Cross-fertilization of QCD and statistical physics

High energy scattering, reaction-diffusion, selective evolution, spin glasses and their connections

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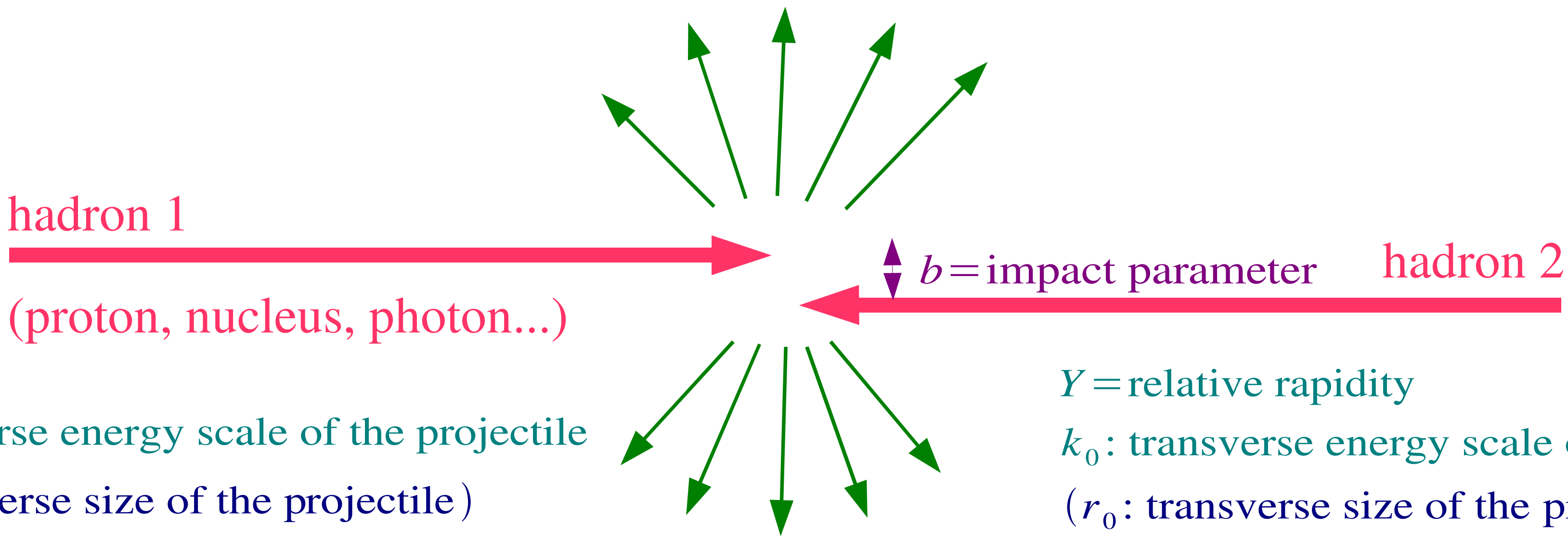
Zakopane, June 4, 2006

High energy QCD



Y = relative rapidity

k_0: transverse energy scale of the target

(r_0: transverse size of the projectile)

k: transverse energy scale of the projectile

(r: transverse size of the projectile)

$$A(Y, \vec{k}) = \int d^2b\, A(b, Y, \vec{k}) = \text{elastic amplitude}$$

$$A(b, Y, \vec{k}) = \text{fixed impact parameter amplitude} \quad \boxed{\leq 1}$$

(High) energy dependence of QCD amplitudes?

The Balitsky equation

Balitsky (1996)

Rapidity evolution of the scattering amplitude:

$\overline{\alpha} = \frac{\alpha_s N_c}{\pi}$

BFKL kernel; acts on transverse coordinates

$$\partial_Y A = \overline{\alpha}\, \chi * \ A$$

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$$\partial_Y A = \overline{\alpha} \chi * (A - \langle T T \rangle)$$

$$\partial_Y \langle T T \rangle = \overline{\alpha} \chi * (\langle T T \rangle - \langle T T T \rangle) + \overline{\alpha} \chi_2 * \langle \mathrm{Tr}(U \overline{U} U \overline{U} U \overline{U}) \rangle$$

$$+\text{source terms}$$

$$\dots$$

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Infinite hierarchy, more complex operators at each step

$$\partial_Y A=\overline{\alpha}\,\chi*(A-\langle T\,T\rangle)$$

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A "mean field" approximation gives the Balitsky-Kovchegov (simpler) equation:

$$\langle T T \rangle = \langle T \rangle \langle T \rangle = A \cdot A \quad \Rightarrow \quad \boxed{\partial_Y A = \overline{\alpha} \chi * (A - A \cdot A)}$$

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How can one solve the Balitsky equation?

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How can one solve the Balitsky equation?

Direct approach too difficult!

Instead, identify the <u>universality class</u> from the physics of the parton model, then apply general results!

Outline

Lecture 1

- ✩ Universality: lessons from condensed matter
- ✩ Stochastic processes: simple examples
- ✩ Reaction-diffusion and traveling wave equations
- ✩ High energy scattering as a reaction-diffusion process

Lecture 2

- ✩ Results on noisy traveling waves
- ✩ Genealogies in selective evolution models
- ✩ A connection to the Parisi theory of spin glasses?

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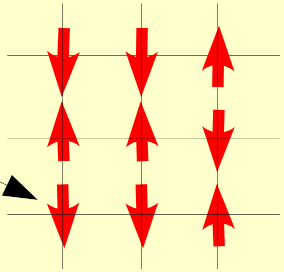
The Ising model

$S_i = \pm 1$



Hamiltonian: $H(\{S_i\}) = -\sum_{i,j} J_{ij} S_i S_j$

$J_{ij} = 1$ if (i, j) are nearest neighbors

Partition function: $Z = \sum_{S_i} e^{-H(\{S_i\})/kT}$

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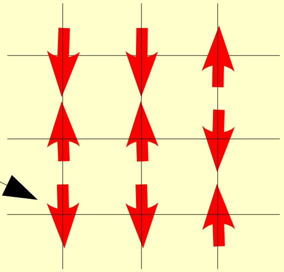

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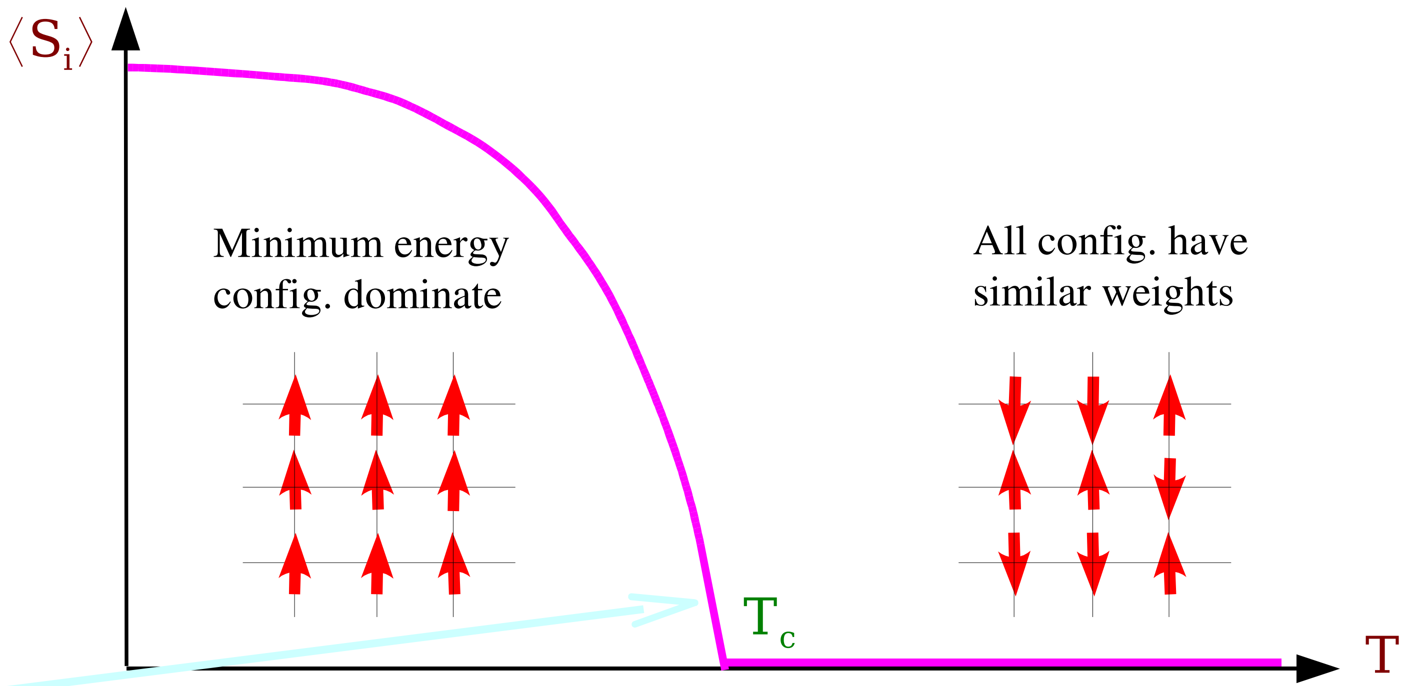

$\langle S_i \rangle \sim (T_c - T)^{\beta}$

$\beta = \frac{1}{8}$

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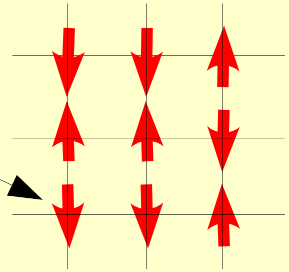

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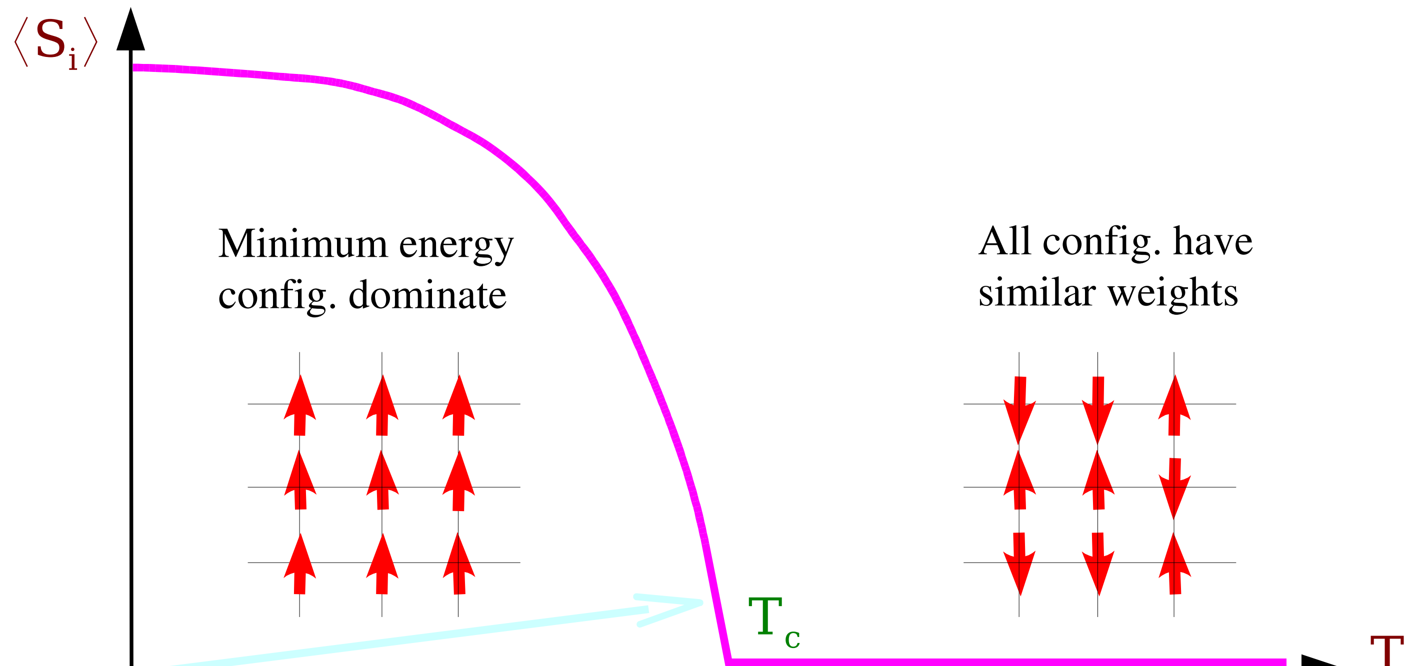

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What is such an (over)simplified model good for?

