Selfgravitation and Stability in Spherical Accretion

Patryk Mach

M. Smoluchowski Institute of Physics, Jagiellonian University
Max Planck Institute for Astrophysics, Garching

Introduction

- Spherical steady accretion: Bondi (1952), Michel (1972) — all effects of selfgravitation are neglected.
- Selfgravity in spherical accretion — simple but nontrivial models, an illustration of what selfgravitation can change in the entire picture.
- Karkowski, Kinasiewicz, Mach, Malec, Świerczyński (2006), Kinasiewicz, Mach, Malec (2006).

Basics

- Spherical symmetry:

$$ds^2 = -N(t,r)^2 dt^2 + \alpha(t,r) dr^2 + R(t,r)^2 \left(d\theta^2 + \sin^2 d\phi^2\right).$$

- Mean Cauchy curvature of two-spheres $t = \text{const}$, $r = \text{const}$

$$k = \text{tr}K' = \frac{2\partial_r R}{R\sqrt{\alpha}}.$$

- Energy-momentum tensor of the perfect fluid

$$\mathbf{T} = (p + \varrho)\mathbf{u} \otimes \mathbf{u} + p\mathbf{g}.$$

- Comoving gauge: $u_r = u_\theta = u_\phi = 0$.
- Define functions

$$U = \frac{\partial_t R}{N}, \quad m(R) = m_{\text{tot}} - 4\pi \int_R^{R_\infty} dR' R'^2 \varrho.$$

Basics

We search for a steady flow:

- Accretion rate

$$\dot{m} = (\partial_t - (\partial_t R)\partial_R)m(R)$$

  for a given R should be constant in time.

- The fluid velocity U, energy density ϱ, sound speed a ($a^2 = dp/d\varrho$) etc. are constant at a given R, i.e.

$$(\partial_t - (\partial_t R)\partial_R)X = 0,$$

  where $X = U, \varrho, a, \ldots$

- Boundary conditions: $|U_\infty| \ll m(R_\infty)/R_\infty \ll a_\infty$.

Basics

Equations for the steady flow:

- Lapse equation

$$N = \frac{kR}{k_\infty R_\infty}\beta(R), \quad \beta(R) = \exp\left(-16\pi \int_R^{R_\infty} \frac{(p+\varrho)dR'}{k^2 R'}\right).$$

- Integrated continuity equation

$$U = \frac{A}{R^2 n},$$

  where A is an integration constant and n the baryonic density ($\mathrm{div}(n\mathbf{u}) = 0$).

General equations:

$$Rk = 2\sqrt{1 - \frac{2m(R)}{R} + U^2}, \quad \partial_R m = 4\pi R^2 \varrho \quad \text{and} \quad N = \frac{Bn}{p+\varrho},$$

where B stands for another integration constatnt.

Equation of state

Assume polytopic equation of state of the form

$$p = K\varrho^{\Gamma}, \quad 1 < \Gamma \leqslant 5/3$$

or

$$p = Kn^{\Gamma}, \quad 1 < \Gamma \leqslant 5/3.$$

Many estimates can be obtained also for general barotropic EOS $p = p(\varrho) = p(\varrho(n))$.

Sonic point

The sonic point is defined as a location where

$$|U| = \frac{1}{2} kRa.$$

Precise information about parameters of the sonic point can provide other important characteristics of the accretion.
In test fluid approximation (Michel's model), i.e., when

$$4\pi \int_{R>2m} dR' R'^2 \varrho \ll m$$

with $p = Kn^\Gamma$, there exist precisely one sonic point, and

$$a_*^2 = \frac{1}{9}\Bigg\{6\Gamma - 7 + 2(3\Gamma - 2)\cos\Bigg[\frac{\pi}{3} + \frac{1}{3}\arccos\Bigg\{\frac{1}{2(3\Gamma-2)^3}\Big(54\Gamma^3 + \\ -351\Gamma^2 - 558\Gamma + 486(\Gamma - 1)a_\infty^2 - 243a_\infty^4 - 259\Big)\Bigg\}\Bigg]\Bigg\}.$$

