Lattice Flavourdynamics

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May 27th – June 5th 2006

Aims of these Lectures

- In these lectures I will not try to:
  - **(i)** give a course on the technology of lattice QCD;
  - **(ii)** try to review all physical quantities which have been (or could be) computed in lattice simulations;
  - **(iii)** present a catalogue of all the latest results. (Some of these will be obsolete in any case by the end of Lattice 2006 in July.)
- Instead I will give an introductory overview of the applications of lattice QCD to phenomenology, so that you will have some feel for:
  - **(i)** which quantities can be calculated on the lattice and which cannot;
  - **(ii)** the precision which might be reached.
- Of course my presentation will necessarily include some theoretical background and many numerical results.

  I will try to embed a discussion of some of the theoretical ideas into the discussion of the phenomenology.

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Introduction to Lattice Phenomenology

- Lattice phenomenology starts with the evaluation of correlation functions of the form:

$$\langle 0|\,O(x_1,x_2,\cdots,x_n)\,|0\rangle = \frac{1}{Z}\int [dA_\mu]\,[d\psi]\,[d\overline{\psi}]\,e^{iS}\,O(x_1,x_2,\cdots,x_n)\ ,$$

where $O(x_1,x_2,\cdots,x_n)$ is a multilocal operator composed of quark and gluon fields and Z is the partition function:

$$Z = \int [dA_\mu]\,[d\psi]\,[d\overline{\psi}]\,e^{iS}\ .$$

- These formulae are written in Minkowski space, whereas Lattice calculations are performed in Euclidean space ($\exp(iS) \to \exp(-S)$ etc.).
- The physics which can be studied depends on the choice of the multilocal operator O.
- The functional integral is performed by discretising space-time and using Monte-Carlo Integration.

Two-Point Correlation Functions

Consider two-point correlation functions of the form:

$$C_2(t) = \int d^3x \, e^{i\vec{p}\cdot\vec{x}} \, \langle 0|\, J(\vec{x},t)\, J^\dagger(\vec{0},0)\, |0\rangle \ ,$$

where J and $J^\dagger$ are any interpolating operators for the hadron H which we wish to study and the time t is taken to be positive.

- We assume that H is the lightest hadron which can be created by $J^\dagger$.
- We take $t > 0$, but it should be remembered that lattice simulations are frequently performed on periodic lattices, so that both time-orderings contribute.

Two-Point Correlation Functions (Cont.)

$$C_2(t) = \int d^3x \, e^{i\vec{p}\cdot\vec{x}} \, \langle 0|\, J(\vec{x},t)\, J^\dagger(\vec{0},0)\, |0\rangle \ ,$$

Inserting a complete set of states $\{|n\rangle\}$:

$$\begin{aligned} C_2(t) &= \sum_n \int d^3x \, e^{i\vec{p}\cdot\vec{x}} \, \langle 0|\, J(\vec{x},t)\, |n\rangle \, \langle n| J^\dagger(\vec{0},0)\, |0\rangle \\ &= \int d^3x \, e^{i\vec{p}\cdot\vec{x}} \, \langle 0|\, J(\vec{x},t)\, |H\rangle \, \langle H| J^\dagger(\vec{0},0)\, |0\rangle + \cdots \end{aligned}$$

where the $\cdots$ represent contributions from heavier states with the same quantum numbers as H.

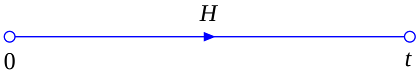
Finally using translational invariance:

$$C_2(t) = \frac{1}{2E}\, e^{-iEt} \left| \langle 0| J(\vec{0},0) |H(p)\rangle \right|^2 + \cdots \ ,$$

where $E = \sqrt{m_H^2 + \vec{p}^{\,2}}$.

