SOLVING SOME GAUGE SYSTEMS AT INFINITE N

G. Veneziano, J.W.

- **Supersymmetric Yang-Mills quantum mechanics**
- **Planar calculus for the Hamiltonian formalism**
- **A very symmetric supersymmetric system**
- **One and two gluinos:**
  the spectrum, the phase transition, the duality – an exact solution
- **F=2,3 $\mapsto$ arbitrary number of gluinos**
- **The strong ('t Hooft) coupling limit – the magic staircase**
- **Lattice equivalencies: XXZ model, q-boson gas**
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mat-ph/0603082

1 SUPERSYMMETRIC QUANTUM MECHANICS AT FINITE N

- **The Hamiltonian (fix $D = d + 1$ and $N = 2, 3, \ldots$)**

$$H = \frac{1}{2} p^i_a p^i_a + \frac{g^2}{4} \epsilon_{abc} \epsilon_{ade} x^i_b x^j_c x^i_d x^j_e + \frac{ig}{2} \epsilon_{abc} \psi^\dagger_a \Gamma^k \psi_b x^k_c,$$

$$i = 1, .., D - 1 \qquad a = 1, ..., N^2 - 1.$$

- **The Hilbert space - basis - occupation number representation - gauge invariance**

$$|\{n^i_a, \eta^j_c\} >= \sum_{contractions} a^{\dagger i}_{c} a^{\dagger j}_{d} a^{\dagger k}_{e} f^{\dagger m}_{b} f^{\dagger n}_{a} \ldots |0> .$$

- **The cutoff**

$$B = \sum_{b,i} a^{\dagger i}_{b} a^i_b < B_{max}.$$

- **Representation of the Hamiltonian in the cut Fock space**
  $< I|H|J > \Rightarrow$ **spectrum** .

- **Increase B_{max} until results converge.**

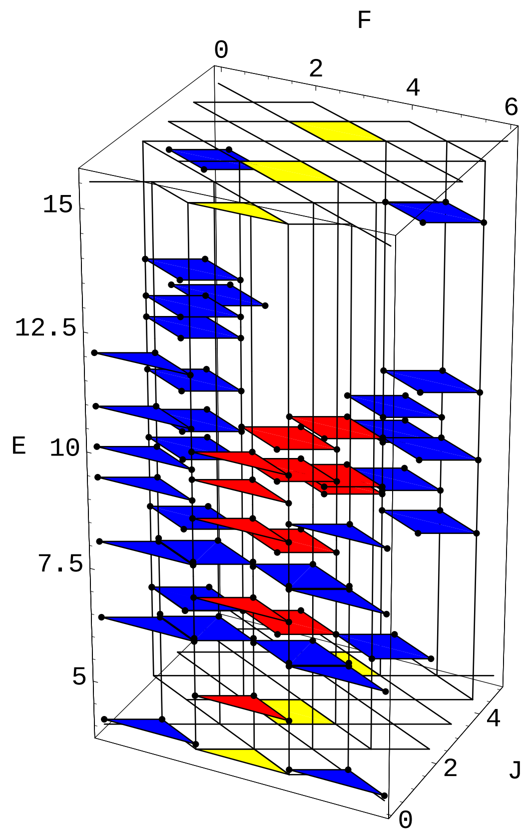


Figure 1: **The spectrum, and its supersymmetry structure, of the three dimensional supersymmetric Yang-Mills quantum mechanics.**

D	2	4	$\ldots$	10
N				
2	$\bullet$	$\bullet$		$\circ$
3	$\bullet$			
.	$\circ$			
.	$\circ$			
.				
∞	$\odot$			M

Table 1:

M. Campostrini, M. Trzetrzelewski, J. Kotanski, P. Korcyl

P. van Baal, R. Janik

G. Veneziano, E.Onofri

2 WHAT IS IT ABOUT ?

- **At large N basis simplifies enormously !**
- **Matrix creation/annihilation operators**

$$a_{ik} = \sqrt{2} a^a T^a_{ik}, \quad f_{ik} = \sqrt{2} f^a T^a_{ik}, \quad i, k = 1, ..., N.$$

- **Gauge invariant elementary building blocks (bricks) for SU(N)**
- **Fermion number** $\Sigma_a {f^\dagger}_a f_a \equiv F = 0$

$$(aa), (aaa), (aaaa), \ldots, (a^N), \quad (.) \equiv Tr[.]$$

- $F = 2$ **for example**

$$(ffa), (fafa), (faafa) ... (fa^{N-1} fa^{N-2}), (ffa...).$$

- **AND all products of lower order bricks !?**

3 PLANAR CALCULUS

- **'t Hooft: Only Feynman planar diagrams contribute to the, leading in N, results (masses, scattering amplitudes, etc.)**
- **Veneziano: Above applies also to states in a Hilbert space !**
- **Matrix elements of simple operators (e.g. 4-th, 6-th order polynomials) can be calculated analytically in the large N limit**

