AdS/CFT and Second Order Viscous Hydrodynamics

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XLVII Course
Zakopane, 20.06.2007

Based on [hep-th/0703243] (MH and Romuald A. Janik)
Roadmap

1. Boost-invariant energy-momentum tensors
   - Bjorken hydrodynamics
   - Dissipative hydrodynamics

2. Einstein and string frame

3. Gravity duals
   - Holography
   - Perfect fluid metric
   - Subasymptotic corrections

4. AdS/CFT and Israel-Stewart theory
   - Third order solution
   - Calculation of relaxation time
   - Discussion

5. Summary and Outlook
• matter created in RHIC is strongly coupled and deconfined
• studying dynamics of $\mathcal{N} = 4$ SYM may be relevant
• let’s use AdS/CFT...
• to study the dynamics on energy-momentum tensor
• ...bearing in mind differences:
  • coupling does not run
  • no hadrons
• simplest dynamics to study
  1D expansion + boost invariance (Bjorken)

From now on we are only within $\mathcal{N} = 4$ SYM plasma!
Boost-invariant energy-momentum tensors

- no dependence on transverse coordinates $x^{2,3}$
- 3 nonzero components $T_{\tau\tau}$, $T_{yy}$, $T_{xx} = T_{x^2x^2} = T_{x^3x^3}$
- boost invariance forces $T_{\mu\nu}(\tau, y) = T_{\mu\nu}(\tau)$
- constraints on energy-momentum $T_{\mu\nu}$ dynamics:
  - conservation $\tau \frac{d}{d\tau} T_{\tau\tau} + T_{\tau\tau} + \frac{1}{\tau^2} T_{yy} = 0$
  - tracelessness $T_{\tau\tau} + \frac{1}{\tau^2} T_{yy} + 2T_{xx} = 0$

- $T_{\mu\nu}$ can be expressed in terms of a single function

$$\epsilon(\tau) = T_{\tau\tau}$$

- $\epsilon(\tau)$ is plasma’s energy density

- perfect fluid case: $\epsilon \sim \frac{1}{\tau^{4/3}}$
Second order viscous hydrodynamics

- Perfect fluid - equation of motion for energy density

\[ \partial_\tau \epsilon = -\frac{\epsilon + p}{\tau} = -\frac{4}{3} \frac{\epsilon}{\tau} \]

- we want to include **dissipative corrections**

- equations of motion in Bjorken regime read

\[ \partial_\tau \epsilon = -\frac{4}{3} \frac{\epsilon}{\tau} + \frac{\Phi}{\tau} \]

\[ \tau_\pi \partial_\tau \Phi = -\Phi + \frac{4}{3} \frac{\eta}{\tau} \]

- in hydrodynamical simulation formula

\[ \tau_\pi^{Boltzmann} = 3 \frac{\eta}{2 \rho} \]

is commonly used

- **assumption:** \( \tau_\pi = r \tau_\pi^{Boltzmann} \) holds for \( \mathcal{N} = 4 \) SYM plasma

- our goal is to calculate coefficient \( r \)
How to determine energy density $\epsilon(\tau)$ and relaxation time $\tau_\pi$ of $\mathcal{N} = 4$ SYM from AdS/CFT correspondence?
String and Einstein frames

Let’s consider scalar field (dilaton) coupled to gravity

• equations of motion

\[ G_{\alpha\beta} = R_{\alpha\beta} + 4g_{\alpha\beta} - \frac{1}{2} \partial_{\alpha} \phi \partial_{\beta} \phi = 0 \]
\[ \Box \phi = 0 \]

• there are two equivalent frames
  • Einstein frame, with \textbf{metric} \( g_E \) entering directly above eqns
  • string frame, with \textbf{rescaled metric} \( g_s = e^{\frac{1}{2} \phi} g_E \)

• differ only by local rescaling, crucial later on

• \textbf{string frame - curved geometry probed by string}
How to extract $\langle T_{\mu\nu} \rangle$ from 5D geometry?