Two-Point Correlation Functions (Cont.)

$$C_2(t) = \frac{1}{2E}\, e^{-iEt} \left| \langle 0 | J(\vec{0},0) | H(p) \rangle \right|^2 + \cdots \;.$$



- In Euclidean space $\exp(-iEt) \to \exp(-Et)$.
- By fitting $C(t)$ to the form above, both the energy (or, if $\vec{p} = 0$, the mass) and the modulus of the matrix element

$$\left| \langle 0 | J(\vec{0},0) | H(p) \rangle \right|$$

  can be evaluated.
- Example: if $J = \bar{u}\gamma^\mu \gamma^5 d$ then the decay constant of the π-meson can be evaluated,

$$\left| \langle 0 | \, \bar{u}\gamma^\mu \gamma^5 d \, | \pi^+(p) \rangle \right| = f_\pi \, p^\mu \;,$$

  (the physical value of $f_\pi \simeq$ is 132 MeV).

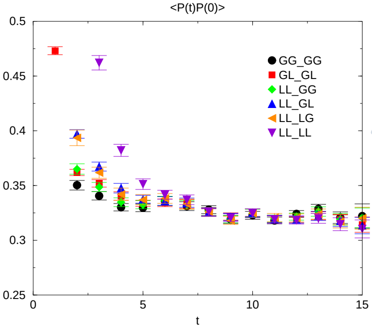
Effective Masses

At zero momentum

$$C_2(t) = \text{Constant} \times e^{-mt}$$

so that it is sensible to define the *effective mass*

$$m_{\text{eff}}(t) = \log\left(\frac{C(t)}{C(t+1)}\right).$$



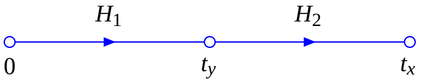
Effective Mass Plot for a Pseudoscalar Meson. UKQCD Collaboration.

Three-Point Correlation Functions

Consider now a three-point correlation function of the form:

$$C_3(t_x,t_y) = \int d^3x\, d^3y\; e^{i\vec{p}\cdot\vec{x}}\; e^{i\vec{q}\cdot\vec{y}}\; \langle 0|\, J_2(\vec{x},t_x)\, O(\vec{y},t_y)\, J_1^\dagger(\vec{0},0)\,|0\rangle\ ,$$

where $J_{1,2}$ may be interpolating operators for different particles and we assume that $t_x > t_y > 0$.

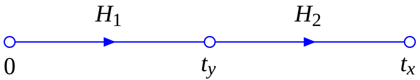


For sufficiently large times t_y and $t_x - t_y$

$$\begin{aligned} C_3(t_x,t_y) \quad &\simeq \quad \frac{e^{-E_1 t_y}}{2E_1}\, \frac{e^{-E_2(t_x-t_y)}}{2E_2}\, \langle 0|J_2(0)|H_2(\vec{p})\rangle \\ &\times \langle H_2(\vec{p})|O(0)|H_1(\vec{p}+\vec{q})\rangle\; \langle H_1(\vec{p}+\vec{q})|J_1^\dagger(0)|0\rangle\ , \end{aligned}$$

where $E_1^2 = m_1^2 + (\vec{p}+\vec{q})^2$ and $E_2^2 = m_1^2 + \vec{p}^{\,2}$.

Three-Point Correlation Functions

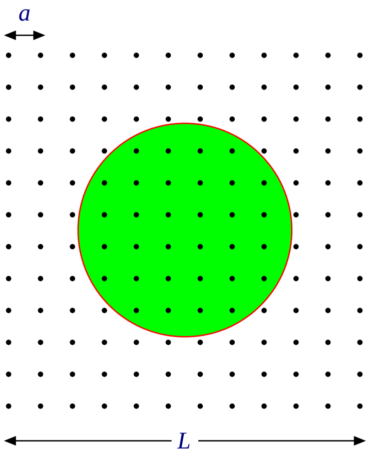


- From the evaluation of two-point functions we have the masses and the matrix elements of the form $|\langle 0|J|H(\vec{p})\rangle|$. Thus, from the evaluation of three-point functions we obtain matrix elements of the form $|\langle H_2|O|H_1\rangle|$.
- Important examples include:
  - $K^0 - \bar{K}^0$ ($B^0 - \bar{B}^0$) mixing. In this case

$$O = \bar{s}\gamma^\mu(1-\gamma^5)d\;\bar{s}\gamma_\mu(1-\gamma^5)d\,.$$

  - Semileptonic and rare radiative decays of hadrons of the form $B \to \pi, \rho$ + leptons or $B \to K^*\gamma$. Now O is a quark bilinear operator such as $\bar{b}\gamma^\mu(1-\gamma^5)u$ or an *electroweak penguin* operator.