Technology: the Wick theorem and

$$[a_{ik}, {a^\dagger}_{jl}] = \delta_{il}\delta_{kj}$$

EXAMPLE 1: A NORM

A state with n gluons in F=0 sector

$$|n >= \frac{1}{\mathcal{N}_n} Tr[(a^\dagger)^n]|0 > .$$

Its norm

$$\begin{aligned} \mathcal{N}_n^2 &= < 0|Tr[a^n]Tr[(a^\dagger)^n]|0 > \\ &= < 0|(12)(23)...(n1)[1'2'][2'3'] \dots [n'1']|0 >, \\ (12) \equiv a_{i_1 i_2}, &\qquad [12] \equiv a^\dagger{}_{i_1 i_2}. \end{aligned}$$

maximal contribution when $(n1)[1'2']$ are contracted $\Rightarrow$

$$1 \times (\text{a single trace}).$$

The next contraction of nearest-neighbors $a^\dagger a \Rightarrow N$.
Continue $n-2$ times.
The last contraction gives N^2.
n **such contributions (cyclic shift under *one* trace).**

$$\mathcal{N}_n^2 = nN^n.$$

- **Planar 1: single trace states give maximal contribution .**

EXAMPLE 2: A MATRIX ELEMENT

$$H_{n+2,n} = g^2 < n+2|Tr[a^\dagger a^\dagger a^\dagger a]|n > .$$

Act with the operator on an initial state

$$\begin{aligned} Tr[a^\dagger a^\dagger a^\dagger a]Tr[(a^\dagger)^n|0> &= [12][23][34](41)[1'2'][2'3']...[n'1']|0> \\ &= n[1'2][23][32'][2'3'][3'4']...[n'1']|0>, \end{aligned}$$

$$\ldots$$

$$H_{n+2,n} = g^2 N\sqrt{n(n+2)}.$$

$$g^2 N = \lambda \qquad \leftrightarrow \qquad {}'\text{t Hooft coupling}$$

- **Planar 2: The leading operators are again single traces.**

4 ONE SUPERSYMMETRIC HAMILTONIAN

$$\begin{aligned} Q &= \sqrt{2}Tr[fa^\dagger(1+ga^\dagger)] = \sqrt{2}Tr[fA^\dagger], \\ Q^\dagger &= \sqrt{2}Tr[f^\dagger(1+ga)a] = \sqrt{2}Tr[f^\dagger A], \\ H = \{Q, Q^\dagger\} &= H_B + H_F. \end{aligned}$$

$$H_B = a^\dagger a + g({a^\dagger}^2 a + a^\dagger a^2) + g^2 {a^\dagger}^2 a^2.$$

$$\begin{aligned} H_F &= f^\dagger f + g(f^\dagger f(a^\dagger + a) + f^\dagger(a^\dagger + a)f) \\ &+ g^2(f^\dagger a f a^\dagger + f^\dagger a a^\dagger f + f^\dagger f a^\dagger a + f^\dagger a^\dagger f a) \end{aligned}$$

LARGE N MATRIX ELEMENTS OF H

F=0 , n=0,1,2,3, ... only H_B contributes.

$$\begin{aligned} < 0, n|H|0, n > &= (1 + \lambda(1 - \delta_{n1}))n, \\ < 0, n+1|H|0, n > = < 0, n|H|0, n+1 > &= \sqrt{\lambda}\sqrt{n(n+1)}. \end{aligned}$$

F=1 , n=0,1,2,3, ... Both, H_B and H_F contribute.

$$\begin{aligned} < 1, n|H|1, n > &= (1 + \lambda)(n+1) + \lambda, \\ < 1, n+1|H|1, n > = < 1, n|H_2|1, n+1 > &= \sqrt{\lambda}(2 + n). \end{aligned}$$

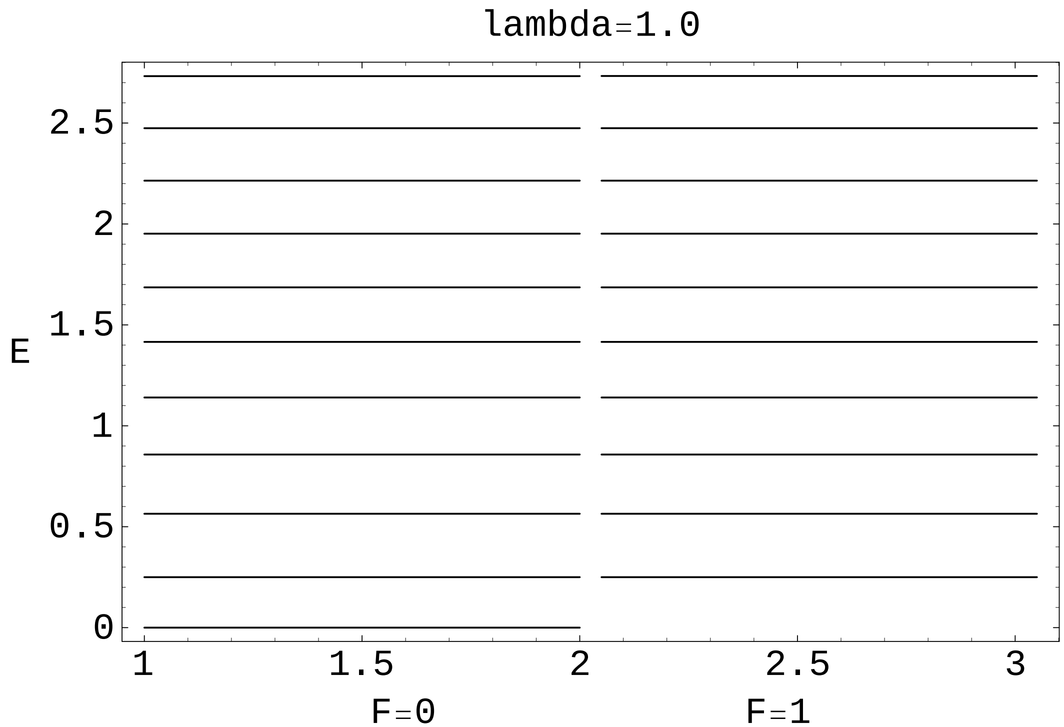


Figure 2: **First 10 energy levels of H in F=0 and F=1 sectors at $\lambda = 1.0$**