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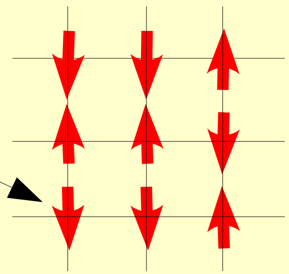

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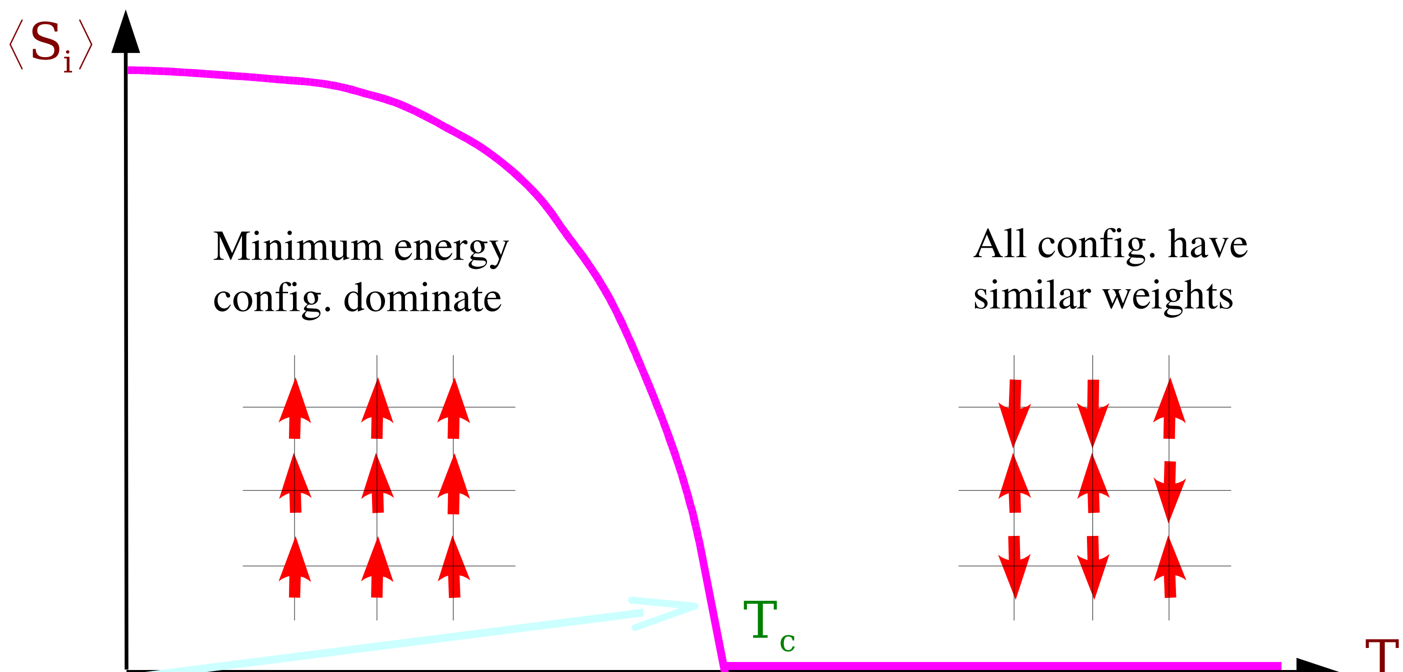

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What is such an (over)simplified model good for?

Critical exponents turn out to be ***universal,*** i.e. insensitive to microscopic details

They are the same in all materials that share some gross properties, like dimensionality, symmetries...

Basic lessons from condensed matter

Common point between *condensed matter* and *high energy scattering*: both have to deal with complex systems

Some measurable quantities can be computed in simple models, and directly taken over to realistic situations: they are said to be ***universal***

Example: critical exponents:
common to 2D ferromagnets, phase transitions...

$$\langle S_i \rangle \sim (T_c - T)^{\beta} \qquad \beta = \frac{1}{8}$$

Counter-example: critical temperature

Reason for universality in Ising: large scale collective effects dominate at the critical point, microscopic details become irrelevant.

Goal: identify the universality class of high energy QCD and the universal observables!

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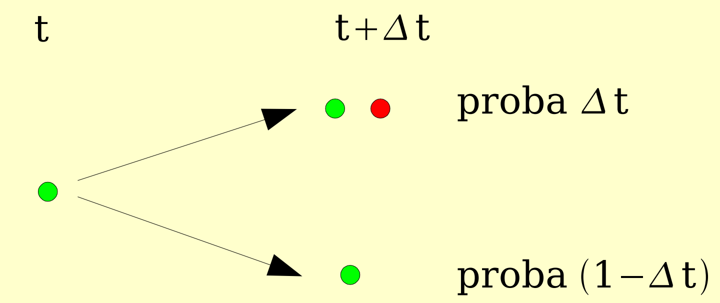
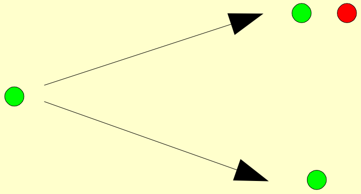
Simple examples of stochastic processes



Simple examples of stochastic processes

t $\qquad t+\Delta t$



proba Δt

proba $(1-\Delta t)$

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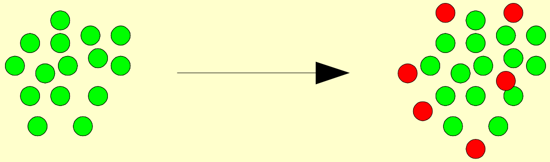
k particles added: k particles split, n-k do not split



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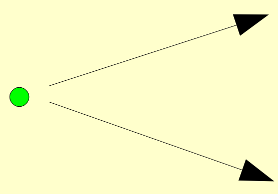

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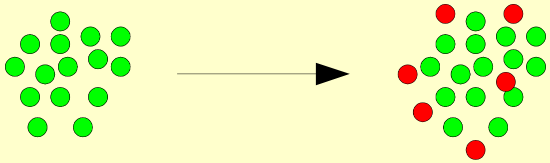

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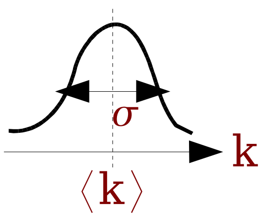

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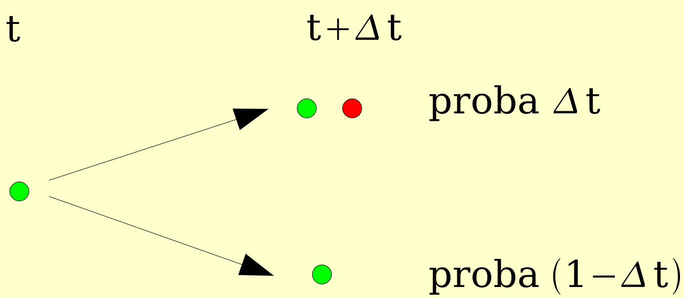

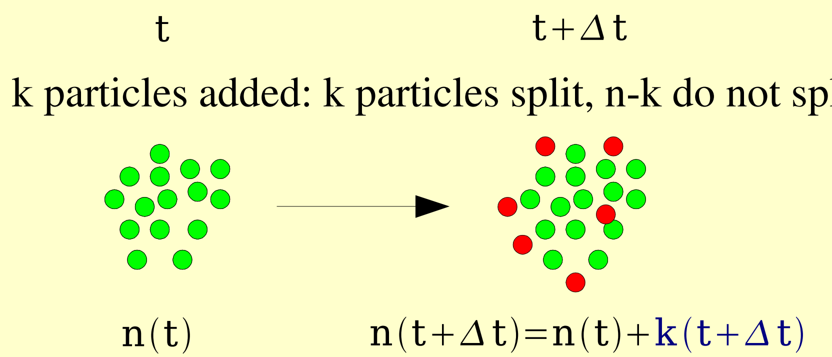

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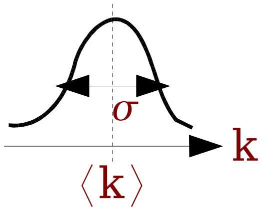

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such that $\sum_t^{t+1}\nu\sim\pm 1$

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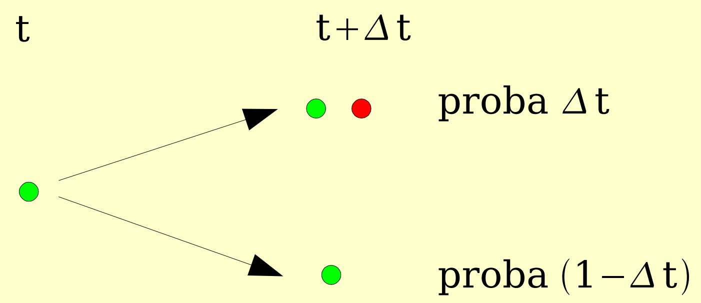

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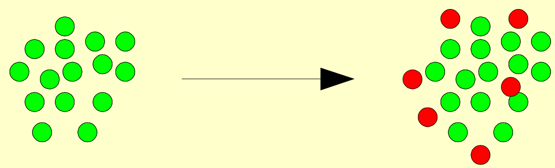

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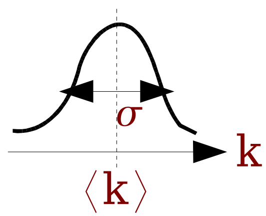

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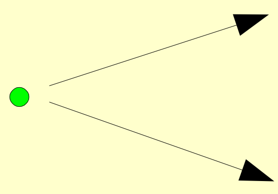