Systematic Uncertainties



We would like

$$L \gg 1\,\mathrm{fm} \quad \text{and} \quad a^{-1} \gg \Lambda_{\mathrm{QCD}}\ .$$

Systematic Uncertainties (Cont.)

- Computing resources limit the number of lattice points which can be included, and hence the precision of the calculation.

  Typically in full QCD we can have about 24 – 32 points in each spatial direction ($O(50)$ points in quenched simulations) and so compromises have to be made.
- **Statistical Errors:** The functional integral is evaluated by Monte-Carlo sampling. The statistical error is estimated from the fluctuations of computed quantities within different clusters of *configurations*.
- The different sources of systematic uncertainty are not independent of each other, so the following discussion is oversimplified.

Systematic Uncertainties (Cont.)

- **Discretization Errors (Lattice Artefacts)**: Current simulations are typically performed with

$$a \sim (0.05 - .125)\,\mathrm{fm} \qquad (0.1\,\mathrm{fm} \simeq 2\,\mathrm{GeV})$$

leading to errors of $O(a\Lambda_{\mathrm{QCD}})$ (with Wilson Fermions) or $O(a^2\Lambda^2_{\mathrm{QCD}})$ for *improved* fermion actions.

The errors can be estimated and reduced by:

  - Performing simulations at several values of a and extrapolating to $a = 0$.
  - *Improvement* (Symanzik), i.e. choosing a discretization of QCD so that the errors are formally smaller.

$$f'(x) = \frac{f(x+a) - f(x)}{a} + O(a) \quad \text{or} \quad f'(x) = \frac{f(x+a) - f(x-a)}{2a} + O(a^2)\,.$$

For example, in this way it is possible to reduce the errors from $O(a)$ for Wilson fermions to ones of $O(a^2)$ by the addition of irrelevant operators.

Systematic Uncertainties (Cont.)

- **Chiral Extrapolations:** Simulations are performed with unphysically heavy u and d quarks and the results are then extrapolated to the chiral limit.

  Wherever possible, we use χPT to guide the extrapolation, but it is still very rare to observe chiral logarithms.

  Today, in general, the most significant source of systematic uncertainty is due to the chiral extrapolation.

	m_q/m_s	m_π (MeV)	m_π/m_ρ
$SU(3)$ Limit	1	690	0.68
Currently Typical	1/2	490	0.55
Impressive	1/4	340	0.42
MILC	1/8	240	0.31
Physical	1/25	140	0.18

  For this reason the results obtained using the MILC Collaboration (using *Staggered* lattice fermions) have received considerable attention. Gradually the challenge set by the MILC Collaboration is being taken up by groups using other formulations of lattice fermions (e.g. Improved Wilson, Twisted Mass, Domain Wall, Overlap).

- $\rho \to \pi\pi$ decays have not been achieved on the lattice up to now.

Systematic Uncertainties (Cont.)

- **Finite Volume Effects:** For the quantities described above the finite-volume errors fall exponentially with the volume, e.g.

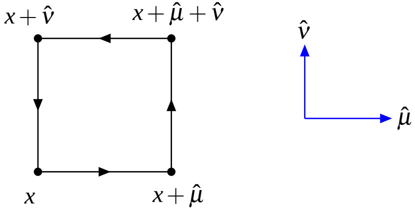
$$\frac{f_{\pi^\pm}(L) - f_{\pi^\pm}(\infty)}{f_{\pi^\pm}(\infty)} \simeq -\frac{6m_\pi^2}{f_\pi^2}\,\frac{e^{-m_\pi L}}{(2\pi m_\pi L)^{3/2}}\,.$$

  Generally these uncertainties are small at the light-quark masses which can be simulated.
  - For two-particle states (e.g. $K \to \pi\pi$ decays) the finite-volume effects decrease as inverse powers of L, and must be removed.
- **Renormalization of Lattice Operators:** From the matrix elements of the bare operators computed in lattice simulations we need to determine matrix elements of operators renormalized in some standard renormalization scheme (such as $\overline{MS}$).
  - For sufficiently large a^{-1} this can be done in perturbation theory, but lattice perturbation theory frequently has large coefficients $\Rightarrow$ large uncertainties ($O(10\%)$).
  - *Non-perturbative renormalization* is possible and frequently implemented, eliminating the need for lattice perturbation theory.