Sonic point

This, in turn, allows us to write

$$\begin{aligned}\dot{m} &= -4\pi N R^2 U(\varrho + p) = -4\pi AB = \\ &= \pi n_\infty m^2 \frac{\Gamma - 1}{\Gamma - 1 - a_\infty^2} \left(\frac{1 + 3a_*^2}{a_*^2}\right)^{\frac{3}{2}} \left(\frac{a_*^2}{a_\infty^2} \frac{\Gamma - 1 - a_\infty^2}{\Gamma - 1 - a_*^2}\right)^{\frac{1}{\Gamma-1}} .\end{aligned}$$

This result can be used to estimate $\dot{m}$ also for more general barotropic EOS.

Selfgravitating vs. test-fluid flow

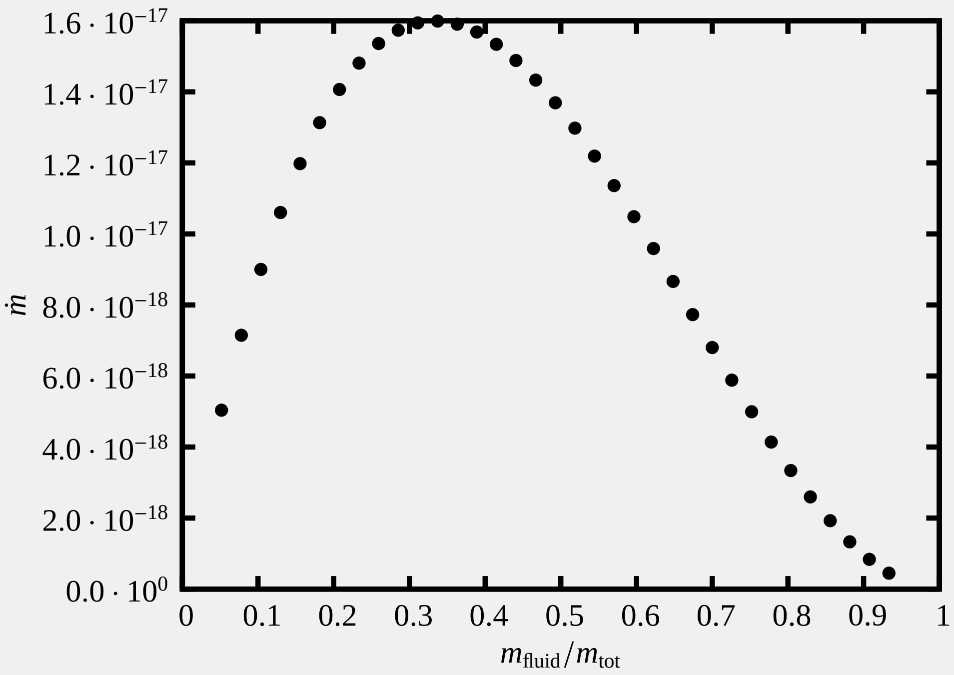
- Consider two models with the same politropic EOS $p = K\varrho^\Gamma$ and asymptotic data ϱ_∞, a_∞ and $|U_\infty| \ll m(R_\infty)/R_\infty \ll a_\infty$: one computed assuming selfgravitation of the fluid and another in test-fluid approximation. For these 2 models the following sonic point parameters: a_*^2, U_*^2 and $m(R_*)/R_*$ can be shown to be respectively the same.
- The accretion rate differs! One can show, that $m(R_*) = m_{\mathrm{tot}} - \gamma\varrho_\infty$, i.e. $m(R_*)$ is, for a fixed total mass m_{tot}, a linear function of ϱ_∞.
- One can also show that the following formula holds

$$\dot{m} = -4\pi m(R_*)^2 \varrho_\infty \frac{R_*^2}{m(R_*)^2} U_* \left(\frac{a_*}{a_\infty}\right)^{\frac{2}{\Gamma-1}} \left(1 + \frac{a_*^2}{\Gamma}\right).$$

Here the whole dependence on ϱ_∞ is contained in $m(R_*)^2\varrho_\infty$. It follows that the maximum of $\dot{m}$ exists for $m(R_*) = 2m_{\mathrm{tot}}/3$ and $m(R_*) \to 0$ for $\varrho_\infty \to 0$ and $m(R_*)/m_{\mathrm{tot}} \to 0$.