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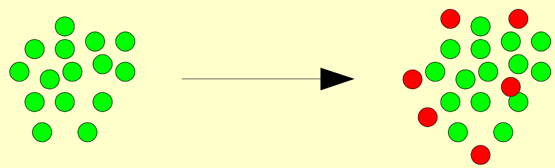

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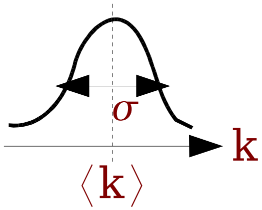

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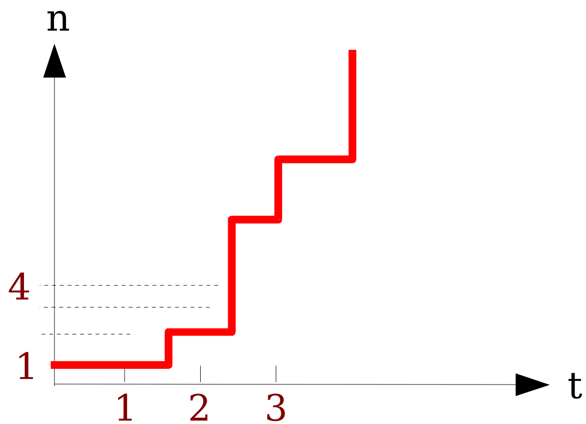

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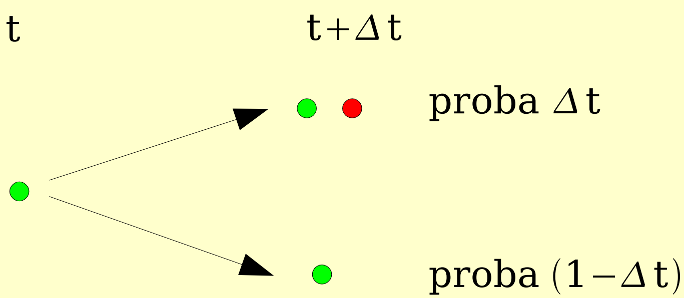

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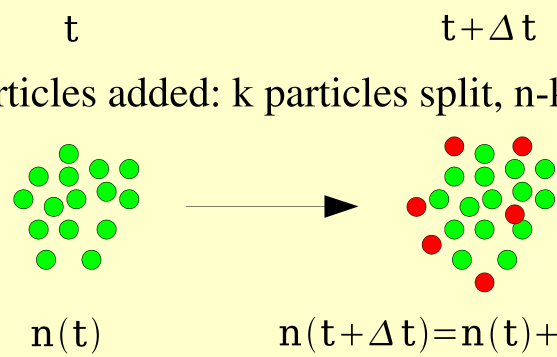

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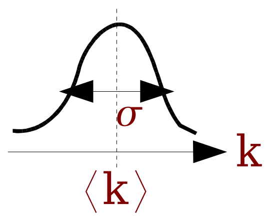

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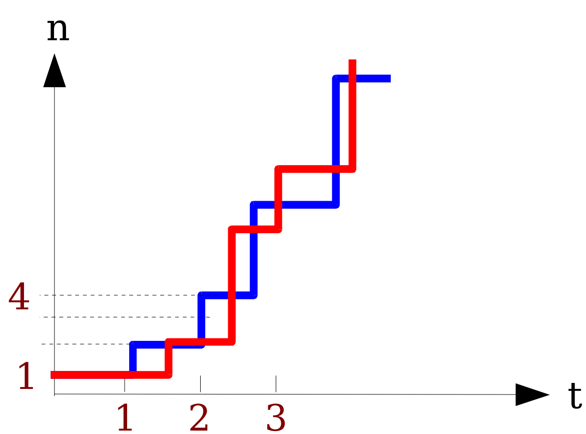

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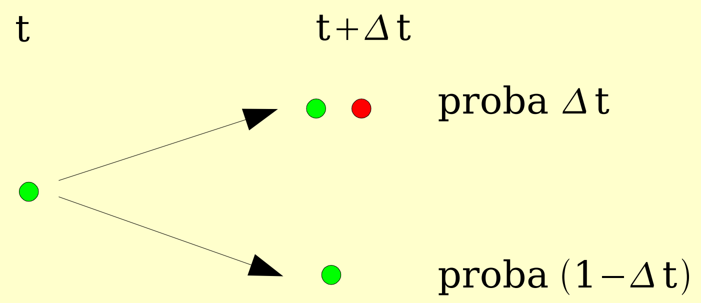

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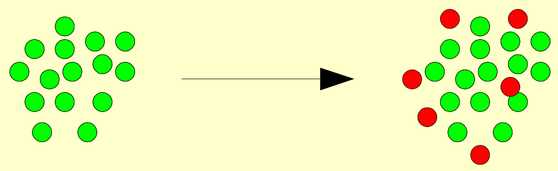

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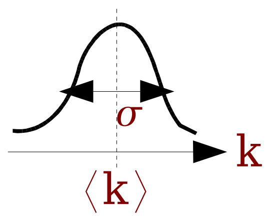

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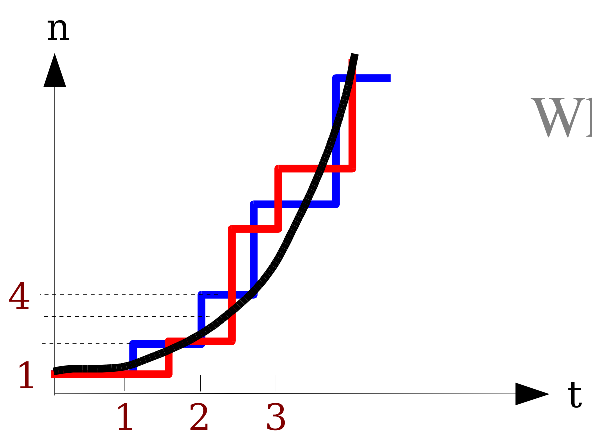

What is, *in average*, the number of particles at time t?

Simple examples of stochastic processes

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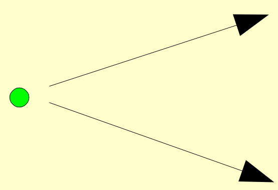

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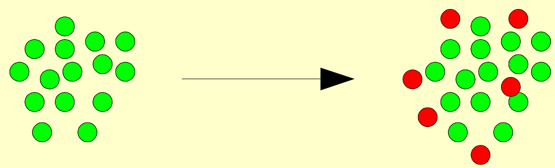

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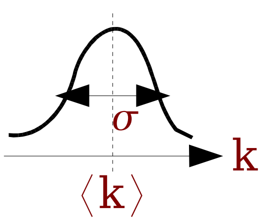
$$\text{proba } P_n(k)=\binom{n}{k}(\Delta t)^k(1-\Delta t)^{n-k}$$

$$\begin{cases}\langle k\rangle = n\,\Delta t\\ \sigma^2=\langle (k-\langle k\rangle)^2\rangle = n\,\Delta t\end{cases}$$

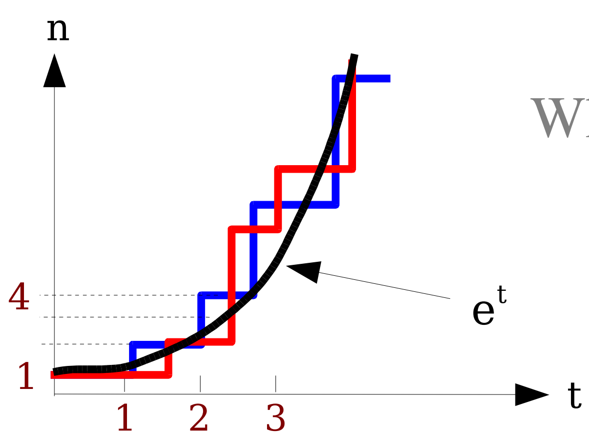


define $\quad \nu=\dfrac{k-\langle k\rangle}{\sigma}\dfrac{1}{\sqrt{\Delta t}}$

such that $\sum_t^{t+1}\nu\sim\pm 1$

$$\begin{cases}\langle \nu\rangle = 0\\ \langle \nu^2\rangle = \dfrac{1}{\Delta t}\end{cases}$$

$$n(t+\Delta t)=n(t)+\Delta t\left(n(t)+\sqrt{n(t)}\,\nu(t+\Delta t)\right) \xrightarrow[\Delta t\to 0]{} \boxed{\frac{dn}{dt}=n+\sqrt{n}\,\nu}$$

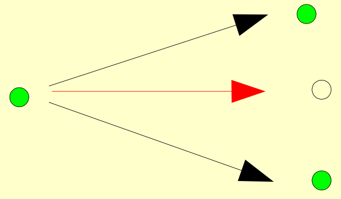


What is, *in average*, the number of particles at time t?

$\langle n(t)\rangle$ obtained by solving the trivial equation $\dfrac{d\langle n\rangle}{dt}=\langle n\rangle$

Simple examples of stochastic processes

t $\qquad t+\Delta t$



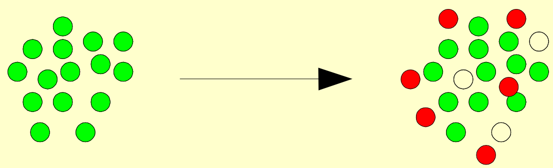
proba Δt

proba $\Delta t \dfrac{n(t)}{N}$

proba $\left(1-\Delta t-\Delta t \dfrac{n(t)}{N}\right)$

t $\qquad t+\Delta t$

k_1 particles added, k_2 particles removed

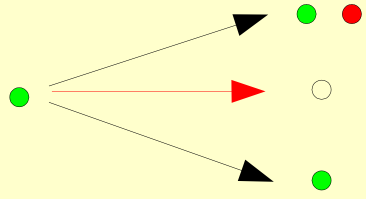


$n(t)$ $\qquad n(t+\Delta t)=n(t)+k_1(t+\Delta t)-k_2(t+\Delta t)$

$$\text{proba } P_n(k_1, k_2)=\binom{n}{k_1\, k_2}(\Delta t)^{k_1}\left(\Delta t \frac{n(t)}{N}\right)^{k_2}\left(1-\Delta t-\Delta t \frac{n(t)}{N}\right)^{n-k_1-k_2}$$

Simple examples of stochastic processes

t $\qquad$ $t+\Delta t$



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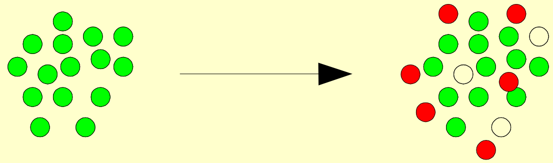