SUSY RESTORATION

- **Supersymmetry is unbroken in this model.**
- **Only breaking was due to the cutoff.**
- **Good test of the planar calculus.**

THE SPECTRUM

- **Well defined system for all values of 't Hooft coupling.**
- **Almost equidistant levels**
- **At $\lambda = 0$ - SUSY harmonic oscillators**
- ***All* levels collapse at $\lambda_c = 1$.**

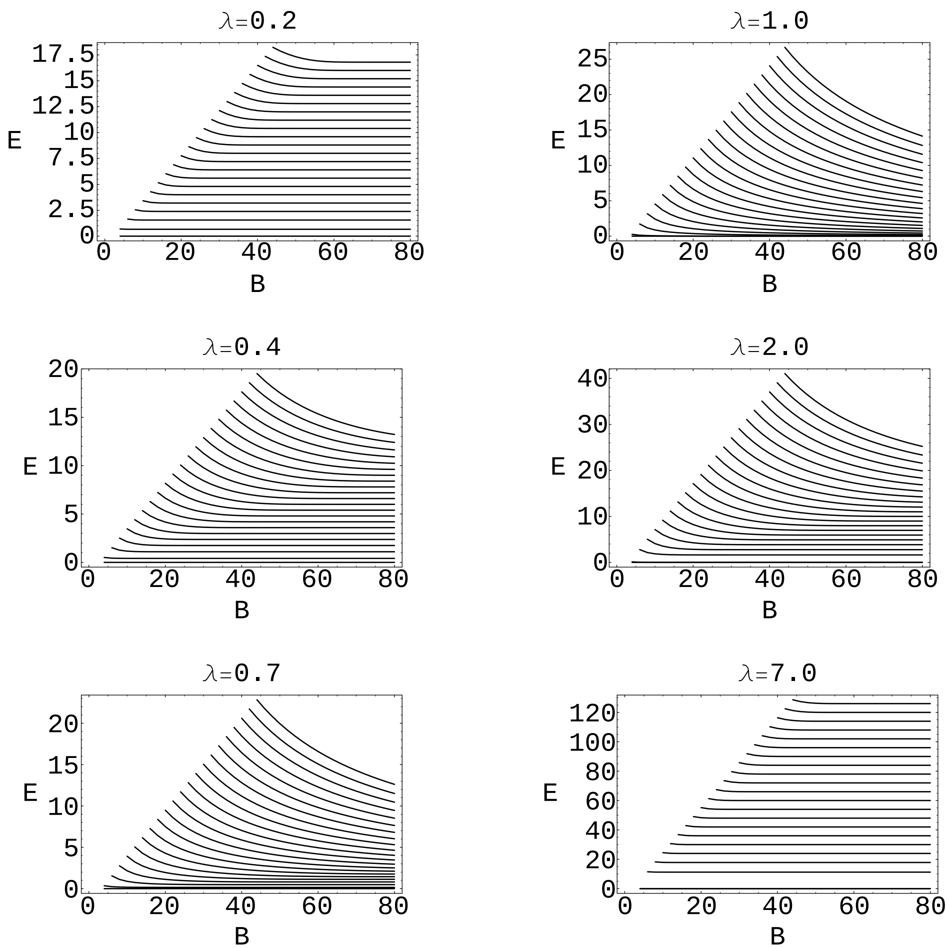
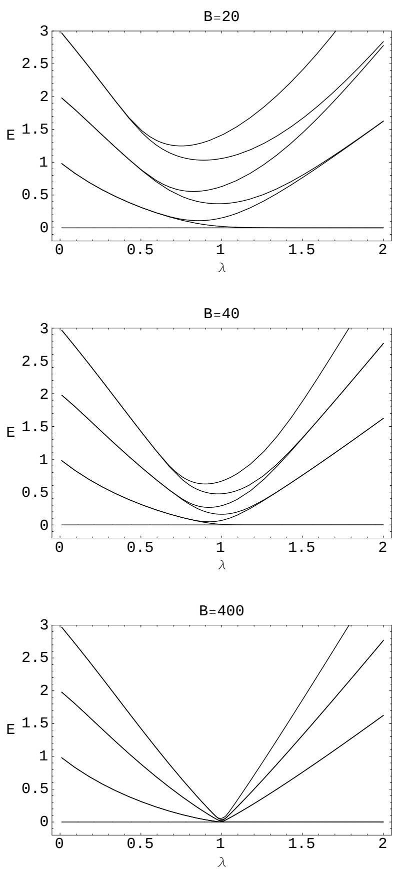


Figure 3: **The cutoff dependence of the spectra of H, in the F=0 sector in a range of λ's**

THE PHASE TRANSITION

- **The critical slowing down**
- **All levels collapse at $\lambda_c = 1$ - the spectrum looses its mass gap - it becomes continuous.**
- **Second ground state with $E = 0$ appears in the strong coupling phase.**
- **Rearrangement of supermultiplets.**
- **Witten index has a discontinuity at λ_c.**
- **The strong - weak duality.**
- **This is not the Gross-Witten phase transition.**