Lattice QCD

- The quark fields are defined on the lattice sites, $\psi^i_\alpha(x_i)$.
- In order to ensure gauge invariance the gauge fields are introduced through *link variables*, defined on the links between two neighbouring points.
  $U_\mu(x_i)$ is the link variable between the points x_i and $x_i + \hat{\mu}$.
- Under a gauge transformation:
  $$\psi(x_i) \to g(x_i)\psi(x_i) \qquad \text{and} \qquad U_\mu(x_i) \to g(x_i)U_\mu(x_i)g^\dagger(x_i+\hat{\mu})\,.$$
  $U_\mu(x_i)$ is the path-ordered exponential of gauge fields between x_i and $x_i + \hat{\mu}$.
- Writing
  $$U_\mu(x_i) = \exp\left\{ig_0 A_\mu(x_i + \frac{\hat{\mu}}{2})a\right\}\,,$$
  Wilson proposed the gauge action
  $$S = \sum_{\mathscr{P}_{\mu\nu}} \mathscr{P}_{\mu\nu} \quad \text{where} \quad \mathscr{P}_{\mu\nu} = \beta\left\{1 - \frac{1}{3}\,\mathrm{Re\,Tr}\left(U_\mu(x)U_\nu(x+\hat{\mu})U^\dagger_\mu(x+\hat{\nu})U^\dagger_\nu(x)\right)\right\}$$
  where $\beta = 6/g_0^2$ and $\mathscr{P}$ is called the *plaquette*.

Gauge Action (Cont.)



"A little suppressed algebra" [Creutz] $\Rightarrow$

$$S = \frac{1}{2}\int d^4x \,\mathrm{Tr}(F_{\mu\nu}F_{\mu\nu}) \quad + \text{terms suppressed by } a^2 .$$

- Gauge invariance is exact.
- In many current simulations an *improved* gauge action is used.

Fermion Actions – The Doubling Problem

- Naive Fermions $\Rightarrow$ Fermion Doubling Problem. The inverse free propagator is

$$m + ia^{-1} \sum_{\mu} \gamma_{\mu} \sin(aq_{\mu}) \,,$$

  where q is the momentum.
- At low momenta this is correct, but there are also similar contributions from $q_{\mu} \simeq \pi/a$ and we have $2^4 = 16$ independent Fermion species.

There is a plethora of Lattice Fermion Actions which overcome this problem:

1. Wilson Fermions (+ improved versions);
2. Staggered Fermions (and modified versions);
3. Twisted Mass QCD;
4. Actions satisfying the Ginsparg-Wilson relation (Domain Wall Fermions, Overlap Fermions, Perfect Actions)
5. $\cdots$

Lattice Fermions (Cont.)

- **Wilson Fermions:** Add irrelevant term to the action ($\propto a\bar{\psi}D^2\psi$) $\Rightarrow$ the inverse propagator is modifies by a term $a^{-1}\sum_\mu\{1-\cos(aq_\mu)\}$.

  The Wilson term breaks the chiral symmetry and induces artefacts of $O(a\Lambda_{\text{QCD}})$.

  The artefacts can be reduced to $O(a^2\Lambda^2_{\text{QCD}})$ by adding further irrelevant operators to the action (and by calculating matrix elements of appropriate *improved* operators). **Symanzik Improvement**.

- **Staggered Fermions:** By a “spin diagonalization” of the γ-matrices the 16 fermion doublers can be reduced to 4. The chiral Ward Identities are still satisfied, the artefacts are of $O(a^2\Lambda^2_{\text{QCD}})$, but we still have four quark *tastes* or 16 *pions*.