We need to adopt Fefferman-Graham coordinates

$$ds^2 = \frac{\tilde{g}_{\mu\nu} dx^\mu dx^\nu + dz^2}{z^2}$$

where

$$\tilde{g}_{\mu\nu} dx^\mu dx^\nu = -e^{a(t,z)} d\tau^2 + \tau^2 e^{b(\tau,z)} dy^2 + e^{c(\tau,z)} dx_\perp^2$$

Near-boundary metric expansion takes the form (only even powers of $z$)

$$\tilde{g}_{\mu\nu} = \tilde{g}_{\mu\nu}^{(0)} + z^2 \tilde{g}_{\mu\nu}^{(2)} + z^4 \tilde{g}_{\mu\nu}^{(4)} + O(z^6)$$

where

- $\tilde{g}_{\mu\nu}^{(0)} = \eta_{\mu\nu}$ (4D Minkowski metric)
- $\tilde{g}_{\mu\nu}^{(2)} = 0$ (consistency condition)
- $\tilde{g}_{\mu\nu}^{(4)} = \frac{N_c^2}{2\pi^2} \langle T_{\mu\nu} \rangle$ (VEV of energy-momentum tensor)

Does the gravity in the bulk specifies uniquely $\langle T_{\mu\nu} \rangle$?
Reproducing 5D geometry from field theory data

Task:
- find a solution of Einstein eqns

\[ R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} + \Lambda \left( = -6 \right) g_{\alpha\beta} = 0 \]
- with boundary conditions

\[ \tilde{g}_{\mu\nu} = g_{\mu\nu}^{0} \left( = \eta_{\mu\nu} \right) + z^{4} g_{\mu\nu}^{(4)} \left( = \frac{N_{c}^{2}}{2\pi^{2}} \langle T_{\mu\nu} \rangle \right) + O(z^{6}) \]

Solution:
- one can iteratively find higher order terms \( \tilde{g}_{\mu\nu}^{(i)} \) (constraints!)
- asymptotic large proper time formula for energy density \( \epsilon \sim \frac{1}{\tau^{s}} \), where \( 0 < s < 4 \) (energy positivity)
- instead of iterating, let’s introduce scaling variable \( \nu = \frac{z}{\tau^{s/4}} \)
- keeping \( \nu \) fixed while \( \tau \to \infty \) reduces Einstein eqns to ODEs
- these can be solved, but how to determine \( s \)?
Regularity of $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ chooses energy density $\epsilon = \frac{e}{\tau^{4/3}}$

Asymptotic geometry looks like

$$ds^2 = \frac{1}{z^2} \left( -\frac{1 - \frac{e}{3} \frac{z^4}{\tau^{4/3}}}{1 + \frac{e}{3} \frac{z^4}{\tau^{4/3}}} d\tau^2 + (1 + \frac{e}{3} \frac{z^4}{\tau^{4/3}})(\tau^2 dy^2 + dx_\perp^2) + dz^2 \right)$$

Similar to standard AdS-Schwarzschild, but with horizon ”moving away”

$$z_0 = \left(\frac{3}{e}\right)^{1/4} \tau^{1/3}$$

**Naive** extraction of thermodynamical quantities

- $T \sim \frac{1}{z_0} \sim \tau^{-1/3}$ (temperature)
- $S \sim \text{AREA} \sim \frac{\tau}{z_0^3} \sim \text{const}$ (entropy per u. rapidity and area$_\perp$)

(Janik, Peschanski [hep-th/0512162])
Subasymptotic solution

- we start with
  \[ a(\tau, z) = a_0(v) + \frac{1}{\tau^{2/3}} a_1(v) + \frac{1}{\tau^{4/3}} a_2(v) + \ldots \]
  where \( v = \frac{z}{\tau^{1/3}} \). Similar relations for \( b(\tau, z) \) and \( c(\tau, z) \)

- rescaling Einstein tensor
  \[ G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} - 6 g_{\alpha\beta} \]
  \[ \tilde{G} = (\tau^{2/3} G_{\tau\tau}, \tau^{4/3} G_{\tau z}, \tau^{-4/3} G_{yy}, \tau^{2/3} G_{xx}, \tau^{2/3} G_{zz}) \]
  leads to systematic expansion in powers of \( \frac{1}{\tau^{2/3}} \)
  \[ \tilde{G} = \tilde{G}_0(v) + \frac{1}{\tau^{2/3}} \tilde{G}_1(v) + \frac{1}{\tau^{4/3}} \tilde{G}_2(v) + \ldots \]