B=20
B=40
B=400
E
λ
0
0.5
1
1.5
2
2.5
3

ANALYTIC SOLUTION

CONSTRUCTION OF THE SECOND GROUND STATE

$$b \equiv \sqrt{\lambda} \tag{1}$$

$$|0\rangle_2 = \sum_{n=1}^{\infty} \left(\frac{-1}{b}\right)^n \frac{1}{\sqrt{n}} |0, n\rangle \ . \tag{2}$$

STRONG/WEAK DUALITY

- **F=0**

$$b\left(E_n^{(F=0)}(1/b) - \frac{1}{b^2}\right) = \frac{1}{b}\left(E_{n+1}^{(F=0)}(b) - b^2\right). \tag{3}$$

- **F=1**

$$b\left(E_n^{(F=1)}(1/b) - \frac{1}{b^2}\right) = \frac{1}{b}\left(E_n^{(F=1)}(b) - b^2\right)$$

4.1 SPECTRUM AND EIGENSTATES

- **The planar basis**

$$|0,n\rangle = \frac{1}{\mathcal{N}_n}(a^{\dagger n})|0\rangle$$

- **A non-orthonormal (but useful) basis:**

$$|B_n\rangle \;=\; \sqrt{n}|n\rangle + b\sqrt{n+1}|n+1\rangle\,.$$

- **The generating function $f(x)$ for the expansion of the eigenstates $|\psi>$ into the $|B_n\rangle$ basis.**

$$f(x) = \sum_{n=0}^{\infty} c_n x^n \qquad \leftrightarrow \qquad |\psi\rangle = \sum_{n=0}^{\infty} c_n |B_n\rangle$$

- **The solution**

$$\begin{aligned} f(x) &= \frac{1}{\alpha}\,\frac{1}{x+1/b}\,F(1,\alpha;1+\alpha;\frac{x+b}{x+1/b}), \quad b<1, \\ f(x) &= \frac{1}{1-\alpha}\,\frac{1}{x+b}\,F(1,1-\alpha;2-\alpha;\frac{x+1/b}{x+b}), \quad b>1, \\ E &= \alpha(b^2-1) \end{aligned}$$

- **The quantization condition**

$$f(0)=0 \;\Rightarrow\; E_n \quad \text{reproduces the numerical eigenvalues of} \quad \langle m|H|n\rangle$$

- **One more check: set $\alpha = 0$ in the $b > 1$ solution.**

$$f_0(x) = \frac{1}{1+bx} \log \frac{b+x}{b-1/b}, \quad b > 1, \tag{4}$$

- **Generates the second vacuum state as it should.**
- ***Cannot* do this for $b < 1$ – there is no such state at weak coupling!**

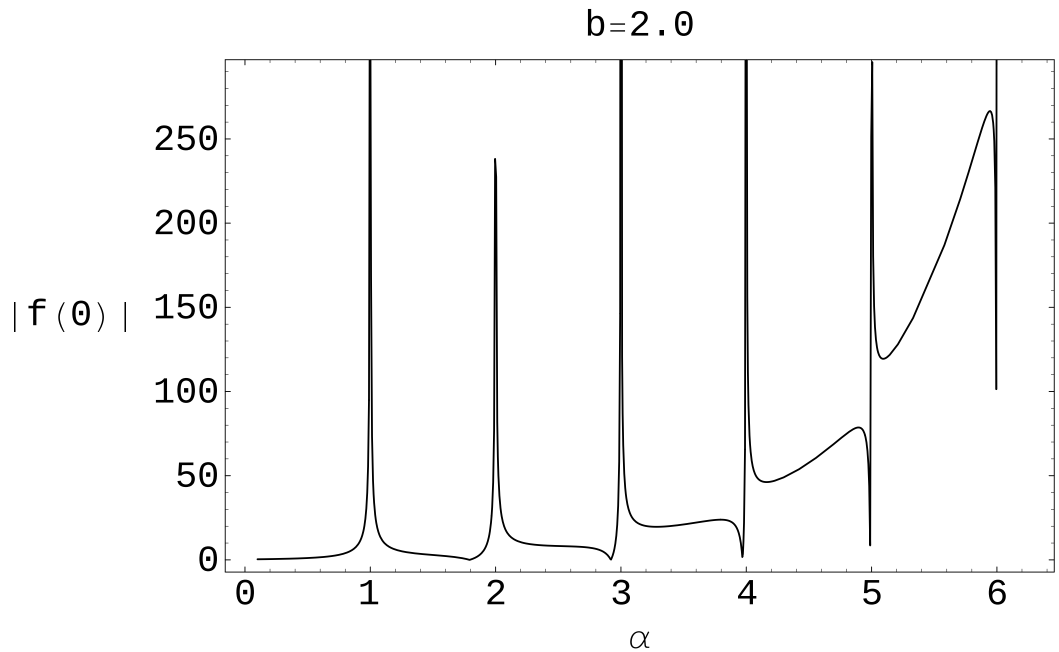


Figure 4: **The absolute value of the LHS of the quantization condition as a function of α. First four zeros are clearly visible. To see higher zeros one needs to increase the α resolution of the plot.**

F=2,3

States are labeled by two integers, n_1, n_2 whose ordering is important modulo a cyclic permutation. Hence we can always take $0 \leq n_1 < n_2$.

