Differentiability does matter...

Curious Spacetime Singularities

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June 16, 2007

- We shall be discussing van Stockum spacetimes of rotating dust. In this case Einstein's Eq. can be solved exactly.
- Without careful analysis a class of solutions may be wrongly interpreted as star-like rotating objects. Bonnor's solution is an example. In fact, global van Stockum flow cannot be considered astrophysically interesting.
- Recently van Stockum flow has been used in a linear approximation by Cooperstock (Cooperstock, Tieu, astro-ph/0507619) as a relativistic model of rotation of spiral galaxies. He claimed the model could be used to explain rotation curves without dark matter. Actually, even better results can be obtained in the framework of Newtonian dynamics at least for spiral galaxies that can not be immersed in a massive spherical halo (Bratek, Jalocha, Kutschera, astro-ph/0611113).

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Asymptotically flat Bonnor spacetime of rotating dust

considered nonsingular – W. B. Bonnor, *Gen. Relativ. Gravit.* **37**, 12, 2245 (2005)

- has a physically acceptable proper energy distribution

$$\mathcal{D} = \frac{\mu\, a^3}{\pi} \cdot \frac{\left(\rho^2 + 4\,(a + |z|)^2\right)}{\left(\rho^2 + (a + |z|)^2\right)^4} e^{-2\Psi}, \quad \mu := \int \rho e^{2\Psi} \mathcal{D} \mathrm{d}\rho \mathrm{d}\phi \mathrm{d}z$$

- Einstein's Eq. imply that $R = 8\pi\mathcal{D}$
  the other curvature invariants are continuous and bounded, as well
- structure functions defining the spacetime geometry

$$K = \frac{\rho^2 \sqrt{8\mu\, a^3}}{\left(\rho^2 + (a + |z|)^2\right)^{3/2}}, \qquad \Psi = \frac{\mu\, a^3}{2} \frac{\rho^2 \left(\rho^2 - 8\,(a + |z|)^2\right)}{\left(\rho^2 + (|z| + a)^2\right)^4}$$

  and the corresponding line element

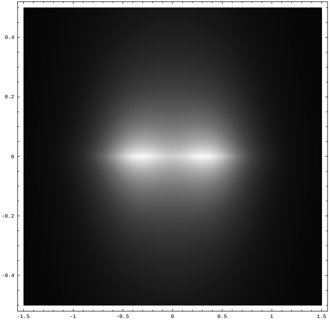
$$\mathrm{d}s^2 = -\mathrm{d}t^2 + 2K(\rho, z)\mathrm{d}t\mathrm{d}\phi + (\rho^2 - K^2(\rho, z))\mathrm{d}\phi^2 + e^{2\Psi(\rho,z)} \left(\mathrm{d}\rho^2 + \mathrm{d}z^2\right)$$

- asymptotic expansion of the line element

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \sqrt{2\, a^3 \mu}\, \frac{4 \sin^2\theta}{r} \mathrm{d}t\mathrm{d}\phi + \mathrm{d}r^2 + r^2 \left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\right)$$

- total angular momentum $J = \sqrt{2\, a^3 \mu}$
- and total mass $M = 0$!!!

Figure: density diagram of proper energy distribution (or of curvature scalar)



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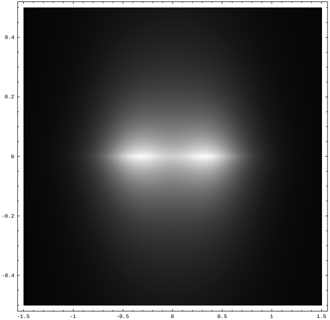

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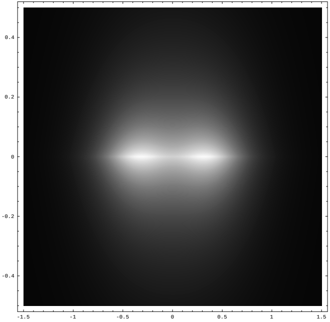

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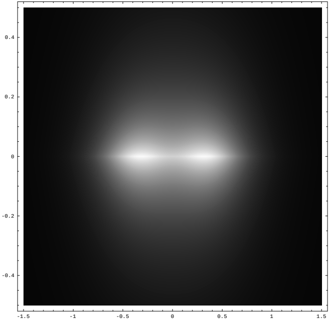