$n(t)$ $\qquad$ $n(t+\Delta t)=n(t)+k_1(t+\Delta t)-k_2(t+\Delta t)$

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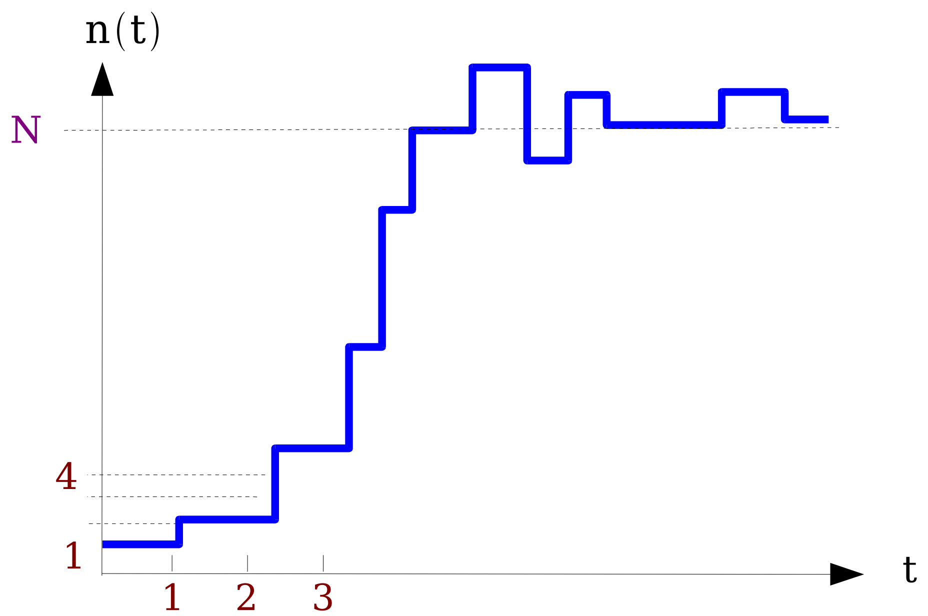
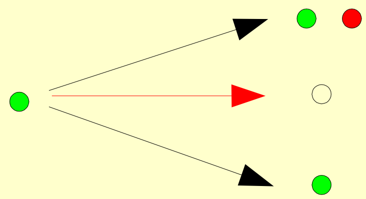
$$\frac{dn}{dt}=n-\frac{n^2}{N}+\sqrt{n\left(1+\frac{n}{N}\right)}\,\nu$$

$$\begin{cases}\langle\nu\rangle=0\\ \langle\nu^2\rangle=\dfrac{1}{dt}\end{cases}$$



Simple examples of stochastic processes

t $\qquad$ $t+\Delta t$



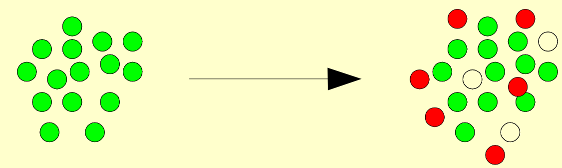
proba Δt

proba $\Delta t \dfrac{n(t)}{N}$

proba $\left(1-\Delta t-\Delta t\dfrac{n(t)}{N}\right)$

t $\qquad$ $t+\Delta t$

k_1 particles added, k_2 particles removed

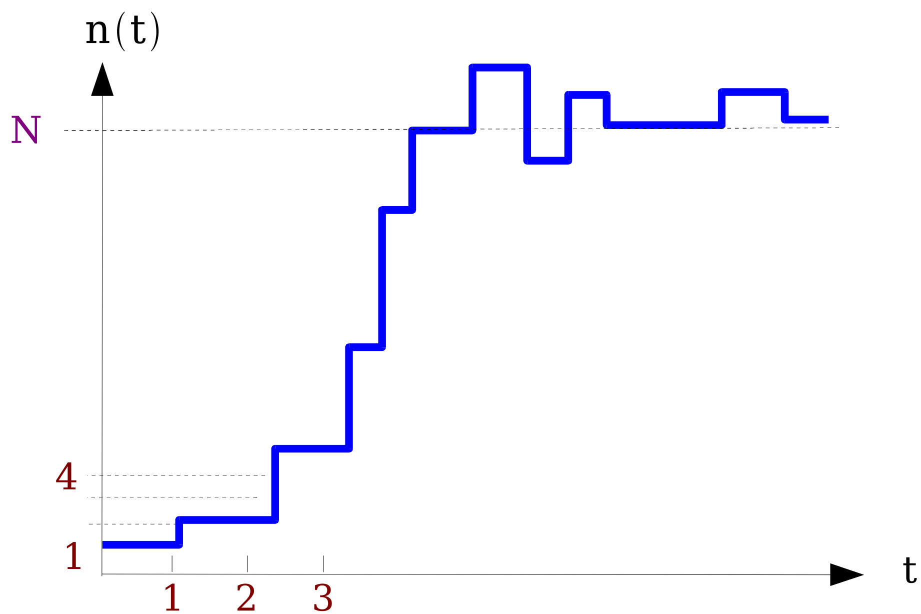


$n(t)$ $\qquad$ $n(t+\Delta t)=n(t)+k_1(t+\Delta t)-k_2(t+\Delta t)$

$$\text{proba } P_n(k_1,k_2)=\binom{n}{k_1\,k_2}(\Delta t)^{k_1}\left(\Delta t\frac{n(t)}{N}\right)^{k_2}\left(1-\Delta t-\Delta t\frac{n(t)}{N}\right)^{n-k_1-k_2}$$

$$\frac{dn}{dt}=n-\frac{n^2}{N}+\sqrt{n\left(1+\frac{n}{N}\right)}\,\nu$$

$$\begin{cases}\langle\nu\rangle=0\\ \langle\nu^2\rangle=\dfrac{1}{dt}\end{cases}$$



$\langle n(t)\rangle$ ***is not*** obtained by solving a trivial equation!

$$\begin{cases}\dfrac{d\langle n\rangle}{dt}=\langle n\rangle-\dfrac{1}{N}\langle n^2\rangle\\ \dfrac{d\langle n^2\rangle}{dt}=\dots\end{cases}$$

...infinite hierarchy!

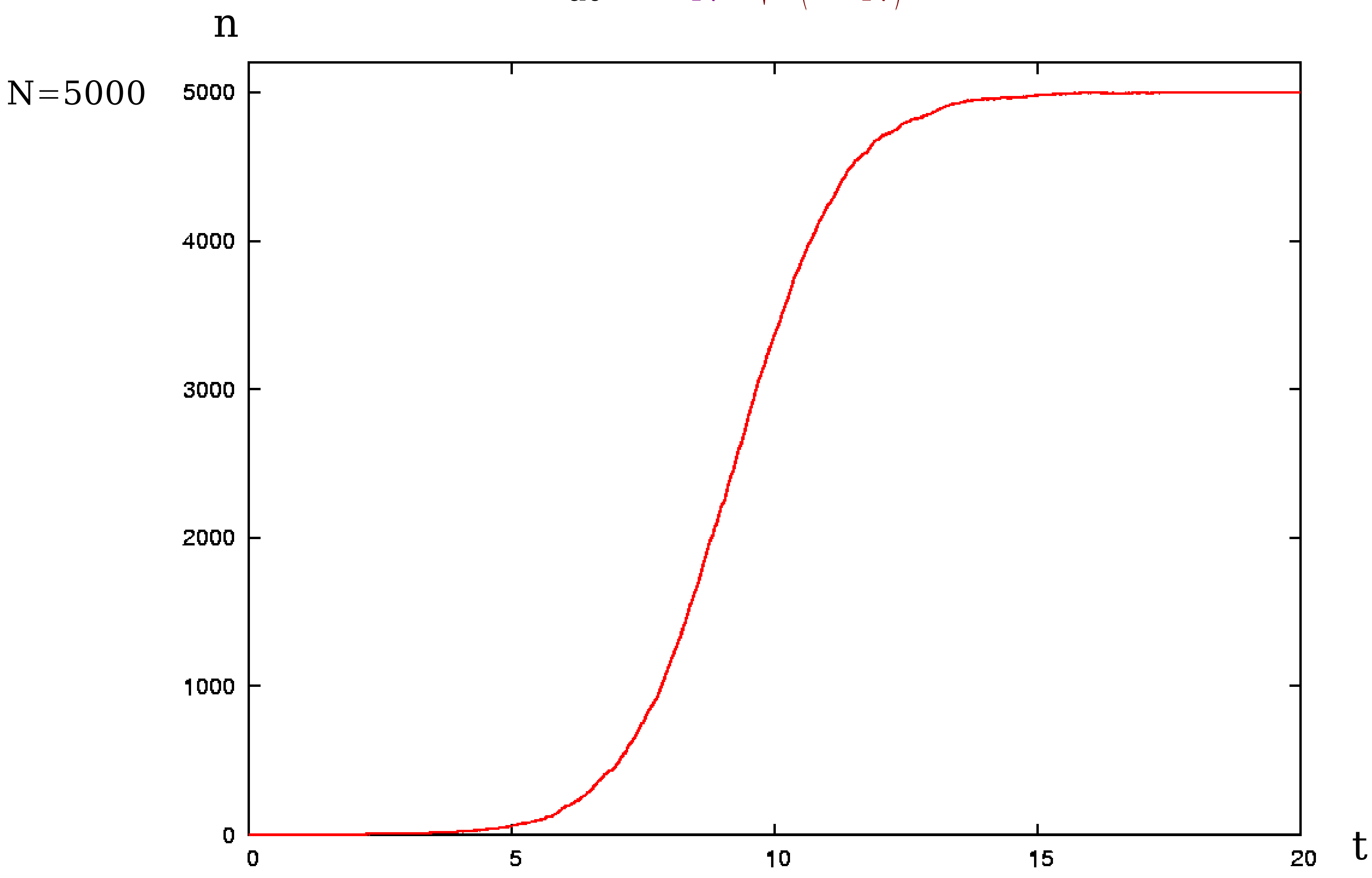
similar to the Balitsky equation in 0D

Mean field approximation: $\dfrac{d\langle n\rangle}{dt}=\langle n\rangle-\dfrac{1}{N}\langle n\rangle^2$