$$\langle n_1, n_2|H|n_1, n_2\rangle = (n_1 + n_2 + 2)(1 + b^2) - b^2(2 - \delta_{n_1,0}) - 2b^2\delta_{n_2,n_1+1},$$

$$\begin{aligned} \langle n_1 + 1, n_2|H|n_1, n_2\rangle &= b(n_1 + 2) = \langle n_1, n_2|H|n_1 + 1, n_2\rangle, \\ \langle n_1, n_2 + 1|H|n_1, n_2\rangle &= b(n_2 + 2) = \langle n_1, n_2|H|n_1, n_2 + 1\rangle. \\ \langle n_1 + 1, n_2 - 1|H|n_1, n_2\rangle &= \langle n_1, n_2|H|n_1 + 1, n_2 - 1\rangle \\ &= 2b^2(1 - 2\delta_{n_2,n_1+1}). \end{aligned}$$

Three fermions: states are labeled by three integers

$$|n_1, n_2, n_3\rangle = \frac{1}{\mathcal{N}_{n_1 n_2 n_3}} Tr[a^{\dagger n_1} f^\dagger a^{\dagger n_2} f^\dagger a^{\dagger n_3} f^\dagger]|0\rangle,$$
$$0 \leq n_1, \qquad n_1 \leq n_2, \qquad n_1 \leq n_3.$$

The Hamiltonian matrix

$$\langle n_1, n_2, n_3|H|n_1, n_2, n_3\rangle = (n_1 + n_2 + n_3 + 3)(1 + b^2) - b^2(3 - \delta_{n_1,0} - \delta_{n_2,0} - \delta_{n_3,0}),$$

$$\langle n_1 + 1, n_2, n_3|H|n_1, n_2, n_3\rangle = b(n_1 + 2)\Delta = \langle n_1, n_2, n_3|H|n_1 + 1, n_2, n_3\rangle, \quad \text{plus cyclic}$$

$$\langle n_1 + 1, n_2 - 1, n_3|H|n_1, n_2, n_3\rangle = b^2\Delta = \langle n_1, n_2, n_3|H|n_1 + 1, n_2 - 1, n_3\rangle, \quad \text{plus cyclic.}$$

where $\Delta = 1/\sqrt{3}$ if $n_1 = n_2 = n_3$ and $\Delta = \sqrt{3}$ if the final state is of this form, otherwise $\Delta = 1$.

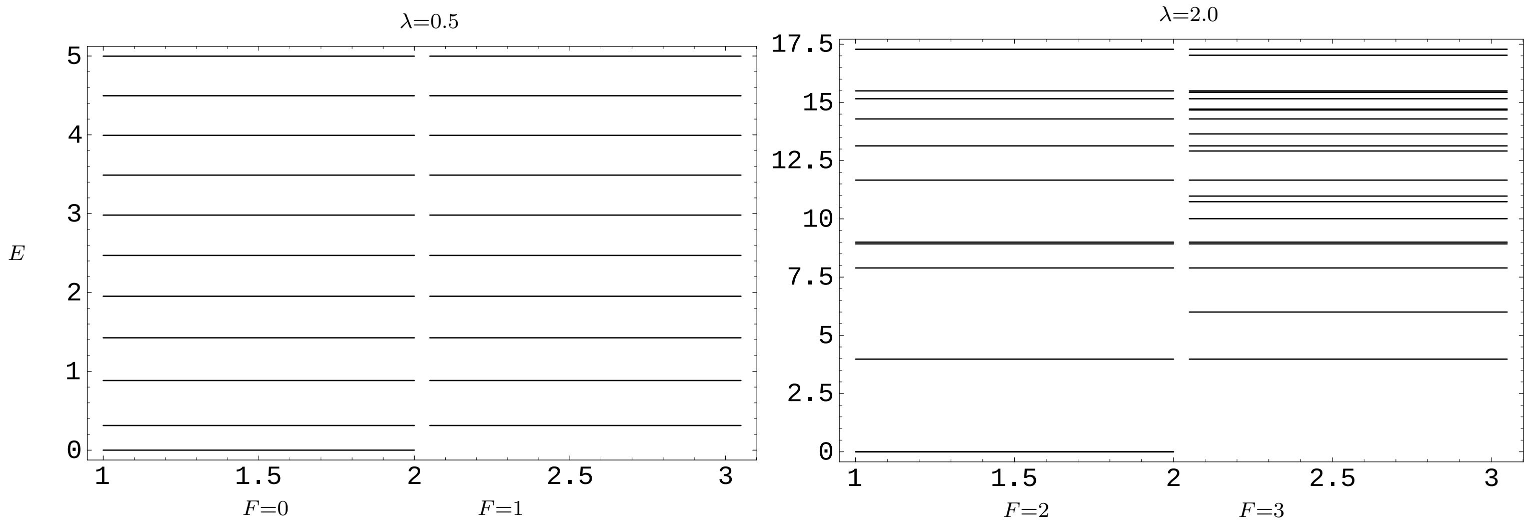


Figure 5: **Low lying bosonic and fermionic levels in the first four fermionic sectors.**

SUPERMULTIPLETS

- **supermultiplets OK**
- **F=(0 - 1) accommodate complete representations of SUSY, but F=(2 - 3) *do not***
- **Richer structure than in 0/1, e.g. not equidistant levels.**

RARRANGEMENT OF F=2 AND F=3 SUPERPARTNERS

- **The phase transition is there, as in 0/1 sectors.**
- **Supermultiplets rearrange across the phase transition point.**
- ***Two new vacua* appear in the strong coupling phase!**
- **The exact construction of both vacua.**

