Selfgravitation and Stability in Spherical Accretion

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Introduction

- Spherical steady accretion: Bondi (1952), Michel (1972) — all effects of selfgravitation are neglected.

- Selfgravity in spherical accretion — simple but nontrivial models, an illustration of what selfgravitation can change in the entire picture.

Basics

- Spherical symmetry:

\[ ds^2 = -N(t, r)^2 dt^2 + \alpha(t, r) dr^2 + R(t, r)^2 \left(d\theta^2 + \sin^2 d\phi^2\right). \]

- Mean Cauchy curvature of two-spheres \( t = \text{const}, \ r = \text{const} \)

\[ k = \text{tr}K' = \frac{2\partial_r R}{R \sqrt{\alpha}}. \]

- Energy-momentum tensor of the perfect fluid

\[ T = (p + \varrho)u \otimes u + pg. \]

- Comoving gauge: \( u_t = u_\theta = u_\phi = 0. \)

- Define functions

\[ U = \frac{\partial_t R}{N}, \quad m(R) = m_{\text{tot}} - 4\pi \int_{R}^{\infty} R' R'^2 \varrho. \]
We search for a steady flow:

- **Accretion rate**

\[ \dot{m} = (\partial_t - (\partial_t R)\partial_R) m(R) \]

for a given \( R \) should be constant in time.

- The fluid velocity \( U \), energy density \( \varrho \), sound speed \( a (a^2 = dp/d\varrho) \) etc. are constant at a given \( R \), i.e.

\[ (\partial_t - (\partial_t R)\partial_R) X = 0, \]

where \( X = U, \varrho, a, \ldots \)

- **Boundary conditions:** \( |U_\infty| \ll m(R_\infty)/R_\infty \ll a_\infty \).
Basics

Equations for the steady flow:

- **Lapse equation**

  \[
  N = \frac{kR}{kR_\infty} \beta(R), \quad \beta(R) = \exp \left( -16\pi \int_R^{R_\infty} \frac{(p + \varrho)dR'}{k^2 R'} \right).
  \]

- **Integrated continuity equation**

  \[
  U = \frac{A}{R^2 n},
  \]

  where \( A \) is an integration constant and \( n \) the baryonic density (\( \text{div}(n\mathbf{u}) = 0 \)).

General equations:

\[
Rk = 2 \sqrt{1 - \frac{2m(R)}{R}} + U^2, \quad \partial_R m = 4\pi R^2 \varrho \quad \text{and} \quad N = \frac{Bn}{p + \varrho},
\]

where \( B \) stands for another integration constant.
Assume polytopic equation of state of the form

\[ p = K \rho^\Gamma, \quad 1 < \Gamma \leq 5/3 \]

or

\[ p = Kn^\Gamma, \quad 1 < \Gamma \leq 5/3. \]

Many estimates can be obtained also for general barotropic EOS \( p = p(\rho) = p(\rho(n)) \).
Sonic point

The sonic point is defined as a location where

\[ |U| = \frac{1}{2} kRa. \]

Precise information about parameters of the sonic point can provide other important characteristics of the accretion.

In test fluid approximation (Michel’s model), i.e., when

\[ 4\pi \int_{R>2m} dR' R'^2 \rho \ll m \]

with \( p = Kn^\Gamma \), there exist precisely one sonic point, and

\[ a^2_* = \frac{1}{9} \left\{ 6\Gamma - 7 + 2(3\Gamma - 2) \cos \left[ \frac{\pi}{3} + \frac{1}{3} \arccos \left( \frac{1}{2(3\Gamma - 2)^3} \left( 54\Gamma^3 + 
- 351\Gamma^2 - 558\Gamma + 486(\Gamma - 1)a^2_\infty - 243a^4_\infty - 259 \right) \right] \right\} \]
Sonic point

