Lie Symmetries of Spherically Symmetric Systems in General Relativity

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June 17, 2007

Lie Symmetries

- Lie groups play an outstanding role in modern mathematical physics. Origins of the theory lie in the development of methods for solving nonlinear differential equations by great mathematicians such as Gustav Jacobi and Sophus Lie.
- Nowdays the Lie symmetries method is a powerfull technique for solving nonlinear differential equations, or at least detecting integrable regimes of systems of differential equations.
- The method is based on looking at differential equations in the geometrical way - more geometrico.

Basic Concepts

Let v be a vector field in space of dependent and independent variables

$$v \equiv \xi(x,u)\partial_x + \sum_{j=1}^{m} \phi_j(x,u)\partial_{u_j}.$$

Let Σ be a system of ordinary differential equations

$$\Sigma = \{\psi_1 = 0, \psi_2 = 0 \ldots, \psi_r = 0\}$$

with one independent variable x and m dependent variables

$$u = (u_1, u_2, \ldots u_m)$$

with derivatives up to $N-th$ order $\left\{u_k^{(n)}, n \leq N\right\}$.

A Computational Recipe-The Main Algorithm

In order to find Lie point symmetries of the system Σ, we consider one-parameter Lie group of transformations

$$x^\star = \Psi\left(x, u, \epsilon\right),$$
$$u^\star = \Phi\left(x, u, \epsilon\right),$$

under which Σ must be invariant. The group action is infinitesimally given by

$$x^\star = x + \epsilon\xi(x, u) + O(\epsilon^2)$$
$$u_j^\star = u_j + \epsilon\phi_j(x, u) + O(\epsilon^2) \quad j = 1, \ldots, m.$$

To determinate the infinitesimals ξ, ϕ_j we require that the previous transformation leaves invariant the space of solutions of Σ:

$$S_\Sigma = \{u : \psi_1 = 0, \psi_2 = 0 \ldots, \psi_r = 0\}.$$

This is equivalent to

$$pr^{(N)}v(\psi_i)|_\Sigma = 0 \quad i = 1, \ldots, r \tag{1}$$

$$pr^{(N)}v = \xi\partial_x + \sum_{k=1}^{m} \phi_k \partial_{u_k} + \sum_{k=1}^{m} \sum_{n=1}^{N} \phi_k^{[n]} \partial_{u_k^{(n)}}$$

is N-th prolongation of the vector field v, $\phi_k^{[n]}$ is defined recursively by

$$\phi_k^{[n+1]} = D_x \phi_k^{[n]} - u_k^{(n+1)} D_x \xi$$

where D_x is the total derivative with respect to x

Examples

The simplest second order equation has eight Lie symmetries

$$u'' = 0$$

Its general Lie symmetry is given by

$$v = (a_1 + a_2x + a_3y + a_4xy + a_5x^2)\frac{\partial}{\partial x} + (a_6 + a_7x + a_8y + a_5xy + a_4y^2)\frac{\partial}{\partial u}$$

Example of a second order differential equation which has no symmetries:

$$u'' = xu + e^{u'} + e^{-u'}$$

Perfect fluid in shearfree motion

Because of the high symmetry of the problem the field equations become very simple: for the line element

$$g = -e^{2\nu(r,t)}dt^2 + e^{2\lambda(r,t)}\left[dr^2 + r^2 d\Omega^2\right] \tag{2}$$

Kustaanheimo and Qvist (1948) showed that for the metric function

$$y(x,t) = e^{-\lambda(r,t)}, \quad x \equiv r^2,$$

they reduce to one ordinary differential equation

$$y'' = F(x)y^2 \tag{3}$$

The second metric function is then

$$e^{\nu(r,t)} = \lambda_{,t} e^{-f(t)}$$

where $f(t)$ is an arbitrary function connected with the freedom of scaling t. The function $F(x)$ depends on the equation of state of the fluid, and mass density μ as well as pressure p can be computed from λ and f.

The symmetry approach

One finds that for given $F(x)$ the equation admits a symmetry if there exists $B(x)$ which satisfies

$$F(\frac{5}{2}B' + c) + BF' = 0$$
$$B''' = 4(dx + e)F$$

where c,d and e are arbitrary constants. One can detect all possible choices of $F(x)$ when there exist symmetries. If for a given F there is no symmetry then Lie method does not help. There are also cases with one or two distinct symmetries. In the first case one can solve exactly the equation under certain conditions, in the second case it is always exactly solvable.

