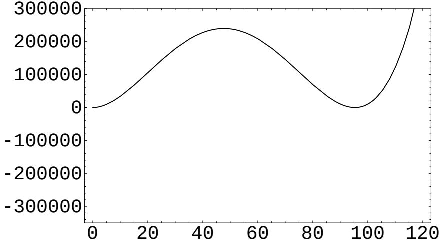
$$R_c = \frac{\sqrt{2\lambda(T)}}{3ET\epsilon(T)}$$

and the action at the critical radius given by

$$S_3 = \frac{64\pi}{81} \frac{ET}{(2\lambda(T))^{3/2}\epsilon(T)^2}$$

Thick wall bubbles

Subsequently, when the temperature drops towards T_o the height of the barrier $V(\phi_M, T)$ becomes small as compared with the depth of the potential at the minimum $-V(\phi_m, T)$. In that case the contribution to F_V from the region $\phi = \phi_M$ is negligible, and the minimal action corresponds to minimizing the surface term F_S. This amounts to a configuration where δR is as large as possible, *i.e.* $\delta R/R = \mathcal{O}(1)$: **thick wall** bubbles



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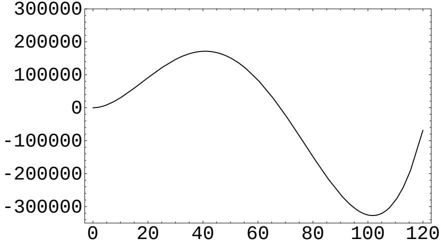

For the case of thick bubbles, $\delta R \sim R$ and the free energy of the bubble can be written as

$$S_3 \sim 2\pi R(\delta\phi)^2 + \frac{4\pi R^3 \langle V \rangle}{3}$$

The critical radius of the bubble obtained as the maximum of the action

$$R_c \sim \frac{\delta\phi}{\sqrt{-2\langle V \rangle}}$$

Thick bubbles

The action at the critical radius is

$$S_3 \sim \frac{(\delta\phi)^3}{\sqrt{-\langle V \rangle}}$$

and for the Standard Model potential

$$S_3 \sim \frac{ET}{\lambda(T)^{3/2}}(1 - \epsilon(T))^{3/2}$$

Bubble nucleation

- Whether the phase transition proceeds through thin or thick wall bubbles depends on how large the bubble nucleation rate is, or how small S_3 is, before thick bubbles are energetically favoured.
- The progress of the phase transition depends on the ratio of the rate of production of bubbles of true vacuum over the expansion rate of the universe. For example if the former remains always smaller than the latter, then the state will be trapped in the supercooled false vacuum. Otherwise the phase transition will start at some temperature T_t by **bubble nucleation**. The probability of bubble formation per unit time per unit volume is given by

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- We have to describe the universe at temperatures close to the electroweak phase transition.
- A homogeneous and isotropic (flat) universe is described by a Robertson-Walker metric which, in comoving coordinates, is given by $ds^2 = dt^2 - a(t)^2 \left(dr^2 + r^2 d\Omega^2\right)$, where $a(t)$ is the scale factor of the universe. The universe expansion is governed by the equation

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3M_{P\ell}^2}\rho$$

- For temperatures $T \sim 10^2\ GeV$ the universe is radiation dominated, and its energy density is given by,

$$\rho = \frac{\pi^2}{30} g(T) T^4, \quad g(T) = g_B(T) + \frac{7}{8} g_F(T)$$

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$$t = \zeta \frac{M_{P\ell}}{T^2}, \quad \zeta = \frac{1}{4\pi}\sqrt{\frac{45}{\pi g}} \sim 3 \times 10^{-2}$$

- The onset of nucleation happens at a temperature T_t such that **the probability for a single bubble to be nucleated within one horizon volume is ~ 1**

$$\int_{T_t}^{\infty} \frac{dT}{T} \left(\frac{2\zeta M_{\rm Pl}}{T}\right)^4 \exp\{-S_3(T)/T\} = \mathcal{O}(1) \ .$$

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- EWBG requires large CP violation and strong first-order phase transition: neither of them is provided by the Standard Model effective potential
- They can be provided in extensions of the SM: e.g. the MSSM
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