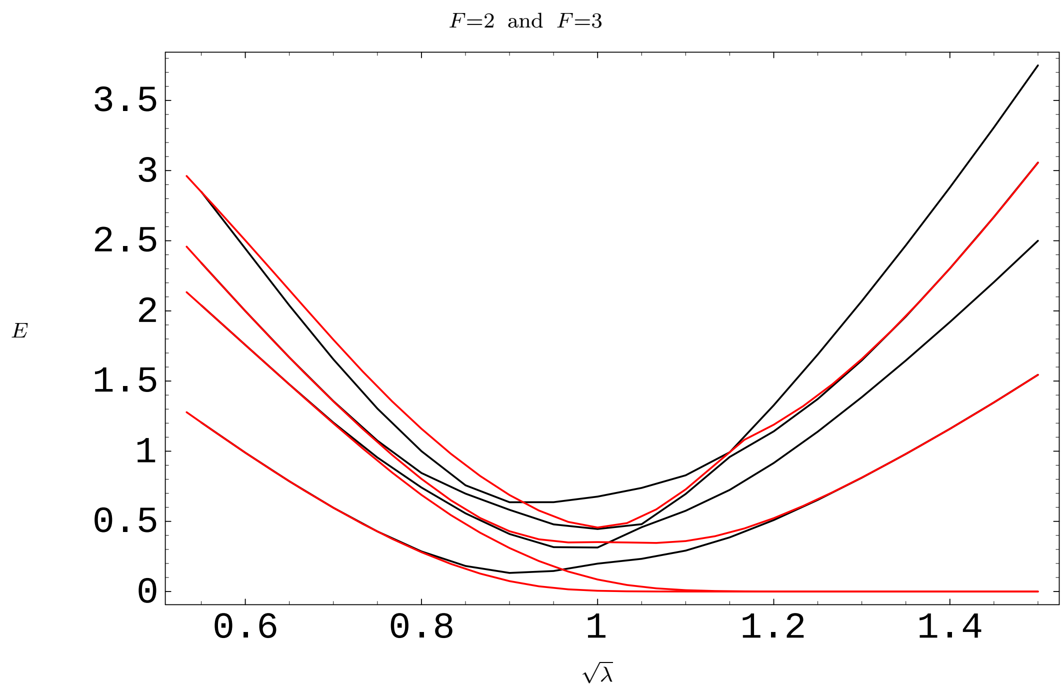


Figure 6: **Rearrangement of the $F = 2$ (red) and $F = 3$ (black) levels while passing through the critical coupling $\lambda_c = 1$.**

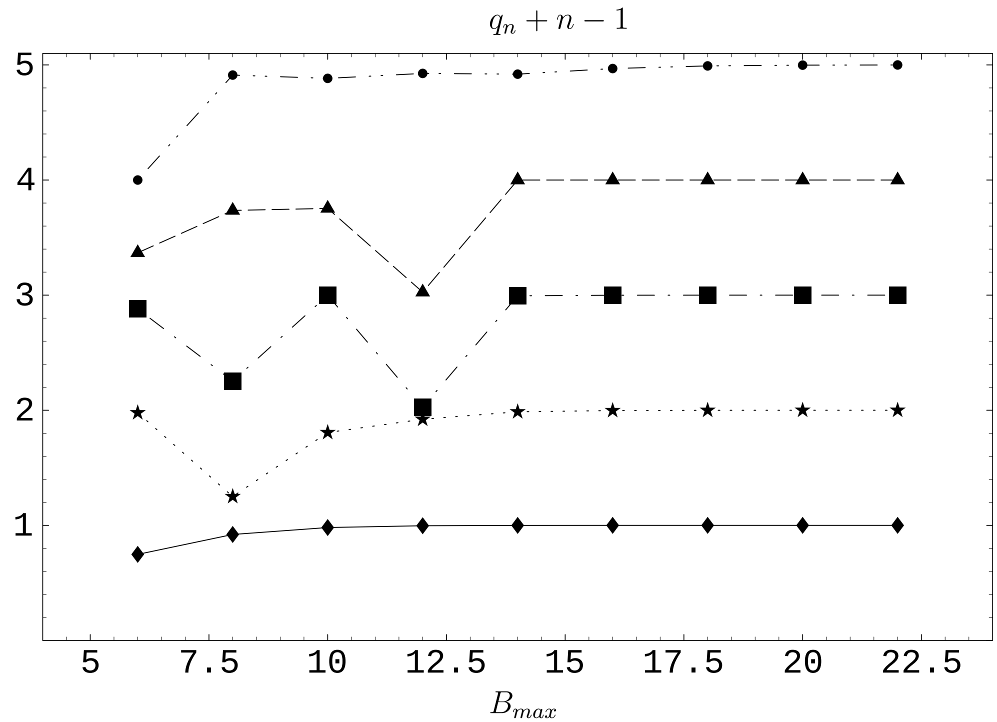


Figure 7: **First five supersymmetry fractions.**

SUPERSYMMETRY FRACTIONS

$$q_{mn} \equiv \sqrt{\frac{2}{E_m + E_n}} < F+1, E_m|Q^{\dagger}|F, E_n > \qquad (5)$$

RESTRICTED WITTEN INDEX

$$W(T,\lambda) = \sum_i (-1)^{F_i} e^{-TE_i}$$

No good when supermultiplets are incomplete (if no SUSY).
New definition - "analytic continuation" into the critical region.

$$W_R(T,\lambda) = \sum_i \left(e^{-TE_i} - e^{-T\bar{E}_i} \right), \quad \bar{E}_i = \frac{\sum_f E_f |q_{fi}|^2}{\sum_f |q_{fi}|^2}$$