  In attempts to reduce the spectrum to a single pion

$$\det[\Delta[U]] \to \det{}^{\frac{1}{4}}[\Delta[U]]$$

  where Δ is the Dirac Operator.

There is a strong polarization in the lattice community concerning the use of staggered fermions.

- S.Dürr (hep-lat/0509026) started his review talk at Lattice2005 with the question:

  Is "staggered QCD" really QCD, or is it just a model of QCD?

  Unphysical *tastes* removed by taking the fermionic Det$^{1/4}$. No proof that this is correct (but growing circumstantial evidence) and no counter example.

  Abstract of Bernard, Golterman & Shamir (hep-lat/0604017):
  We show that the use of the fourth-root trick [...] corresponds to a non-local theory at $a \neq 0$, but argue that the non-local behaviour is likely to go away in the continuum limit.

- Staggered Chiral Perturbation theory has to include the a-dependence and the extrapolation has many parameters (e.g. over 50 for f_π).
  The $m \to 0$ limit cannot be taken before the $a \to 0$ limit.

  Bernard hep-lat/0603011

- Renormalization is performed perturbatively.
- It would be nice if the results of the extrapolations and procedures were confirmed by other groups.

Lattice Fermions (Cont.)

- **Chiral Fermions:** Much work is being devoted to developing algorithms for lattice fermions which have a continuum-like chiral symmetry even at finite lattice spacing.

  Ginsparg Wilson Relation

$$\left\{\gamma^5,\Delta\right\} = 2a\Delta\gamma^5\Delta \quad \Rightarrow \quad \text{Invariance under}$$

$$\psi \to \psi + \varepsilon\gamma^5(1-a\Delta)\psi \quad \text{and} \quad \bar\psi \to \bar\psi + \varepsilon\,\bar\psi(1-a\Delta)\gamma^5$$

  Lüscher - hep-lat/9802011

- Formulations satisfying the Ginsparg-Wilson relation are ultimately likely to be the preferred ones.

Evaluation of the Masses of the Light Quarks

- There are a number of ways being used to determine the quark masses. These should give identical results if the systematic errors (in particular, lattice artefacts) are negligible.
- One standard method exploits the axial Ward identity

$$\partial_\mu A^\mu = (m_1 + m_2) P$$

where A and P are the axial current and pseudoscalar density corresponding to quarks with masses $m_{1,2}$:

$$A^\mu(x) = \bar{\psi}_1(x)\gamma^\mu\gamma^5\psi_2(x) \qquad \text{and} \qquad P(x) = \bar{\psi}_1(x)\gamma^5\psi_2(x)\,.$$

For illustration here let me take $m_1 = m_2 \equiv m_q$.

Evaluation of the Masses of the Light Quarks (Cont)

$$\partial_\mu A^\mu = 2m_q P$$

- Calculate the two-point correlation function

$$\langle 0 \,|\, P(t) P^\dagger(0) \,|\, 0 \rangle = \frac{Z_P^2}{2m_P} \left\{ \exp(-m_P t) + \exp(-m_P (L_t - t)) \right\} .$$

  - m_P is the mass of the pseudoscalar meson.
  - Z_P is the matrix elements $\langle 0|P(0)|P\rangle$.
- The use of axial or vector Chiral Ward identities is particularly useful if chiral symmetry is not exact in the lattice formulation being used.

Evaluation of the Masses of the Light Quarks (Cont)

$$\partial_\mu A^\mu = 2m_q P$$

$$\langle 0 | P(t) P^\dagger(0) | 0 \rangle = \frac{Z_P^2}{2m_P} \{\exp(-m_P t) + \exp(-m_P(L_t - t))\} .$$

Also calculate

$$\langle 0 | A_4(t) P^\dagger(0) | 0 \rangle = \frac{Z_A Z_P}{2m_P} \{\exp(-m_P t) - \exp(-m_P(L_t - t))\} .$$

In this way we obtain:

$$m_q^{(0)\,\mathrm{AWI}} \equiv \frac{m_P Z_A}{2Z_P} .$$

Now we would like the mass in some standard renormalization scheme, and the axial current and pseudoscalar density are both multiplicatively renormalizable. The renormalization constants can be fixed and we obtain the masses.