similar to the Balitsky-Kovchegov equation

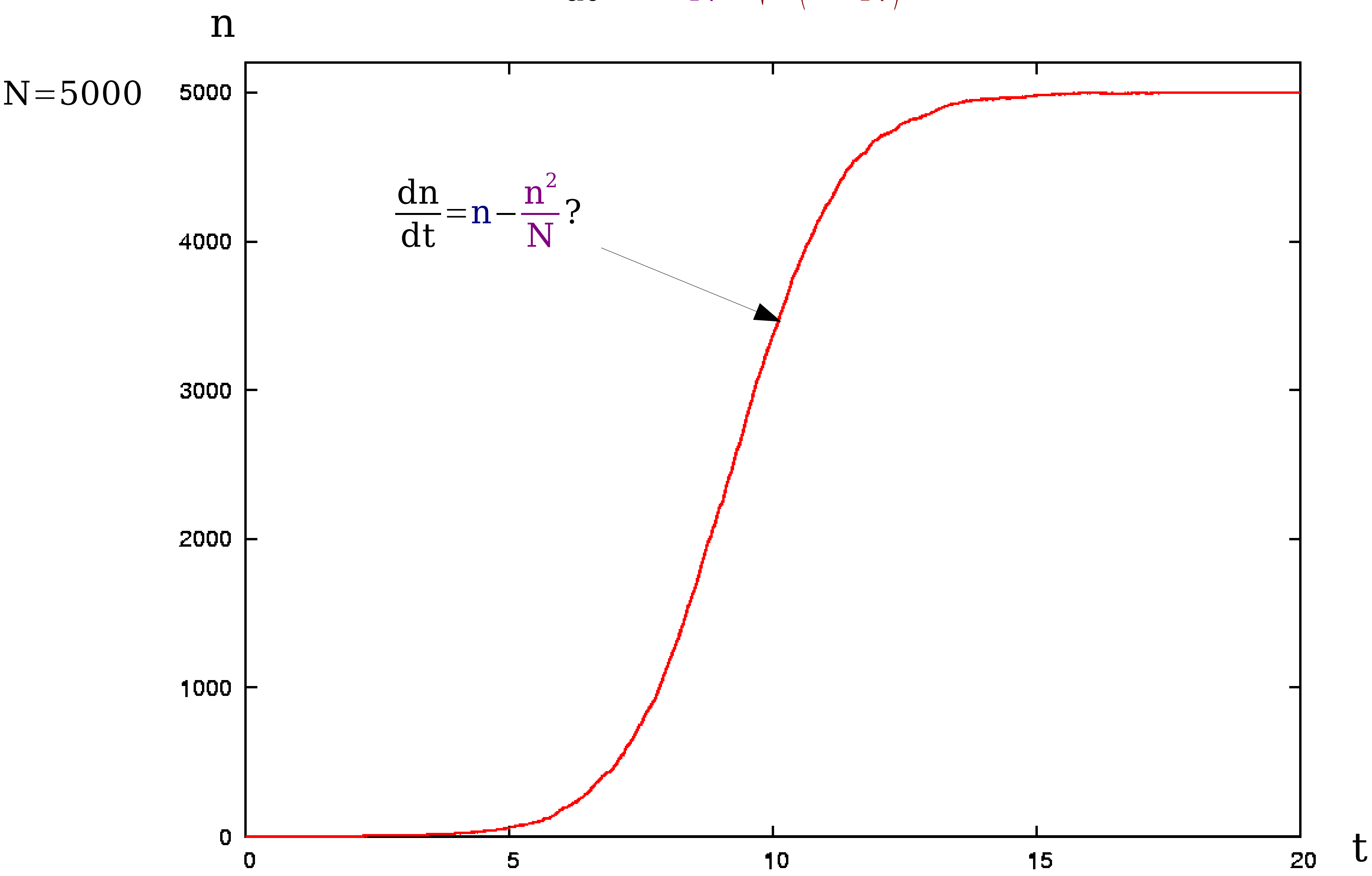
Numerical illustration

$$\frac{dn}{dt} = n - \frac{n^2}{N} + \sqrt{n\left(1+\frac{n}{N}\right)}\,\nu$$



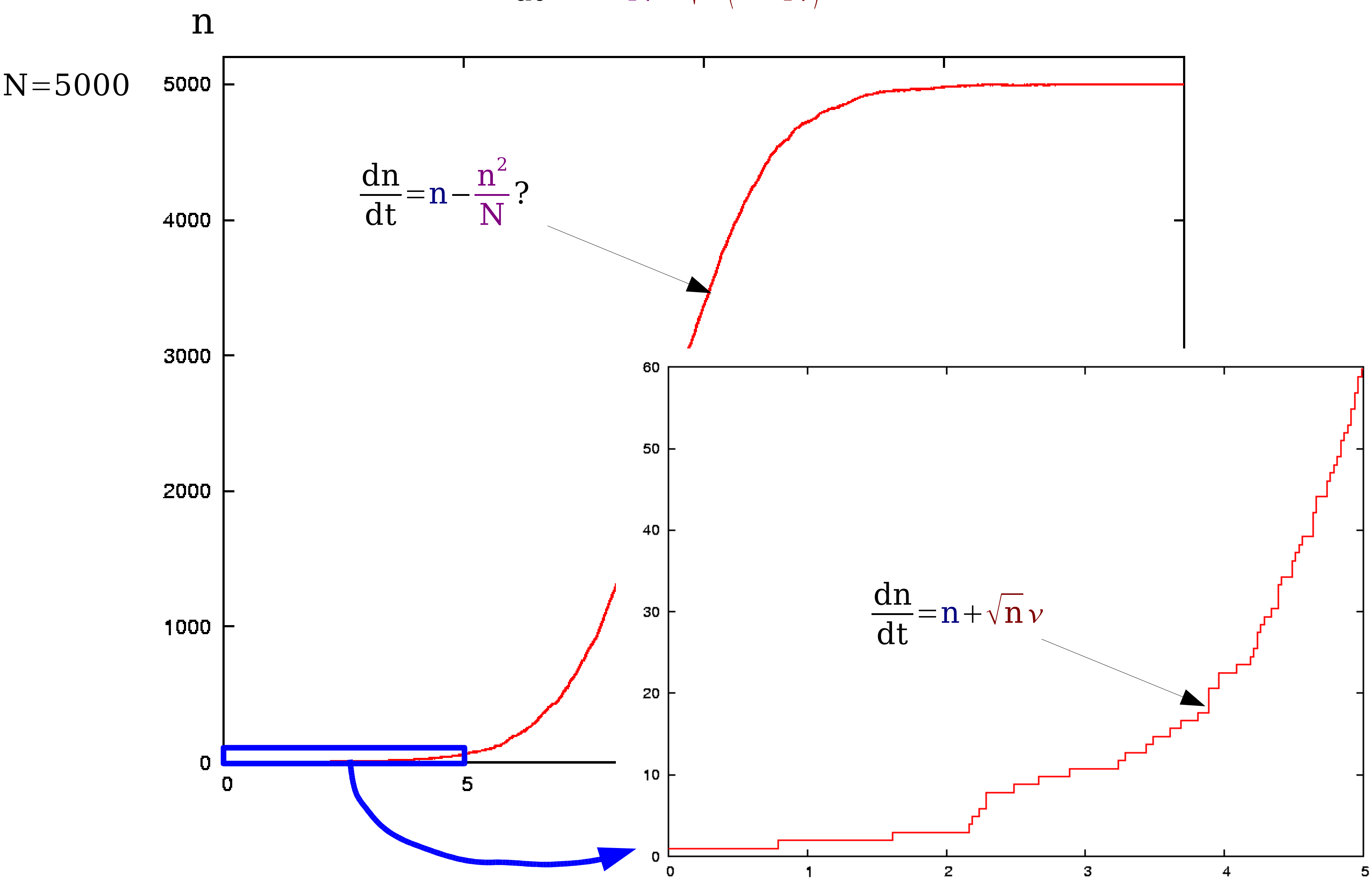
Numerical illustration

$$\frac{dn}{dt} = n - \frac{n^2}{N} + \sqrt{n\left(1+\frac{n}{N}\right)}\,\nu$$



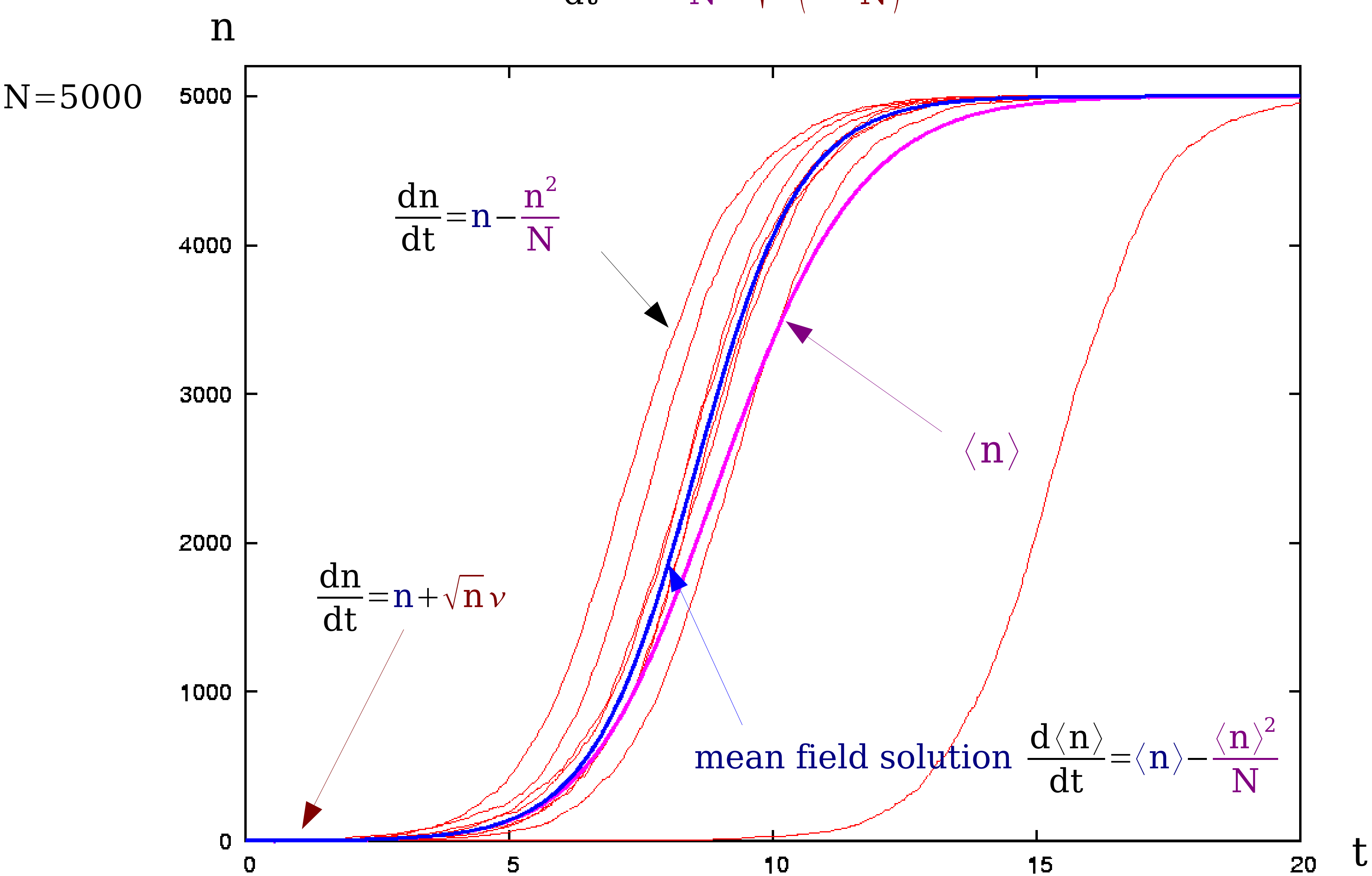
Numerical illustration

$$\frac{dn}{dt} = n - \frac{n^2}{N} + \sqrt{n\left(1+\frac{n}{N}\right)}\,\nu$$



Numerical illustration

$$\frac{dn}{dt}=n-\frac{n^2}{N}+\sqrt{n\left(1+\frac{n}{N}\right)}\,\nu$$



Summary of the part on simple stochastic processes

We have considered a model that evolve according to nonlinear stochastic differential equations of the form

$$\frac{dn}{dt} = n - \frac{n^2}{N} + \sqrt{n\left(1+\frac{n}{N}\right)}\,\nu$$

For the nonlinearity, $\langle n \rangle$ does not obey a closed equation, but an infinite hierarchy of equations of the Balitsky type. A direct resolution is difficult, and the mean field solution completely fails!