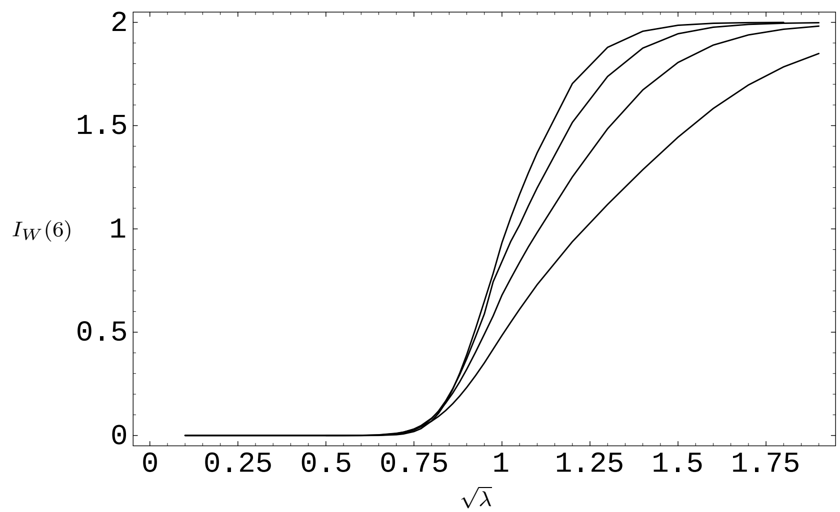


Figure 8: **Behaviour of the restricted Witten index, at $T = 6$, around the phase transition.**

5 ARBITRARY F

States with F fermions are labeled by F bosonic occupation numbers (configurations).

$$|n\rangle = |n_1, n_2, \ldots, n_F\rangle = \frac{1}{\mathcal{N}_{\{n\}}} Tr(a^{\dagger^{n_1}} f^\dagger a^{\dagger^{n_2}} f^\dagger \ldots a^{\dagger^{n_F}} f^\dagger)|0\rangle$$

- **Cyclic shifts give the same state**
- **Pauli principle $\longrightarrow$ some configurations are not allowed, e.g.**

$$\{n, n\}, \qquad \text{or} \qquad \{2, 1, 1, 2, 1, 1\}$$

- **Degeneracy factors**

THE STRONG COUPLING LIMIT

$$
\begin{aligned}
H_{strong} &= \lim_{\lambda\to\infty} \frac{1}{\lambda} H = \\
Tr(f^\dagger f) + & \quad \frac{1}{N}[Tr(a^{\dagger 2} a^2) + Tr(a^\dagger f^\dagger a f) + Tr(f^\dagger a^\dagger f a)].
\end{aligned} \tag{6}
$$

- **It conserves both** F **and** $B = n_1 + n_2 + \ldots + n_F$**.**
- **Still has exact supersymmetry.**
- H_{strong} **is the *finite* matrix in each** (F, B) **sector (c.f. a map of all sectors).**
- **The SUSY vacua are only in the sectors with even F and** $(F, B = F \pm 1)$ **– the magic staircase**

11	1	1	6	26	91	...	...	...	...	...	**16796**
10	1	1	5	22	73	201	497	1144	...	...	...
9	1	1	5	19	55	143	335	715	**1430**	...	**4862**
8	1	1	4	15	42	99	212	429	809	1430	2424
7	1	1	4	12	30	66	**132**	247	**429**	715	1144
6	1	1	3	10	22	42	76	132	217	335	497
5	1	1	3	7	**14**	26	**42**	66	99	143	201
4	1	1	2	5	9	14	20	30	43	55	70
3	1	1	**2**	4	**5**	7	10	12	15	19	22
2	1	1	1	2	3	3	3	4	5	5	5
1	**1**	1	**1**	1	1	1	1	1	1	1	1
0	1	1	0	1	0	1	0	1	0	1	0
B / F	0	1	2	3	4	5	6	7	8	9	10

Table 2: **Sizes of gauge invariant bases in the (F,B) sectors.**

- **The magic staircase ⇒ there are always two SUSY vacua at finite λ (in the strong coupling phase).**

6 q-BOSON GAS

- **A one dimensional, periodic lattice with length F.**
- **A boson at each lattice site a_i, $i = 1, ..., F$**
- **The new Hamiltonian**

$$H = B + \sum_{i=1}^{F} \delta_{N_i,0} + \sum_{i=1}^{F} b_i b_{i+1}^\dagger + b_i b_{i-1}^\dagger, \tag{7}$$

where $N_i = a^\dagger{}_i a_i$ and $B = n_1 + n_2 + ... + n_F$.

- **The $b_i^\dagger$ (b_i) operators create (annihilate) one quantum *without* the usual $\sqrt{n}$ factors – *assisted* transitions.**

$$b^\dagger|n\rangle = |n+1\rangle, \quad b|n\rangle = |n-1\rangle, \quad b|0\rangle \equiv 0,$$
$$[b, b^\dagger] = \delta_{N,0} \tag{8}$$

- **This Hamiltonan conserves B.**
- **It is also invariant under lattice shifts U.**
- **The spectrum of above H, in the sector with $\lambda_U = -1$, exactly coincides with the spectrum of H_{strong}, for even F and any B.**

- **q-bosons: the b and $b^\dagger$ c/a operators satisfy the q-deformed harmonic oscillator algebra**

$$[b, b^\dagger] = q^{-2N} \tag{9}$$

with $q \to \infty$.