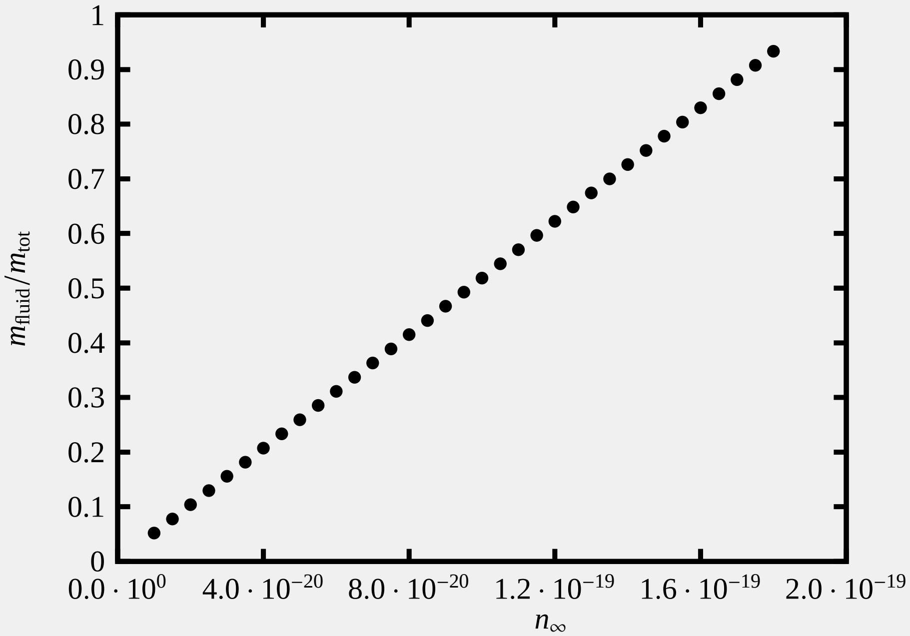
Examples

Numerical solutions for polytropic EOS of both types p–ϱ and p–n agree with above considerations.



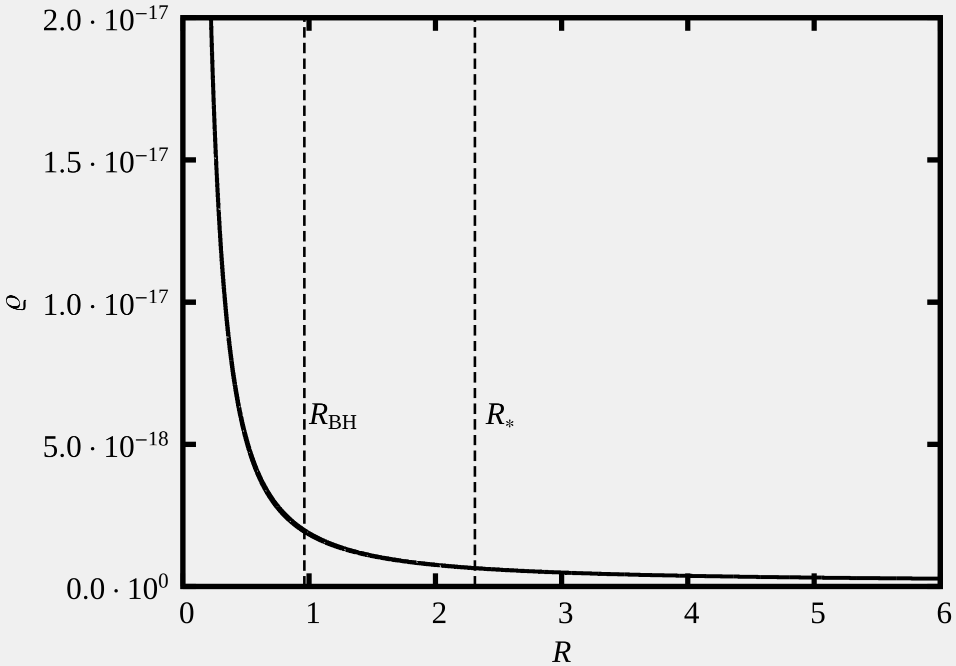
EOS: $p = Kn^\Gamma$. Models with $\Gamma = 1.4$, $a_\infty^2 = 0.1$ and different n_∞.