- Time translation symmetry + Axial symmetry

$$\Rightarrow \qquad \exists\,\xi \quad \& \quad \exists\,\eta, \qquad (\textit{Killing vectors})$$

- these symmetries should commute

$$[\xi,\eta]=0 \quad \Leftrightarrow \quad \xi^\nu\partial_\nu\eta^\mu=\eta^\nu\partial_\nu\xi^\mu$$

- Asymptotic Flatness
- Dust matter (pressureless perfect fluid)

$$T_{\mu\nu}=\mathcal{D}u_\mu u_\nu$$

- Our Aim Find spacetimes of dust moving along trajectories of the time translation Killing field

$$u^\mu=Z\xi^\mu, \quad Z^{-1}=\sqrt{-\xi^\alpha\xi_\alpha}$$

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van Stockum spacetimes – an equivalent definition

another formulation 1937: W.J. van Stockum, *Proc. Roy. Soc. Edin.* **57**, 135 (1937)

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- the unique algebraic decomposition of $\nabla_\mu u_\nu$
  $h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$ (projector), $u^\mu u_\mu = -1$
  - the symmetric and traceless part: shear tensor (local distorsions)

$$\sigma_{\mu\nu} = \frac{1}{2}\left(\nabla_\alpha u_\beta + \nabla_\beta u_\alpha\right) h^\alpha{}_\mu h^\beta{}_\nu - \frac{1}{3}\nabla_\alpha u^\alpha h_{\mu\nu}$$

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Characteristics of van Stockum flow

via algebraic decomposition of $\nabla_\mu u_\nu$ for a general flow

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Theorem (on the line element of stationary and axially symmetric spacetime)

1. *Asymptotic flatness* $\Rightarrow$ $\eta = 0$ *on the symmetry axis at least for radii sufficiently large*

   $\Rightarrow$ *i)* $\eta_{[\alpha}\xi_{\beta}\xi_{\mu;\nu]} = 0$ *and* $\xi_{[\alpha}\eta_{\beta}\eta_{\mu;\nu]} = 0$ *at least at one point*

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3. *commutativity of symmetries*

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structure functions V, K, Ψ, Λ *depend only on* ρ *and* z

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1. $T_{\mu\nu} = \mathcal{D}Z^2\xi_\mu\xi_\nu, \qquad \xi^\mu\mathcal{D}_{,\mu} = 0 = \eta^\mu\mathcal{D}_{,\mu}$
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1. Let $E^{\mu}{}_{\nu} := R^{\mu}{}_{\nu} - \frac{1}{2}R\delta^{\mu}{}_{\nu} - 8\pi T^{\mu}{}_{\nu}$, then $E^{\rho}{}_{\rho} = 0$ and $E^{\rho}{}_{z} = 0$ yield

$$\Psi_{,\rho} = \frac{K_{,z}^2 - K_{,\rho}^2}{4\rho}, \qquad \Psi_{,z} = -\frac{K_{,\rho}K_{,z}}{2\rho}$$

2. integrability condition $\Psi_{,\rho z} = \Psi_{,z\rho}$ imposes on K the elliptic constraint

$$\mathcal{L}K = 0, \qquad \mathcal{L} = \partial_\rho^2 - \frac{1}{\rho}\partial_\rho + \partial_z^2$$

3. then, the other components of $E^{\mu}{}_{\nu}$, but $E^{t}{}_{t}$ and $E^{t}{}_{\phi}$, vanish identically.
4. The latter two will also vanish for $\mathcal{C}^2$ solutions if only

$$\mathcal{D} = e^{-2\Psi}\frac{K_{,\rho}^2 + K_{,z}^2}{8\pi\rho^2} \geqslant 0$$

- the elliptic constraint very easy to solve
- energy density positive definite

too beautiful to be true?... Yes

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1. Let $E^{\mu}{}_{\nu} := R^{\mu}{}_{\nu} - \frac{1}{2}R\delta^{\mu}{}_{\nu} - 8\pi T^{\mu}{}_{\nu}$, then $E^{\rho}{}_{\rho} = 0$ and $E^{\rho}{}_{z} = 0$ yield

$$\Psi_{,\rho} = \frac{K_{,z}^2 - K_{,\rho}^2}{4\rho}, \qquad \Psi_{,z} = -\frac{K_{,\rho}K_{,z}}{2\rho}$$