See Shoshi, Xiao (2005)

However, there is a simple factorization at the level of individual realizations:

If N is large enough, realizations evolve first through the ***stochastic <u>but</u> linear equation***

$$\frac{dn}{dt} = n + \sqrt{n}\,\nu$$

until n is large enough for the noise term to be small, and continues evolving through the ***nonlinear <u>but</u> deterministic equation***

$$\frac{dn}{dt} = n - \frac{n^2}{N} \qquad \text{when } n \gg 1 .$$

Then, $\langle n \rangle$ is obtained from the averaging of many such realizations

Outline

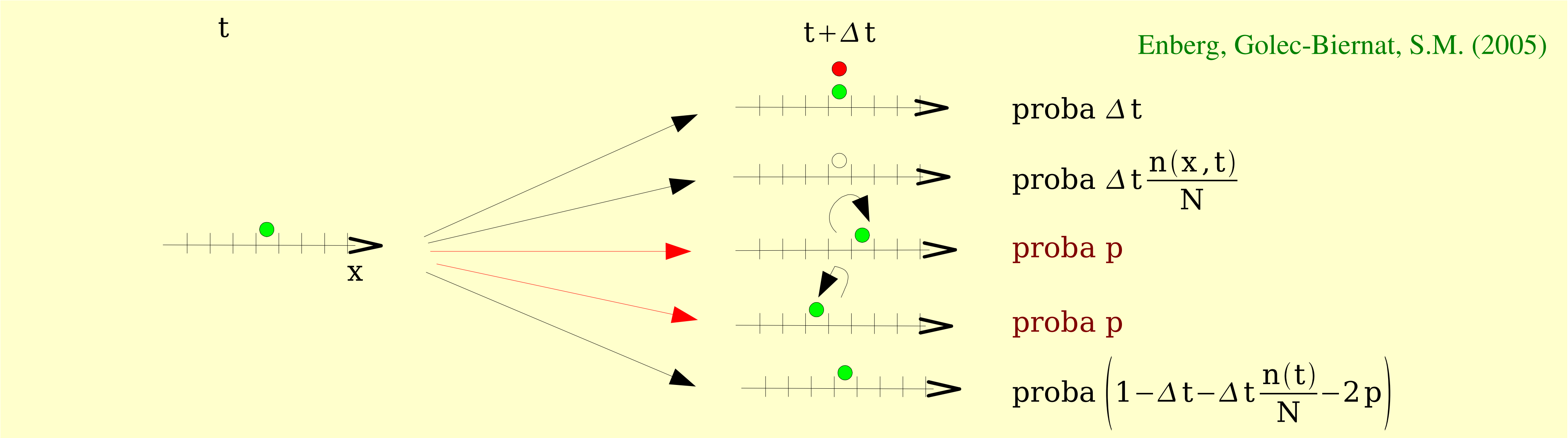
Lecture 1

- Universality: lessons from condensed matter
- Stochastic processes: simple examples
- Reaction-diffusion and traveling wave equations
- High energy scattering as a reaction-diffusion process

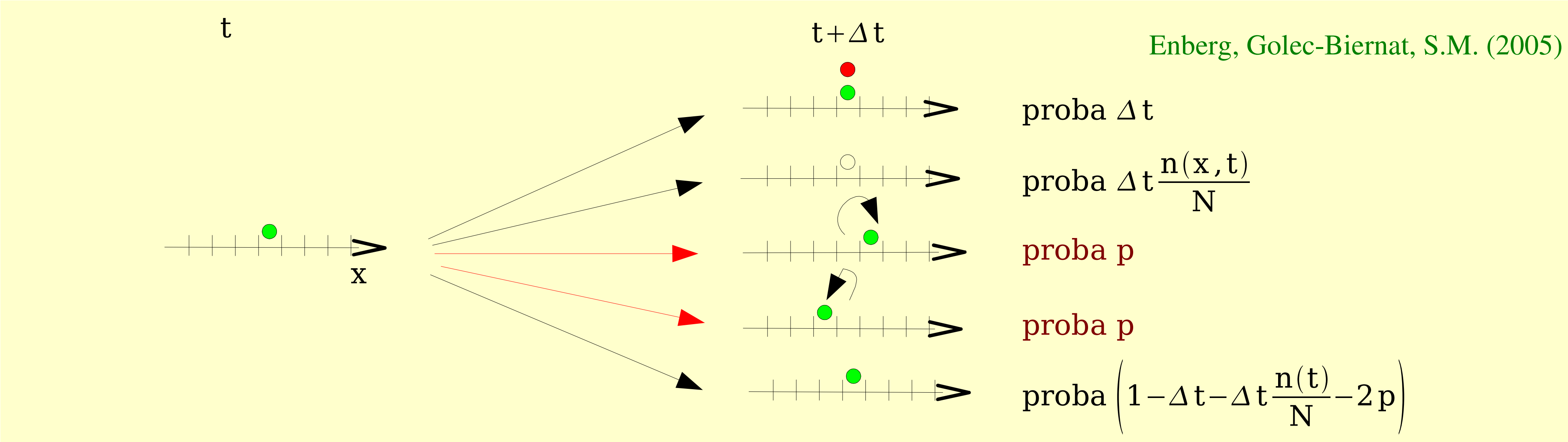
Lecture 2

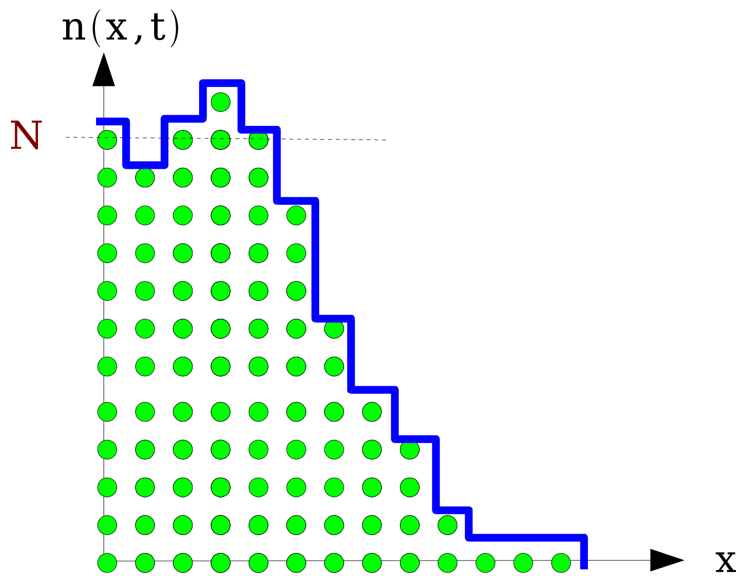
- Results on noisy traveling waves
- Genealogies in selective evolution models
- A connection to the Parisi theory of spin glasses?