Weyl Conformal Gravity

The lagrangian density is

$$\mathcal{L} = \frac{1}{3}R^2 - R_{\mu\nu}R^{\mu\nu} = \frac{1}{2}\left\{L_{GB} - C_{\alpha\beta\mu\nu}C^{\alpha\beta\mu\nu}\right\}$$

where $L_{GB} = R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu} - 4R_{\mu\nu}R^{\mu\nu} + R^2$, the Gauss-Bonnet term, is a total divergence in four dimensional spaces, i.e. it doesn't contribute to the field equations. The Bach equations which follow for the lagrangian (4) read

$$2\nabla^{\alpha}\nabla^{\beta}C_{\alpha\ \ \nu\beta}^{\ \ \mu} + R^{\alpha\beta}C_{\alpha\ \ \nu\beta}^{\ \ \mu} = B^{\mu}{}_{\nu} = 0$$

where $B^{\mu}{}_{\nu}$ is the Bach tensor. the isotropic coordinates

$$g = -e^{2\nu(x)}d\tau^2 + e^{2(\nu-\alpha(x))}\left[dx^2 + x^2 d\Omega^2\right] \qquad (4)$$

The the only nonvanishing components of the Bach tensor in the static spherically symmetric case are

$$
\begin{aligned}
B^0{}_0 &= \frac{e^{4(\alpha-\nu)}}{3}\left\{2\alpha'''' + \left(4\alpha' + \frac{8}{x}\right)\alpha''' + 3\alpha''^2 - 2\alpha'^2\alpha'' - \alpha'^4 \right. \\
&\qquad \left. + \frac{2\alpha'^3 + 22\alpha'\alpha''}{x} + \frac{5\alpha'^2}{x^2}\right\} \\
B^1{}_1 &= \frac{e^{4(\alpha-\nu)}}{3}\left\{2\left(\alpha' - \frac{1}{x}\right)\alpha''' - \alpha''^2 + 2\alpha'^2\alpha'' - \alpha'^4 + \frac{6\alpha'^3}{x} \right. \\
&\qquad \left. - \frac{4\alpha'' + 9\alpha'^2}{x^2} + \frac{4\alpha'}{x^3}\right\} \\
B^2{}_2 &= \frac{e^{4(\alpha-\nu)}}{3}\left\{-\alpha'''' - 3\left(\alpha' + \frac{1}{x}\right)\alpha''' - \alpha''^2 + \alpha'^4 - \frac{11\alpha'\alpha'' + 4\alpha'^3}{x} \right. \\
&\qquad \left. + \frac{2\alpha'' + 2\alpha'^2}{x^2} - \frac{2\alpha'}{x^3}\right\} \\
B^3{}_3 &= B^2{}_2
\end{aligned}
$$

By direct inspection one sees that there is a simple Lie symmetry

$$v = d_1 \frac{\partial}{\partial \alpha} + d_2 \frac{\partial}{\partial \nu}$$

which appears to be the conformal symmetry of the field equations. With help of this symmetry one can find the general solution

$$g = -a(x) b^2(x) d\tau^2 + \frac{1}{a(x)} dx^2 + x^2 d\Omega^2$$

where

$$\begin{aligned} a(x) &= c_1 + \frac{c_2}{x} + c_3 x + c_4 x^2 \text{, with a constraint} \quad 1 - c_1^2 + 3c_2 c_3 = 0 \\ b &= const \end{aligned}$$

Higher – Derivative – Gravities (Lagrange picture)

& Einstein-matter fields systems (Hamilton pictures)

Lagrange picture

Jordan frame

$\mathcal{L} = \tilde{R} + a\tilde{R}^2 + b\tilde{R}_{\mu\nu}\tilde{R}^{\mu\nu}$

$\tilde{g}$ satisfies 4$^{\text{th}}$ order field equations.

Legandre transformation

Hamilton picture

Helmholtz-Jordan frame

$\mathcal{H} = \frac{d}{4(ad+b)}\chi^2 + \frac{1}{4b}\pi_{\mu\nu}\pi^{\mu\nu}$

$\{\tilde{g}, \chi, \pi\}$ satisfy 2$^{\text{nd}}$ order field eqs

$\{g, \Phi_{\mu\nu}\}$ satisfy2$^{\text{nd}}$ order field equations

$\mathcal{H} = \ldots\ldots$

Einstein frame

Jordan and Helmholtz-Jordan Frames

We restrict only to the case when $a = \frac{1}{3}$ and $b = -1$, which describes in Hamilton picture spin-2 field interacting to the standard Einsten gravity. Euler – Lagrange equations read