- **The q-Bose gas was considered non-soluble (Bogoliubov) ... until now.**

7 THE XXZ MODEL

The one dimensional chain of Heisenberg spins

$$H^{(\Delta)}_{\mathrm{XXZ}} \;=\; -\frac{1}{2}\sum_{i=1}^{L}\left(\sigma_i^x\sigma_{i+1}^x + \sigma_i^y\sigma_{i+1}^y + \Delta\;\sigma_i^z\sigma_{i+1}^z\right)$$

- **Our planar system, at strong coupling, is equivalent to the XXZ chain with**

$$L = F + B, \quad S^z = \sum_{i=1}^{L} s_i^z = F - B, \quad \text{and} \quad \Delta = \pm\frac{1}{2}$$

- **Riazumov-Stroganv conjecture: for odd L and $S^z = \pm 1$ there exists an eigenstate with known, simple eigenvalue $E = \frac{L}{12}$**
- **$\Rightarrow$ the R-S states are the SUSY vacua of H_{SC} !**

8 BETHE ANSATZ

- **The XXZ model is soluble by the Bethe Ansatz**
- **The existence of the magic staircase can be proven using BA**
- **Even more: there is the hidden supersymmetric structure in the Heisenberg chain.**

Bethe phase factors for the first three magic sectors

F=4,B=3, 5×5

$$x = \frac{1}{2} + \frac{i\sqrt{3}}{2}$$

F=4,B=5, 14×14

$$x = \frac{1}{64}\left(16 + i\sqrt{2}\sqrt{15+\sqrt{33}}(7-\sqrt{33}) - 4\sqrt{-16(3+\sqrt{33}) - i2\sqrt{2}\sqrt{15+\sqrt{33}}(9+\sqrt{33})}\right)$$

$$y = \frac{1}{64}\left(16 + i\sqrt{2}\sqrt{15+\sqrt{33}}(7-\sqrt{33}) + 4\sqrt{-16(3+\sqrt{33}) - i2\sqrt{2}\sqrt{15+\sqrt{33}}(9+\sqrt{33})}\right)$$

F=6,B=5, 42×42

$$x = \frac{1}{72}\left(36 + i\sqrt{2}\sqrt{11+\sqrt{13}}(7+\sqrt{13}) - 6\sqrt{2}\sqrt{6(-3+\sqrt{13}) + i\sqrt{2}\sqrt{11+\sqrt{13}}(-5+\sqrt{13})}\right)$$

$$y = \frac{1}{72}\left(36 + i\sqrt{2}\sqrt{11+\sqrt{13}}(7+\sqrt{13}) + 6\sqrt{2}\sqrt{6(-3+\sqrt{13}) + i\sqrt{2}\sqrt{11+\sqrt{13}}(-5+\sqrt{13})}\right)$$

9 FROM N=3,4,5 TO INFINITY

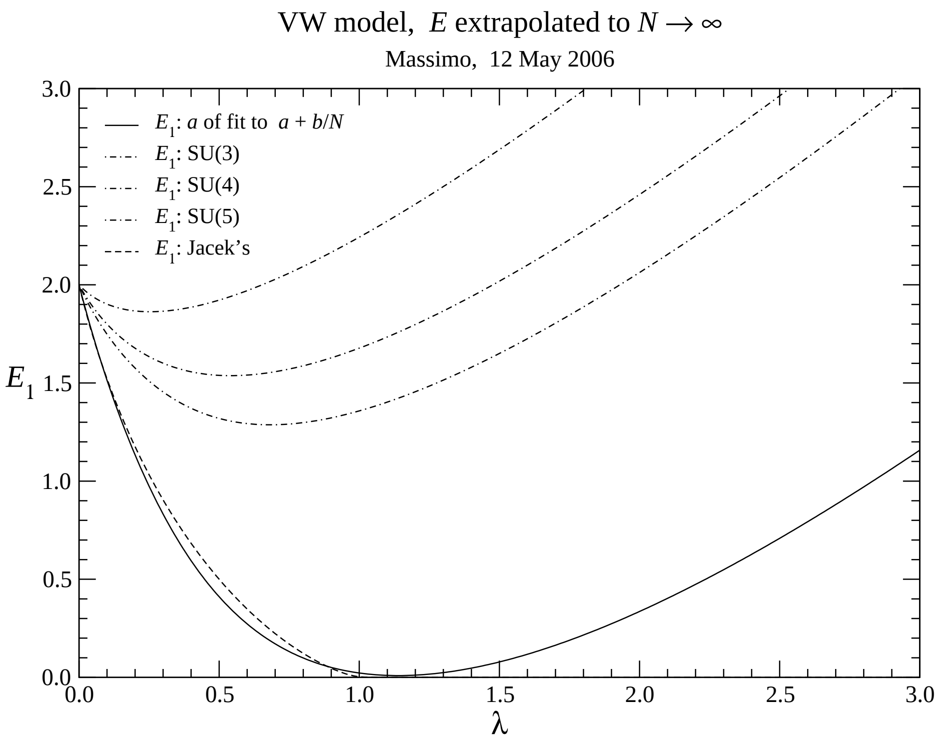


Figure 9: **Behaviour of the restricted Witten index, at $T = 6$, around the phase transition.**

10 THE FUTURE

- **Supersymmetric Yang-Mills Quantum Mechanics in d=3 (QFT at $V = 1^3$)**
- **Supersymmetric Yang-Mills Quantum Mechanics in d=9 (QFT at $V = 1^3$) − M**
- **QCD in tiny volumes**