2. integrability condition $\Psi_{,\rho z} = \Psi_{,z\rho}$ imposes on K the elliptic constraint

$$\mathcal{L}K = 0, \qquad \mathcal{L} = \partial_\rho^2 - \frac{1}{\rho}\partial_\rho + \partial_z^2$$

3. then, the other components of $E^{\mu}{}_{\nu}$, but $E^{t}{}_{t}$ and $E^{t}{}_{\phi}$, vanish identically.
4. The latter two will also vanish for $\mathcal{C}^2$ solutions if only

$$\mathcal{D} = e^{-2\Psi}\frac{K_{,\rho}^2 + K_{,z}^2}{8\pi\rho^2} \geqslant 0$$

- the elliptic constraint very easy to solve
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example

- $U = -\frac{GM}{r}$ is a smooth solution of $\nabla^2 U = 0$ for $r \neq 0$
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Theorem

$\mathcal{I}$ is nonempty. There are no asymptotically flat van-Stockum spacetimes with globally positive definite energy. Asymptotically flat van-Stockum spacetimes must contain curvature singularities with negative active masses.

Proof.

- Let's suppose that $\mathcal{I} = \emptyset$
- inside a ball $\mathcal{B}_{\mathrm{R}} \subset \mathbb{R}^3$ bounded by a two-sphere $\mathcal{S}_{\mathrm{R}}$ of radius R and centered at the origin

$$\int_{\mathcal{B}_{\mathrm{R}}} \mathcal{D}e^{2\Psi}\rho \mathrm{d}\rho\wedge\mathrm{d}\phi\wedge\mathrm{d}z \overset{\mathcal{L}K=0}{=} \frac{1}{8\pi}\int_{\mathcal{S}_{\mathrm{R}}} \frac{K}{\rho}\left(K_{,z}\mathrm{d}\rho - K_{,\rho}\mathrm{d}z\right)\wedge\mathrm{d}\phi \equiv \frac{1}{8\pi}\int_{\mathcal{S}_{\mathrm{R}}} \frac{K\,\partial_r K}{\sin\theta}\mathrm{d}\phi\wedge\mathrm{d}\theta$$

in virtue of the Stokes theorem, provided $(K^2)_{,r} = o(\sin\theta)$, $(r\sin\theta = \rho,\ r\cos\theta = z)$

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1. Total mass of asymptotically flat van Stockum spacetimes is zero

$$M=-\frac{1}{8\pi}\lim_{R\to\infty}\int_{\mathcal{S}_R}\frac{1}{2}\sqrt{-g}\epsilon_{\alpha\beta\mu\nu}\nabla^{\mu}\xi^{\nu}\mathrm{d}x^{\alpha}\wedge\mathrm{d}x^{\beta}\equiv\lim_{R\to\infty}\frac{1}{8\pi}\int_{\mathcal{S}_R}\frac{K\,\partial_r K}{\sin\theta}\mathrm{d}\phi\wedge\mathrm{d}\theta=0$$

2. For any smooth Ψ and K satisfying $\Psi_{,\rho}=\frac{K_{,z}^2-K_{,\rho}^2}{4\rho}$ and $\Psi_{,z}=-\frac{K_{,\rho}K_{,z}}{2\rho}$
   - the trace of spatial stresses

   $$\widetilde{\mathcal{S}}=T_{\mu\nu}\left(u^{\mu}u^{\nu}+g^{\mu\nu}\right)=-e^{-2\Psi}\frac{K_{,\rho}}{16\pi\rho}\mathcal{L}K,$$

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   $$\widetilde{\mathcal{D}}=T_{\mu\nu}u^{\mu}u^{\nu}=\mathcal{D}-\widetilde{\mathcal{S}}$$

   - Tolman's active mass density on a hypersurface of constant time

   $$\widetilde{\mathcal{D}}_T=(8\pi)^{-1}R^{t}{}_{\mu}\xi^{\mu}=\mathcal{D}+e^{-2\Psi}\frac{K}{8\pi\rho^2}\mathcal{L}K$$

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   $$\widetilde{R}=R^{\mu}{}_{\mu}=8\pi\left(\mathcal{D}-2\widetilde{\mathcal{S}}\right)$$

3. In the regularity region $\mathbb{R}^3\setminus\mathcal{I}$:

$$\widetilde{\mathcal{D}}=\widetilde{\mathcal{D}}_T=D=(8\pi)^{-1}\widetilde{R},\quad\text{and}\quad\widetilde{\mathcal{S}}=0,$$

like for dust, and Einstein's equations are equivalent to our reduced set

On nonexistence... Additional sources of negative active mass

required to concord with the full set of Einstein's Eq.