$$G^{\mu}{}_{\nu} + \Big\{ -\Box R^{\mu}{}_{\nu} + 2R_{\alpha}{}^{\mu}{}_{\nu}{}^{\beta} R^{\alpha}{}_{\beta} + \frac{1}{2} R^{\alpha}{}_{\beta} R^{\beta}{}_{\alpha} \delta^{\mu}{}_{\nu} + \frac{1}{6}(\Box R - R^2)\delta^{\mu}{}_{\nu} + \frac{1}{3}\nabla^{\mu}\nabla_{\nu} R + \frac{2}{3} R R^{\mu}{}_{\nu} \Big\} = 0$$

i.e. *Einsten tensor* $+$ *Bach tensor* $= 0$.
We point out that trace of this system implies vanishing of the Ricci scalar :

$$R = 0$$

Let v be a vector field on an open subset $M \subset \Re \times \Re^2$

$$v = \xi(x, \nu, \lambda)\frac{\partial}{\partial x} + \psi(x, \nu, \lambda)\frac{\partial}{\partial \nu} + \phi(x, \nu, \lambda)\frac{\partial}{\partial \lambda}$$

where ξ, ϕ, ψ are real functions on M. The second prolongation (see [2]) of v is a vector field in :

$$pr^{(2)}v = v + \sum_J \psi_J\left(x, \nu^{(2)}, \lambda^{(2)}\right)\frac{\partial}{\partial \nu^J} + \sum_J \phi_J\left(x, \nu^{(2)}, \lambda^{(2)}\right)\frac{\partial}{\partial \lambda^J}$$

where

$$\psi_J\left(x, \nu^{(2)}, \lambda^{(2)}\right) = D^J\left(\psi - \xi\nu'\right) + \xi\nu^{(J+1)}$$
$$\phi_J\left(x, \nu^{(2)}, \lambda^{(2)}\right) = D^J\left(\phi - \xi\lambda'\right) + \xi\lambda^{(J+1)}$$

D^J is J-th total derivative and J is integer $1 \leq J \leq 2$.

$$\left(\frac{\nu^{'}}{x} - \frac{1}{x^2}\right)\lambda^{''} + \frac{\nu^{'3} + \nu^{'2}\lambda^{'} - 2\nu^{'}\lambda^{'2}}{x} + \frac{-\frac{3}{2}\nu^{'2} + \nu^{'}\lambda^{'} + \frac{1}{2}\lambda^{'2}}{x^2}$$
$$+\left(1 - e^{2\lambda}\right)\left(\frac{\lambda^{'}}{x^3} - \frac{1}{x^4}\right) + e^{2\lambda}\left(\frac{\nu^{'}}{x} + \frac{1 - e^{2\lambda}}{2x^2}\right) = 0$$
$$-2e^{-2\lambda}\left\{\nu^{''} + \nu^{'2} - \nu^{'}\lambda^{'} + 2\frac{\nu^{'} - \lambda^{'}}{x} + \frac{1}{x^2}\right\} + \frac{2}{x^2} = 0$$

Conditions of being v a symmetry generator are:

$$pr^{(2)}v(\psi_1) = 0$$
$$pr^{(2)}v(\psi_2) = 0$$

Solving the above system of partial differential equations one obtains that the only one symmetry vector field is

$$v = const\,\frac{\partial}{\partial\nu}$$

which can be found at first sight since there is no variable ν in the equations of the system but only $\nu^{'}$ and $\nu^{''}$. This is insufficient to make a reduction procedure to the system. The second order system not possesing Lie point symmetries may be rich in nonlocal symmetries providable one route for integration.

Summary

- When a direct approach to differential equations fails one can use the Lie symmetry method, which is probably the most powerful one. In case of perfect fluid or conformal gravity existence of symmetries appears to be very helpful for solving those difficult equations.
- In Jordan Frame the field equations have trivial Lie point symmetries, but one expects there exist non-trivial generalized Lie Backlund symmetries, however they are much more difficult to detect.
- The field equations in Jordan frame turn out to be a slight generalization of the Klein-Gordon equations for the traceless part of the Ricci tensor $S^{\mu}{}_{\nu} = R^{\mu}{}_{\nu} - \frac{1}{4}R\delta^{\mu}{}_{\nu}$ + coupling to the conformal curvature.
- Therefore we hope to understand what is the dynamics of the spin-2 field in Helmholtz-Jordan frame at least close to the Minkowski spacetime
- From algebraic structure of the field equations we understand why we did not detect non-trivial Lie symmetries.
- We know from the Painleve analysis that there are more general integrable cases than only Schwarzschild solutions

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