This, in turn, allows us to write

\[
\dot{m} = -4\pi NR^2 U(q + p) = -4\pi AB = \\
\pi n_{\infty} m^2 \left( \frac{\Gamma - 1}{\Gamma - 1 - a_{\infty}^2} \right) \left( \frac{1 + 3a_{*}^2}{a_{*}^2} \right)^{\frac{\Gamma}{2}} \left( \frac{a_{*}^2 \Gamma - 1 - a_{\infty}^2}{a_{\infty}^2 \Gamma - 1 - a_{*}^2} \right)^{\frac{1}{\Gamma - 1}}.
\]

This result can be used to estimate \( \dot{m} \) also for more general barotropic EOS.
Selfgravitating vs. test-fluid flow

- Consider two models with the same poltropic EOS \( p = K \rho^\Gamma \) and asymptotic data \( \rho_\infty, a_\infty \) and \( |U_\infty| \ll m(R_\infty)/R_\infty \ll a_\infty \): one computed assuming selfgravitation of the fluid and another in test-fluid approximation. For these 2 models the following sonic point parameters: \( a_*^2, U_*^2 \) and \( m(R_\ast)/R_\ast \) can be shown to be respectively the same.

- The accretion rate differs! One can show, that \( m(R_\ast) = m_{\text{tot}} - \gamma \rho_\infty \), i.e. \( m(R_\ast) \) is, for a fixed total mass \( m_{\text{tot}} \), a linear function of \( \rho_\infty \).

- One can also show that the following formula holds

\[
\dot{m} = -4\pi m(R_\ast)^2 \rho_\infty \frac{R_\ast^2}{m(R_\ast)^2 U_*} \left( \frac{a_*}{a_\infty} \right)^{2-\Gamma} \left( 1 + \frac{a_*^2}{\Gamma} \right).
\]

Here the whole dependence on \( \rho_\infty \) is contained in \( m(R_\ast)^2 \rho_\infty \). It follows that the maximum of \( \dot{m} \) exists for \( m(R_\ast) = 2m_{\text{tot}}/3 \) and \( m(R_\ast) \to 0 \) for \( \rho_\infty \to 0 \) and \( m(R_\ast)/m_{\text{tot}} \to 0 \).
Examples

Numerical solutions for polytropic EOS of both types $p\varrho$ and $p\cdot n$ agree with above considerations.

EOS: $p = Kn^\Gamma$. Models with $\Gamma = 1.4$, $a_\infty^2 = 0.1$ and different $n_\infty$. 

![Graph showing the relationship between $m_{\text{fluid}}/m_{\text{tot}}$ and $n$]
Same as before. Here $m_{\text{fluid}} \equiv m_{\text{tot}} - m_{\text{BH}} \approx m_{\text{tot}} - m(R_*)$. 
Examples

Boundary conditions:
$R_\infty = 10^4,$
$n_\infty = 0.1 \cdot 10^{-18},$
$a_\infty^2 = 0.1,$
$\Gamma = 1.4.$

Sonic point parameters: $R_* = 2.318,$ $a_*^2 = 0.15116,$ $|U_*| = 0.3225,$ $m(R_*)/m_{\text{tot}} = 0.4814.$ Horizon: $R_{\text{BH}} = 0.9627,$ $m_{\text{fluid}}/m_{\text{tot}} = 0.5186.$
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Stability of selfgravitating fluids

- Newtonian case (simplicity).
- Lagrangian approach.
- Stability of Bondi accretion has been analyzed by Balasz (1972). Correct but inconclusive approach — too stringent understanding of the notion of linearized stability.
- Using Lagrangian variables one can reproduce the Eulerian stability result obtained for non selfgravitating fluids by Moncrief (1980).
Stability of selfgravitating fluids

- Equations:

\[
\partial_t U + U \partial_R U = - \frac{\partial_R \rho}{\rho} \frac{m(R)}{R^2},
\]

\[
\partial_t \rho = - \frac{1}{R^2} \partial_R \left( R^2 \rho U \right).
\]

- Introduce \( \zeta(r, t) = \Delta R(r, t) \) — deviation from the particle position in the unperturbed flow.