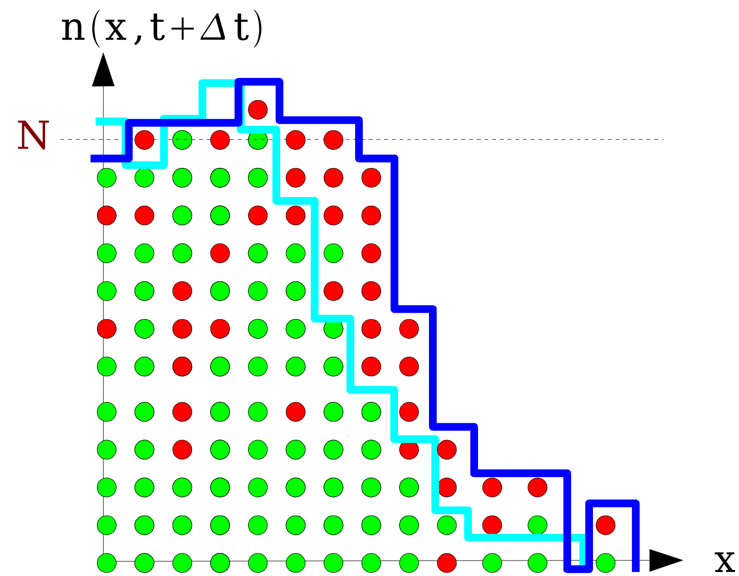
Reaction-diffusion



Reaction-diffusion

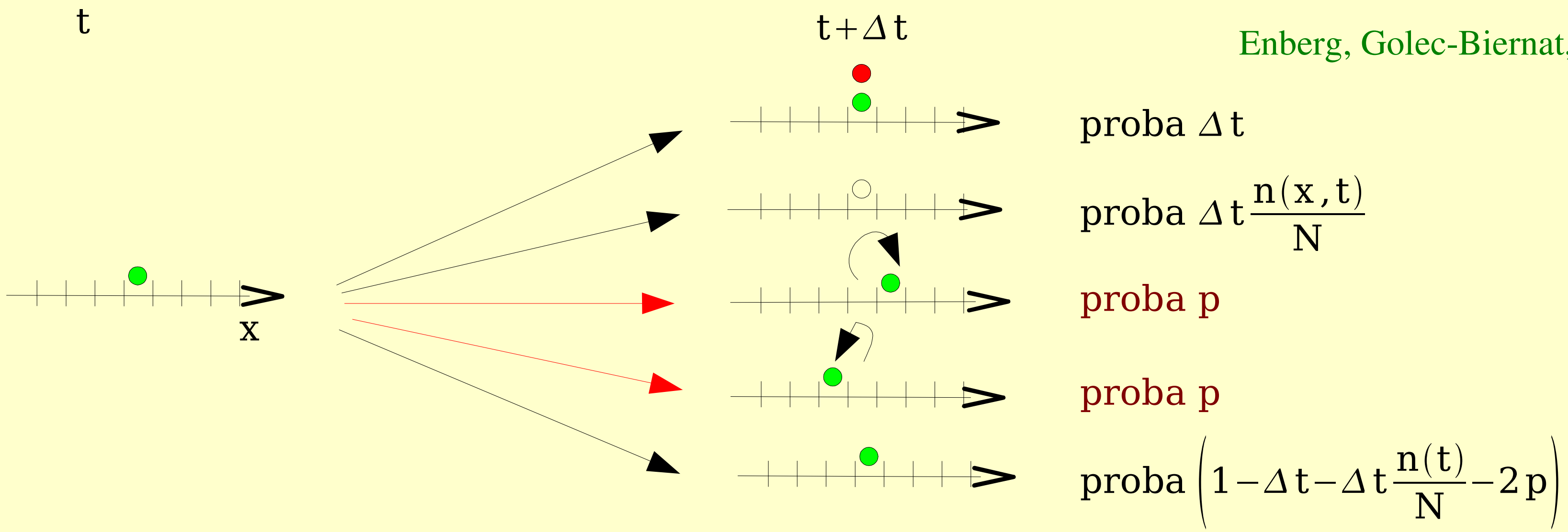


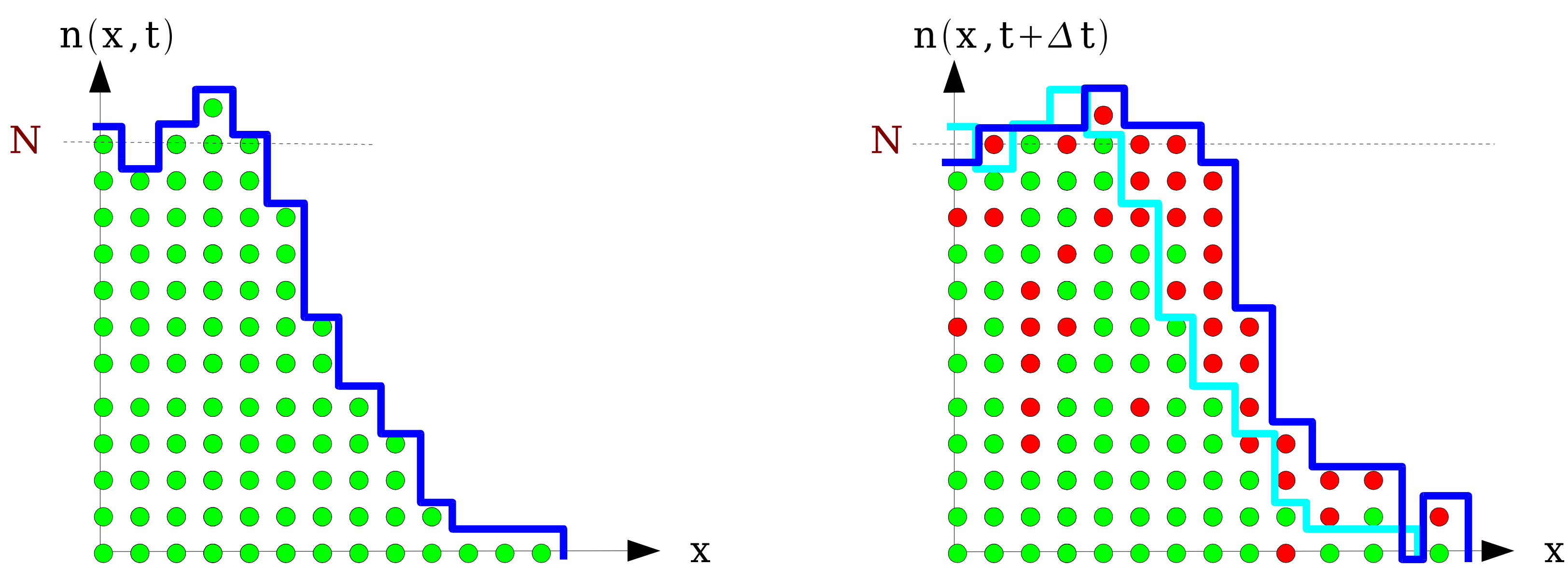




Reaction-diffusion

Enberg, Golec-Biernat, S.M. (2005)

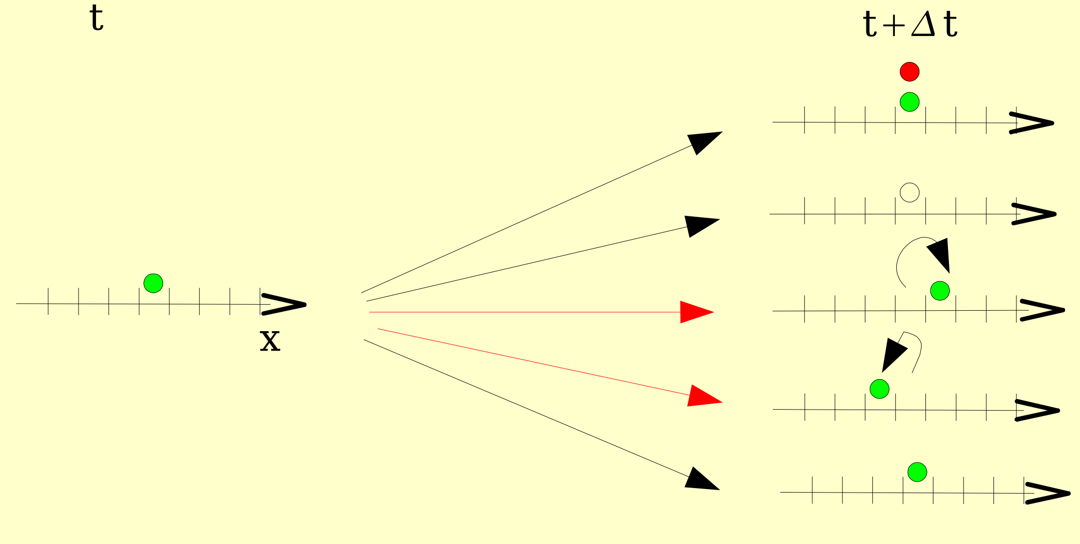




$$n(x,t+\Delta t)=n(x,t)+p(n(x+\Delta x,t)+n(x-\Delta x,t)-2n(x,t))+\Delta t\,u(x,t)-\Delta t\frac{n^2(x,t)}{N}+\Delta t\sqrt{n}\,\nu(x,t+\Delta t)$$

Reaction-diffusion

Enberg, Golec-Biernat, S.M. (2005)



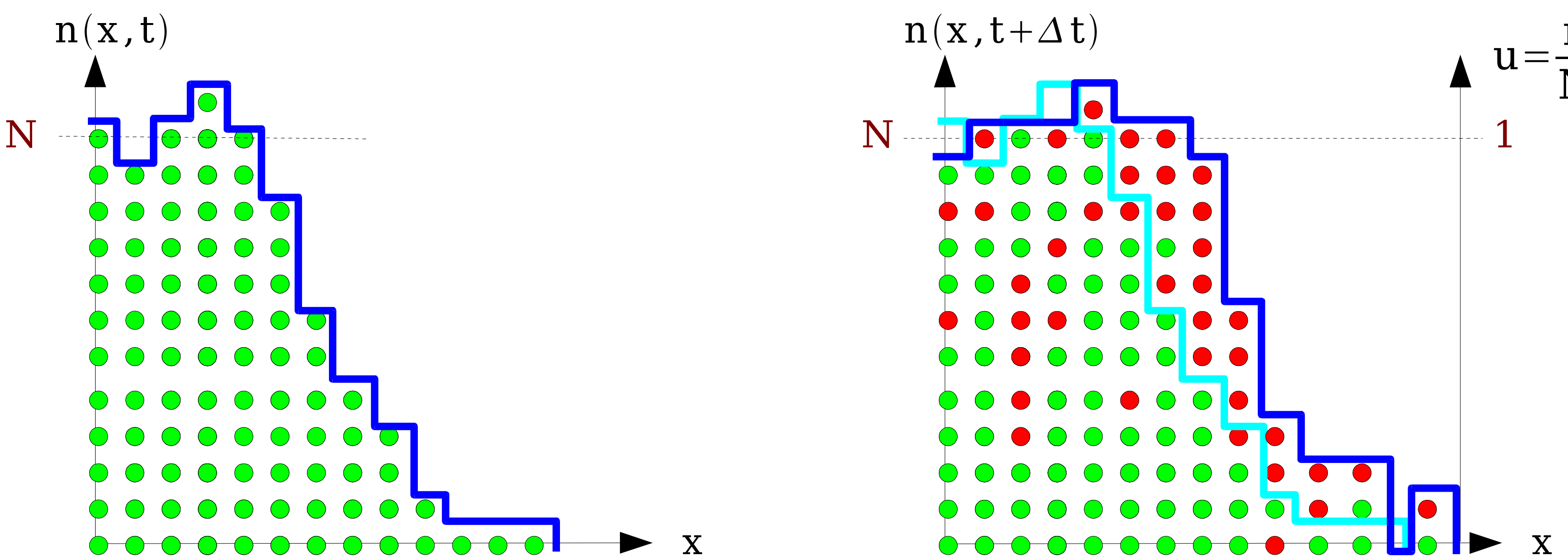
proba Δt

proba $\Delta t \dfrac{n(x,t)}{N}$

proba p

proba p

proba $\left(1-\Delta t-\Delta t\dfrac{n(t)}{N}-2p\right)$



$u=\dfrac{n}{N}$

$$u(x,t+\Delta t)=u(x,t)+p\big(u(x+\Delta x,t)+u(x-\Delta x,t)-2u(x,t)\big)+\Delta t\, u(x,t)-\Delta t\, u^2(x,t)+\Delta t\sqrt{\frac{u}{N}}\,\nu(x,t+\Delta t)$$

Traveling wave equations

Reaction-diffusion

$$u(x,t+\Delta t)=u(x,t)+p(u(x+\Delta x,t)+u(x-\Delta x,t)-2u(x,t))+\Delta t\, u(x,t)-\Delta t\, u^2(x,t)+\Delta t\sqrt{\frac{u}{N}}\nu(x,t+\Delta t)$$

Paradigm evolution equation: the sF-KPP equation

Fisher;
Kolmogorov, Petrovsky, Piscounov (1937)

$$\partial_t u=\partial_x^2 u+u-u^2+\sqrt{\frac{u}{N}(1-u)}\,\nu$$

$$\chi(\gamma)=\gamma^2+1$$

nonlinear function: u^2

Traveling wave equations

Reaction-diffusion

$$u(x,t+\Delta t)=u(x,t)+p(u(x+\Delta x,t)+u(x-\Delta x,t)-2u(x,t))+\Delta t\, u(x,t)-\Delta t\, u^2(x,t)+\Delta t\sqrt{\frac{u}{N}}\,\nu(x,t+\Delta t)$$

Paradigm evolution equation: the sF-KPP equation

Fisher;
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$$\boxed{\partial_t u=\partial_x^2 u+u-u^2+\sqrt{\frac{u}{N}(1-u)}\,\nu}$$