Examples



Same as before. Here $m_{\rm fluid} \equiv m_{\rm tot} - m_{\rm BH} \approx m_{\rm tot} - m(R_*)$.

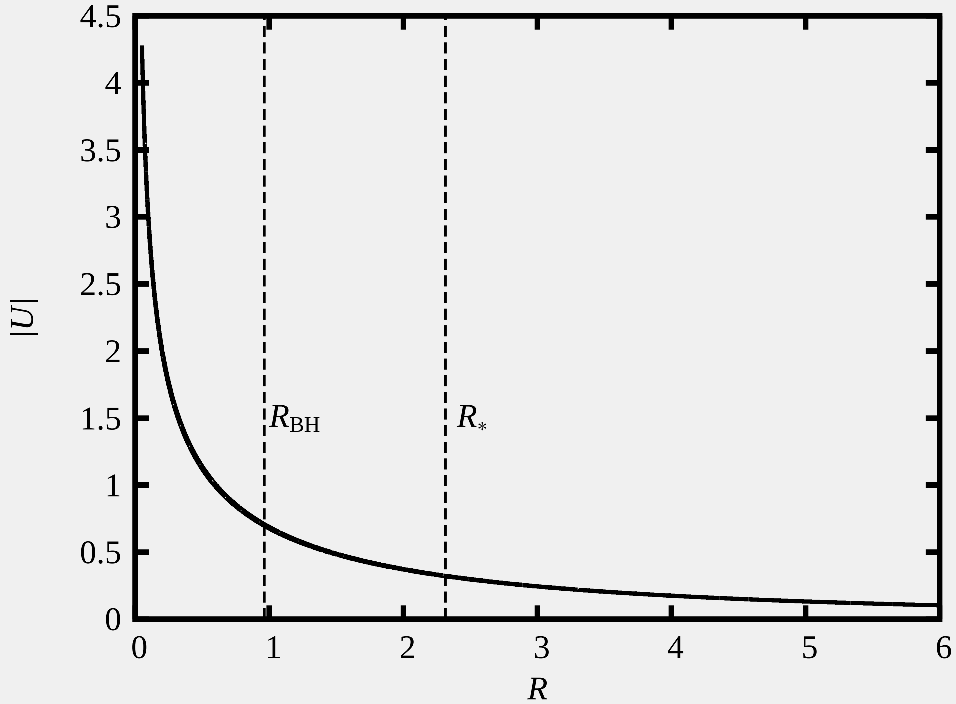
Examples



Boundary conditions:
$R_\infty = 10^4$,
$n_\infty = 0.1 \cdot 10^{-18}$,
$a_\infty^2 = 0.1$,
$\Gamma = 1.4$.

Sonic point parameters: $R_* = 2.318$, $a_*^2 = 0.15116$, $|U_*| = 0.3225$, $m(R_*)/m_{\rm tot} = 0.4814$. Horizon: $R_{\rm BH} = 0.9627$, $m_{\rm fluid}/m_{\rm tot} = 0.5186$.