Vector Ward Identities can also be used.

Recent Compilation of (Unquenched) Lattice Results

Reference	m_s	$\hat{m}$
HPQCD, MILC and UKQCD	$76 \pm 3 \pm 7$ MeV	$2.8 \pm .1 \pm .3$ MeV
HPQCD, MILC and UKQCD Update including 2-loop Z's	$86 \pm 3 \pm 4$ MeV	$3.2 \pm .1 \pm .2$ MeV *
CP-PACS & JLQCD (K-input)	80.4$\pm$1.9 MeV	3.05$\pm$.06 MeV
CP-PACS & JLQCD (Φ-input)	89.3$\pm$2.9 MeV	3.04$\pm$.06 MeV
SPQR (VWI)	111 ± 6 MeV	$4.8 \pm .5$ MeV
SPQR (AWI)	103 ± 9 MeV	$4.5 \pm .5$ MeV
QCDSF & UKQCD	$119 \pm 5 \pm 8$ MeV	$4.7 \pm .2 \pm .3$ MeV
Alpha	97 ± 22 MeV	—

* - My estimate from results for $\hat{m}_u$ and $\hat{m}_d$.

Non-Perturbative Renormalization

Since $m\overline{\psi}\psi$ is renormalization group invariant, the determination of $Z_m(a\mu)$ is equivalent to the determination of $Z_S(a\mu)$, where S is the scalar density.

Here we will consider the NPR of $S=\bar{\psi}(x)\psi(x)$, but the method is applicable to other composite operators relevant for weak matrix elements and hadronic structure.

In lattice simulations we compute

$$\langle f|S_B(a)|i\rangle\,,$$

whereas we would like to know

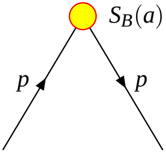
$$\langle f|S_R(\mu)|i\rangle\,,$$

in some standard renormalization scheme R.

The long distance physics is the same in both.

Define the renormalized operator $S_R(\mu)$ to be the one whose matrix element between quark states, at some scale $p^2 = \mu^2$ and in some gauge (the Landau gauge say) is the tree-level one. We compute

$$\langle p|S_B(a)|p\rangle =$$



and determine the renormalization constant $Z_S(a\mu)$ by requiring that

$$Z_S(a\mu)\,\langle p|S_B(a)|p\rangle_{p^2=\mu^2} = \text{tree level value.}$$

The renormalized operator

$$S_R^{\text{RI Mom}}(\mu) \equiv Z_S(a\mu)\,S_B(a)$$

is independent of the regularization (RI) and can be used in (continuum) studies of hadronic physics.

Other renormalization conditions that the MOM one can of course be applied.

Non-Perturbative Renormalization (Cont.)

- To go from RI Mom scheme to any other standard scheme (such as the $\overline{\mathrm{MS}}$ scheme) only requires continuum perturbation theory.
- We require
$$\Lambda^2_{\mathrm{QCD}} \ll p^2 \ll a^{-2}$$
and this *window* is small, in practice.
- By calculating the matrix element between quark states on a sequence of lattices with decreasing a (and hence smaller volumes) and matching, it is possible to eliminate the constraint $p^2 \gg \Lambda^2_{\mathrm{QCD}}$. This procedure is called *step scaling.*

Light-Quark Masses – Summary

My current summary of lattice results for the light-quark masses is:

$$\begin{aligned}\frac{1}{2}\left(m_u^{\overline{\rm MS}}(2\,{\rm GeV})+m_d^{\overline{\rm MS}}(2\,{\rm GeV})\right) &= (3.8\pm 0.8)\,{\rm MeV}\\ m_s^{\overline{\rm MS}}(2\,{\rm GeV}) &= (95\pm 20)\,{\rm MeV}\,.\end{aligned}$$

These are the values which I submitted to the PDG review of quark masses (written together with A.Manohar).

Many of the unquenched simulations are new and I'm confident that the errors will decrease for the next PDG review.