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However,

- For regularized profiles K^{ϵ} we get

$$\lim_{\epsilon\to 0}\int_{\mathcal{I}_\delta\supset\mathcal{I}}\widetilde{S}\neq 0$$

-

$$\widetilde{R}=8\pi\left(\mathcal{D}-2\widetilde{\mathcal{S}}\right)\qquad\Rightarrow\qquad\int_{\mathbb{R}^3}\widetilde{R}\neq\int_{\mathbb{R}^3\setminus\mathcal{I}}\widetilde{R}=8\pi\int_{\mathbb{R}^3}\mathcal{D},$$

$$\Rightarrow\qquad\widetilde{R}=8\pi\mathcal{D}+\gamma_{\mathcal{I}}$$

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An example – Asymptotically flat Bonnor spacetime with an embedded layer of negative active mass

considered nonsingular – W. B. Bonnor, *Gen. Relativ. Gravit.* **37**, 12, 2245 (2005)

- has a physically acceptable proper energy distribution

$$\mathcal{D} = \frac{\mu a^3}{\pi} \cdot \frac{\left(\rho^2 + 4\,(a + |z|)^2\right)}{\left(\rho^2 + (a + |z|)^2\right)^4} e^{-2\Psi}, \quad \mu := \int \rho e^{2\Psi} \mathcal{D} \mathrm{d}\rho \mathrm{d}\phi \mathrm{d}z$$

- and the following structure functions

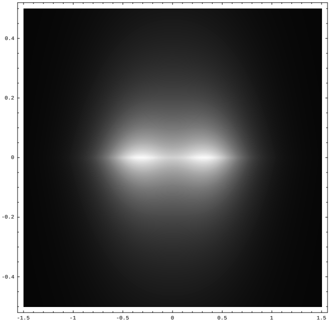
$$K = \frac{\rho^2 \sqrt{8\mu a^3}}{\left(\rho^2 + (a + |z|)^2\right)^{3/2}}, \qquad \Psi = \frac{\mu a^3}{2} \frac{\rho^2 \left(\rho^2 - 8\,(a + |z|)^2\right)}{\left(\rho^2 + (|z| + a)^2\right)^4}$$

- asymptotic expansion of the corresponding line element

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \sqrt{2\,a^3 \mu}\,\frac{4 \sin^2 \theta}{r} \mathrm{d}t \mathrm{d}\phi + \mathrm{d}r^2 + r^2 \left(\mathrm{d}\theta^2 + \sin^2 \theta \mathrm{d}\phi^2\right)$$

- total angular momentum $J = \sqrt{2\,a^3 \mu}$
- and total mass $M = 0$

Figure: density diagram of proper energy distribution (or of curvature scalar)



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$$K_B^{\epsilon}(\rho, z) = \sqrt{8a^3\mu}\cdot\rho^2\cdot\left((a+\sqrt{z^2+\epsilon^2})^2+\rho^2\right)^{-3/2}, \qquad a>0, \quad \mu>0$$

- K_B^{ϵ} is globally $\mathcal{C}^{\infty}$
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the latter equality holds identically for any $\mathcal{C}^2$ asymptotically flat profiles as then

$$8\pi\sqrt{-g}\widetilde{D}_T \mathrm{d}\rho\wedge\mathrm{d}\phi\wedge\mathrm{dz} = \mathrm{d}\left(\rho^{-1}KK_{,z}\mathrm{d}\rho\wedge\mathrm{d}\phi + \rho^{-1}KK_{,\rho}\mathrm{d}\phi\wedge\mathrm{dz}\right)$$

- Since $\int_{\mathbb{R}^3\setminus\mathcal{I}} D = \int_{\mathbb{R}^3} D = \mu \neq \mu/2 = (8\pi)^{-1}\int_{\mathbb{R}^3}\widetilde{R}$, again, the curvature scalar is a distribution and it may be smooth and bounded only outside a measure zero set

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