- Perturbation of velocity: \( \Delta U = \partial^L_t = (\partial_t + U \partial_R) \zeta \).

- Perturbation of density (follows from \( \Delta m(R(r = \text{const})) = 0 \) or continuity equation)

\[
\Delta \rho = -\rho \left( \frac{2\zeta}{R} + \partial_R \zeta \right).
\]
Stability of selfgravitating fluids

- Main equation:

\[
\left(\partial_t^L\right)^2 \zeta = \frac{2m(R)\zeta}{R^3} + \frac{1}{Q} \partial_R \left(a^2 Q \left(\partial_R \zeta + \frac{2\zeta}{R}\right)\right) - \frac{2\zeta}{Q} \partial_R p.
\]

- Standard way (Balazs): try to find solutions of the form

\[
\zeta(R(r), t) = \exp(i\omega t)\zeta(R(r)),
\]

where \(\omega^2\) is positive and modulus \(\zeta(R(r))\) is time independent. This cannot be done!

- Instead, define the energy

\[
E = \int_V dV Q \left(\frac{1}{2} (\partial_t \zeta)^2 + \frac{1}{2} (\partial_R \zeta)^2 \left(a^2 - U^2\right) + \frac{\zeta^2}{R^2} \left(a^2 - \frac{m}{R} - R \partial_R a^2\right)\right),
\]

where \(V\) is an annulus between \(R_*\) and \(R_\infty\).
Stability of selfgravitating fluids

- Is this energy positive?
- One can show that

\[ E = \tilde{E} - \left[ 4\pi R \zeta^2 \varrho \left( a^2 - \frac{m}{2R} \right) \right]^{R_{\infty}}_R, \]

where

\[ \tilde{E} = \frac{1}{2} \int_V dV \left( X^2 + Y^2 \right) - 2\pi \int_V dV \zeta \varrho^2, \]

and

\[ X = \sqrt{\varrho} \partial_t \zeta, \]

\[ Y = \sqrt{\varrho} \left( \frac{2a^2 R - m}{R^2 \sqrt{a^2 - U^2}} \zeta + \sqrt{a^2 - U^2} \partial_R \zeta \right). \]

- \( \tilde{E} \) may be negative due to the last term appearing in the selfgravitating case.
Stability of selfgravitating fluids

- We compute $\partial_t E$ to get

$$
\partial_t E = - \int_V dV \zeta^2 \frac{\partial_t m(R)}{R^3} + 
+ 4\pi \left[ R^2 q \left( \partial_t \zeta \partial_R \zeta \left( a^2 - U^2 \right) - U (\partial_t \zeta)^2 \right) \right]_{R=0}.
$$

- One can show that the boundary terms are negative definite. Energy $E$ of perturbations cannot grow for critical flow. Unstable behaviour is still possible as $E$ is not necessarily positive.

- For test fluids

$$
E = \tilde{E} = \frac{1}{2} \int_V dV \left( X^2 + Y^2 \right)
$$

is positive and $\partial_t E \leq 0$. This excludes exponential growth of $X$, $Y$ and long-term exponential growth of linear modes $\zeta$. The modulus $|\zeta(R)|$ may depend on time.
Stability of Selfgravitating fluids

- The absence of exponentially growing linear modes does not guarantee that the perturbed solution will be always close to background solution.
- This kind of stability means that the evolving perturbations can be bounded by initial solutions in a suitable sense.
- The linear instability means that the “strength” of the evolving perturbation does not depend on the “strength” of the initial perturbation rather than the perturbation grows infinitely.
- Generalisation onto general-relativistic case is possible.
Summary

- Spherical accretion provides a simple playground in which one can observe different effects caused by selfgravity of the accreting fluid.
- Many properties of those solutions can by obtained by analytical means.
- Steady, selfgravitating solutions are relatively easy to be obtained numerically (ordinary differential equations) and can serve as tests for more sophisticated numerical schemes.
- The stability of the selfgravitating solutions needs to be investigated carefully. Many issues (e.g. stability of subsonic accretion flows) remain unclear.