- solving problem in perturbative manner:
  \[ \tilde{G}_i = 0 \]
• **first order** \( \left( \frac{1}{\tau^{2/3}} \right) \) solution reads

\[
a_1(v) = 2\eta_0 \frac{(9+v^4)v^4}{9-v^8}
\]
\[
b_1(v) = -2\eta_0 \frac{v^4}{3+v^4} + 2\eta_0 \log \frac{3-v^4}{3+v^4}
\]
\[
c_1(v) = -2\eta_0 \frac{v^4}{3+v^4} - \eta_0 \log \frac{3-v^4}{3+v^4}
\]

• giving **regular Riemann squared** \( \mathcal{R}^2 \)

• second order solution = lengthy formulas with two parameters
  • \( \eta_0 \) (viscosity coefficient)
  • \( C \) (needed to calculate relaxation time)

• Riemann squared takes the form

\[
\mathcal{R}^2 = \# + \frac{1}{\tau^{4/3}} \frac{\text{polynomial in } v, \eta_0 \text{ and } C}{(3-v^4)^4(3+v^4)^6}
\]

• is nonsingular only for

\[
\eta_0 = \frac{1}{2^{1/4} 3^{3/4}} \quad \text{(Janik [hep-th/0710144])}
\]

• \( C \) cannot be determined in this order
Third order solution

- Riemann squared up to this order gives $\Re^2 = \# + \frac{1}{\tau^2} \left( \text{polynomial in } \nu \text{ and } C \right) + 8 \frac{\tau^{1/2} 3^{3/4} \log (3^{1/4} - \nu)}{\nu - 3^{1/4}} + \ldots$

- cancelation of 4th order pole at $\nu = 3^{1/4}$ fixes $C = -\frac{17 + 6 \log 2}{3^{1/2}}$

- logarithmic singularity survives

- idea = in string frame dilaton contribution cancels singularity

- indeed the case for $\phi = \frac{1}{\tau^2} \frac{1}{14} 2^{1/2} 3^{3/4} \log \frac{3 - \nu^4}{3 + \nu^4}$

- regularity restored, but in the string frame
Relaxation time in $\mathcal{N} = 4$ SYM

- we assume
  \[ \tau_\pi = r \tau_\pi^{\text{Boltzmann}} = \frac{3}{2} \frac{r \eta}{p} \]
- in the leading order
  \[ \epsilon \sim \frac{1}{\tau^{4/3}} \quad \text{and} \quad \eta \sim \frac{1}{\tau} \quad \text{so let's write} \]
  \[ \eta = A \epsilon^{3/4} \]
- goal = determine $A$ and $r$
- second order dissipative hydrodynamics gives
  \[ -\frac{4A \epsilon^{3/4}}{\tau} + \frac{4\epsilon}{3} + \tau \epsilon' + \frac{21Ar \epsilon'}{2\epsilon^{1/4}} + \frac{9Ar \tau \epsilon''}{2\epsilon^{1/4}} = 0 \]
- vanishing of this expression for energy density
  \[ \epsilon = \frac{1}{\tau^{4/3}} - \frac{\sqrt{2}}{3^{3/4}\tau^2} + \frac{1 + 2 \log 2}{12\sqrt{3}\tau^{8/3}} \]
  requires
  \[ r = \frac{1 - \log 2}{9} \quad \text{and} \quad A = \frac{1}{\sqrt{2}3^{3/4}} \]
- relaxation times than takes the form
  \[ \tau_\pi = \frac{1 - \log 2}{6\pi T} \quad (T - \text{temperature}) \]
• nontrivial dilaton profile leads to
  \[ \text{tr } F^2 < 0 \]

• this means that
  \[ \langle \vec{E}^2 \rangle \neq \langle \vec{B}^2 \rangle \]

• in fact, magnetic modes dominate

• relaxation time is almost 30 x shorter than weak coupling approximation
Summary

• studying 1D expansion of strongly coupled plasma using AdS/CFT

• regularity of dual geometry chooses the physical behavior

• results are consistent with second order dissipative hydrodynamics

Perspectives

• applying dynamical horizons framework to calculate the entropy (work in progress)

• determining short time behavior from geometry regularity (work in progress)

• generalizing dynamics to less symmetric situation