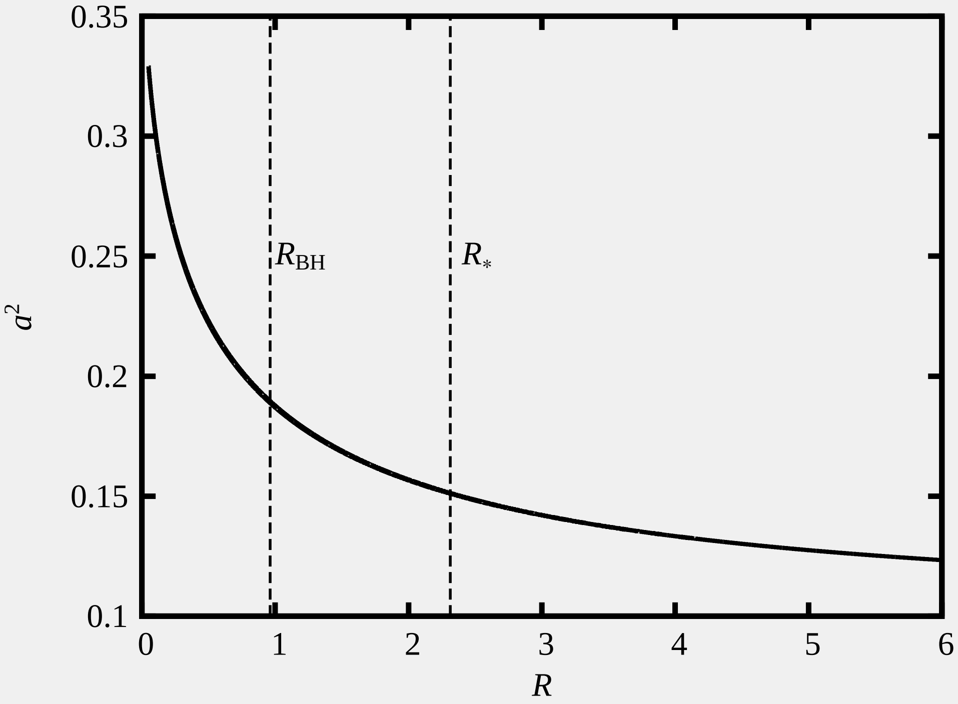
Examples



Boundary conditions:
$R_\infty = 10^4$,
$n_\infty = 0.1 \cdot 10^{-18}$,
$a^2_\infty = 0.1$,
$\Gamma = 1.4$.

Sonic point parameters: $R_* = 2.318$, $a^2_* = 0.15116$, $|U_*| = 0.3225$, $m(R_*)/m_{\rm tot} = 0.4814$. Horizon: $R_{\rm BH} = 0.9627$, $m_{\rm fluid}/m_{\rm tot} = 0.5186$.

Examples



Boundary conditions:
$R_\infty = 10^4$,
$n_\infty = 0.1 \cdot 10^{-18}$,
$a^2_\infty = 0.1$,
$\Gamma = 1.4$.

Sonic point parameters: $R_* = 2.318$, $a^2_* = 0.15116$, $|U_*| = 0.3225$, $m(R_*)/m_{\rm tot} = 0.4814$. Horizon: $R_{\rm BH} = 0.9627$, $m_{\rm fluid}/m_{\rm tot} = 0.5186$.

Stability of selfgravitating fluids

- Newtonian case (simplicity).
- Lagrangian approach.
- Stability of Bondi accretion has been analized by Balasz (1972). Correct but inconclusive approach — too stringent understanding of the notion of linearized stability.
- Using Lagrangian variables one can reproduce the Eulerian stability result obtained for non selfgravitating fluids by Moncrief (1980).

Stability of selfgravitating fluids

- Equations:

$$\begin{aligned}\partial_t U + U\partial_R U &= -\frac{\partial_R p}{\varrho} - \frac{m(R)}{R^2},\\ \partial_t \varrho &= -\frac{1}{R^2}\partial_R\left(R^2\varrho U\right).\end{aligned}$$

- Introduce $\zeta(r,t) = \Delta R(r,t)$ — deviation from the particle position in the unperturbed flow.
- Perturbation of velocity: $\Delta U = \partial_t^L = (\partial_t + U\partial_R)\zeta$.
- Perturbation of density (follows from $\Delta m(R(r = \text{const})) = 0$ or continuity equation)

$$\Delta\varrho = -\varrho\left(\frac{2\zeta}{R} + \partial_R\zeta\right).$$

Stability of selfgravitating fluids

- Main equation:

$$\left(\partial_t^L\right)^2 \zeta = \frac{2m(R)\zeta}{R^3} + \frac{1}{\varrho}\partial_R\left(a^2\varrho\left(\partial_R\zeta + \frac{2\zeta}{R}\right)\right) - \frac{2\zeta}{\varrho}\partial_R p.$$

- Standard way (Balazs): try to find solutions of the form $\zeta(R(r), t) = \exp(i\omega t)\zeta(R(r))$, where ω^2 is positive and modulus $\zeta(R(r))$ is time independent. This cannot be done!
- Instead, define the energy

$$E = \int_V dV\varrho\left(\frac{1}{2}(\partial_t\zeta)^2 + \frac{1}{2}(\partial_R\zeta)^2\left(a^2 - U^2\right) + \frac{\zeta^2}{R^2}\left(a^2 - \frac{m}{R} - R\partial_R a^2\right)\right),$$

where V is an annulus between R_* and R_∞.

Stability of selfgravitating fluids

- Is this energy positive?
- One can show that

$$E = \tilde{E} - \left[4\pi R\zeta^2\varrho\left(a^2 - \frac{m}{2R}\right)\right]_{R_*}^{R_\infty},$$

where

$$\tilde{E} = \frac{1}{2}\int_V dV\left(X^2 + Y^2\right) - 2\pi\int_V dV\zeta^2\varrho^2$$

and

$$X = \sqrt{\varrho}\,\partial_t\zeta,$$
$$Y = \sqrt{\varrho}\left(\frac{2a^2R - m}{R^2\sqrt{a^2 - U^2}}\zeta + \sqrt{a^2 - U^2}\,\partial_R\zeta\right).$$

- $\tilde{E}$ may be negative due to the last term appearing in the selfgravitating case.

Stability of selfgravitating fluids

- We compute $\partial_t E$ to get

$$\partial_t E = - \int_V dV \zeta^2 \frac{\partial_t m(R)}{R^3} + + 4\pi \left[R^2 \varrho \left(\partial_t \zeta \partial_R \zeta \left(a^2 - U^2 \right) - U \left(\partial_t \zeta \right)^2 \right) \right]_{R_*}^{R_\infty} .$$

- One can show that the boundary terms are negative definite. Energy E of perturbations cannot grow for critical flow. Unstable behaviour is still possible as E is not necessarily positive.
- For test fluids

$$E = \tilde{E} = \frac{1}{2} \int_V dV \left(X^2 + Y^2 \right)$$

  is positive and $\partial_t E \leqslant 0$. This excludes exponential growth of X, Y and long-term exponential growth of linear modes ζ. The modulus $|\zeta(R)|$ may depend on time.

Stability of Selfgravitating fluids

- The absence of exponentially growing linear modes does not guarantee that the perturbed solution will be always close to background solution.
- This kind of stability means that the evolving perturbations can be bounded by initial solutions in a suitable sense.
- The linear instability means that the “strength” of the evolving perturbation does not depend on the “strength” of the initial perturbation rather than the perturbation grows infinitely.
- Generalisation onto general-relativistic case is possible.

Summary

- Spherical accretion provides a simple playground in which one can observe different effects caused by selfgravity of the accreting fluid.
- Many properties of those solutions can by obtained by analytical means.
- Steady, selfgravitating solutions are relatively easy to be obtained numerically (oridinary differential equations) and can serve as tests for more sophisticated numerical schemes.
- The stability of the selfgravitating solutions needs to be investigated carefully. Many issues (e.g. stability of subsonic accretion flows) remain unclear.