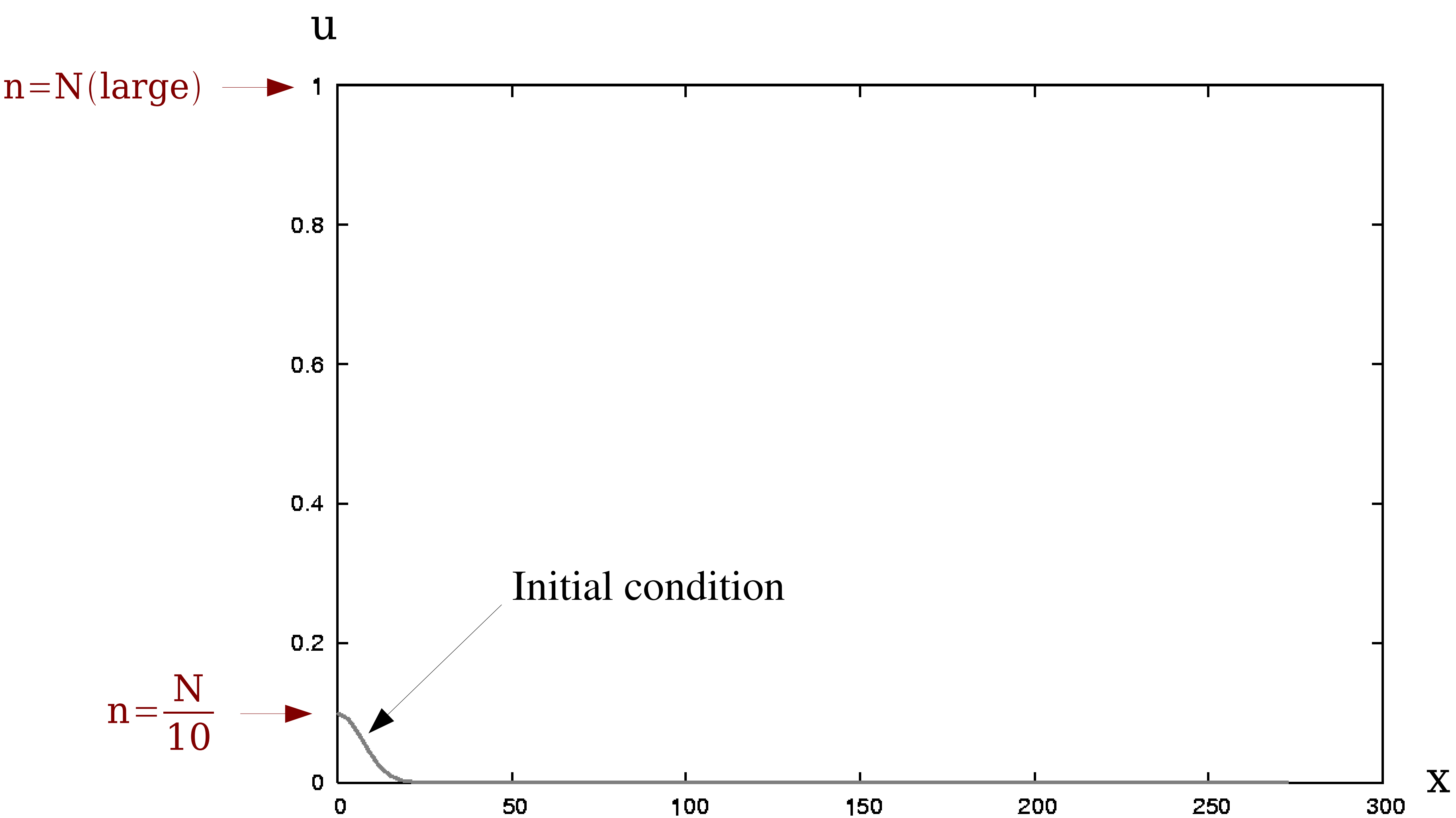
$$\chi(\gamma)=\gamma^2+1$$

$\text{nonlinear function}: u^2$

General structure of evolution equations for such processes

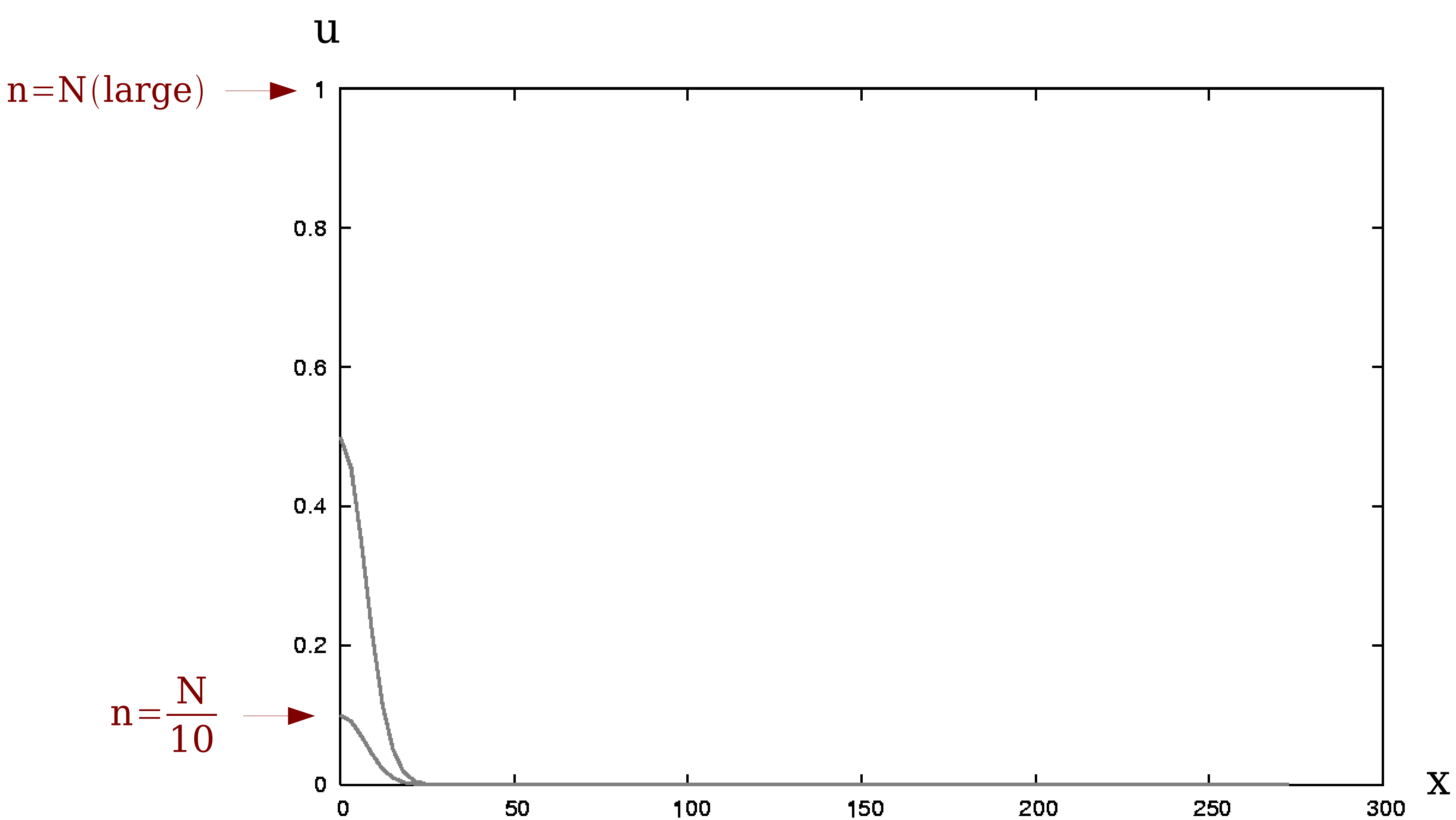
$$\partial_t u=\left[\begin{array}{c}\chi(-\partial_x)u\\ \text{encodes diffusive growth of u}\end{array}\right]-\left[\begin{array}{c}\text{nonlinear function of u}\\ \text{compensates the growth of u near 1}\end{array}\right]+\left[\text{noise of order }\sqrt{\frac{u}{N}}\right]$$

Traveling wave equations: solutions



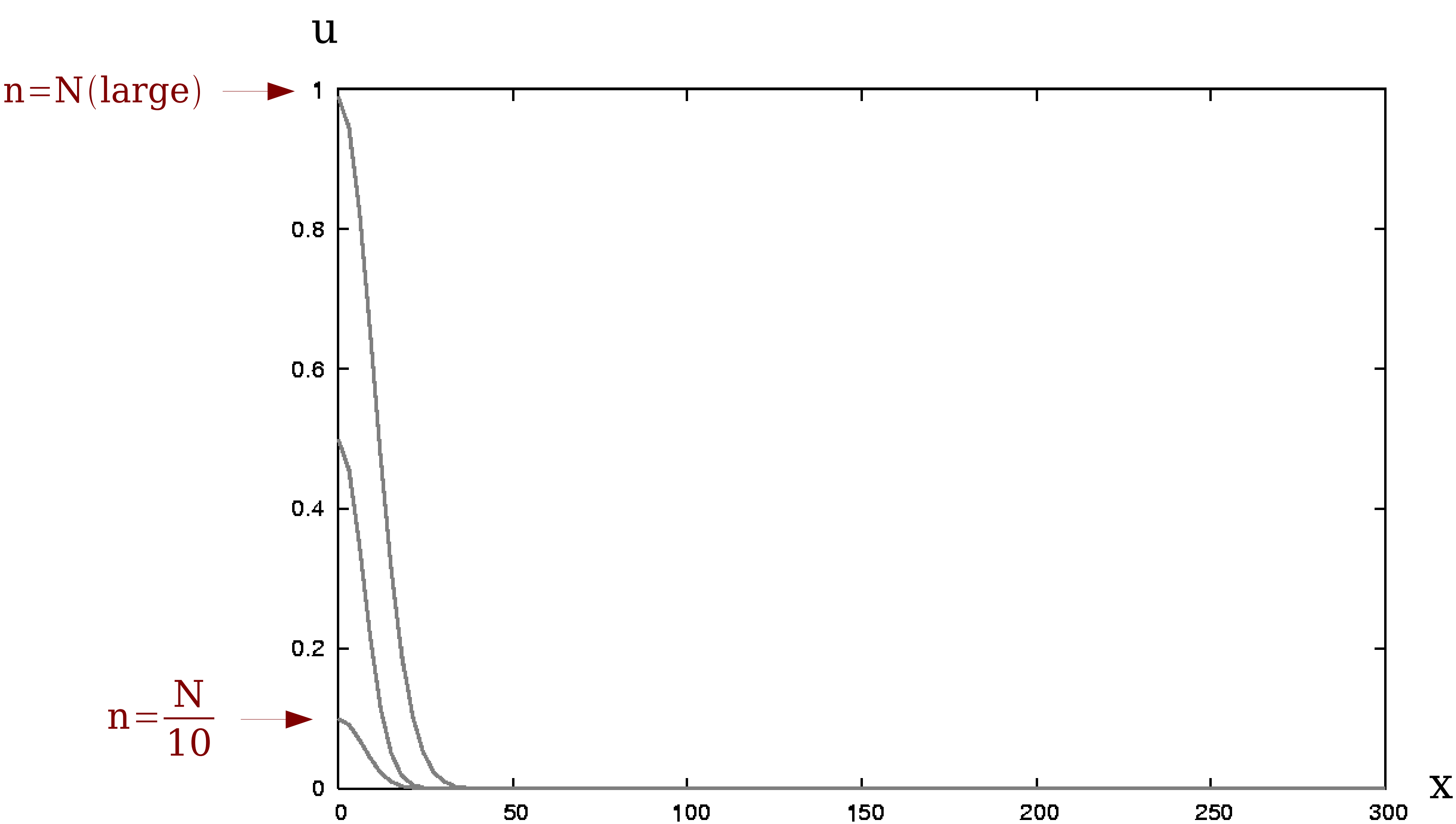
$$\partial_t u = \left[\begin{array}{c} \chi(-\partial_x) u \\ \text{encodes diffusive growth of u} \end{array} \right] - \cancel{\left[\begin{array}{c} \text{nonlinear function of u} \\ \text{compensates the growth of u near 1} \end{array} \right]} + \left[\text{noise of order } \sqrt{\frac{u}{N}} \right]$$

Traveling wave equations: solutions



$$\partial_t u = \left[\begin{array}{c}\chi(-\partial_x)u \\ \text{encodes diffusive growth of u}\end{array}\right] - \cancel{\left[\begin{array}{c}\text{nonlinear function of u} \\ \text{compensates the growth of u near 1}\end{array}\right]} + \left[\text{noise of order } \sqrt{\frac{u}{N}}\right]$$

Traveling wave equations: solutions



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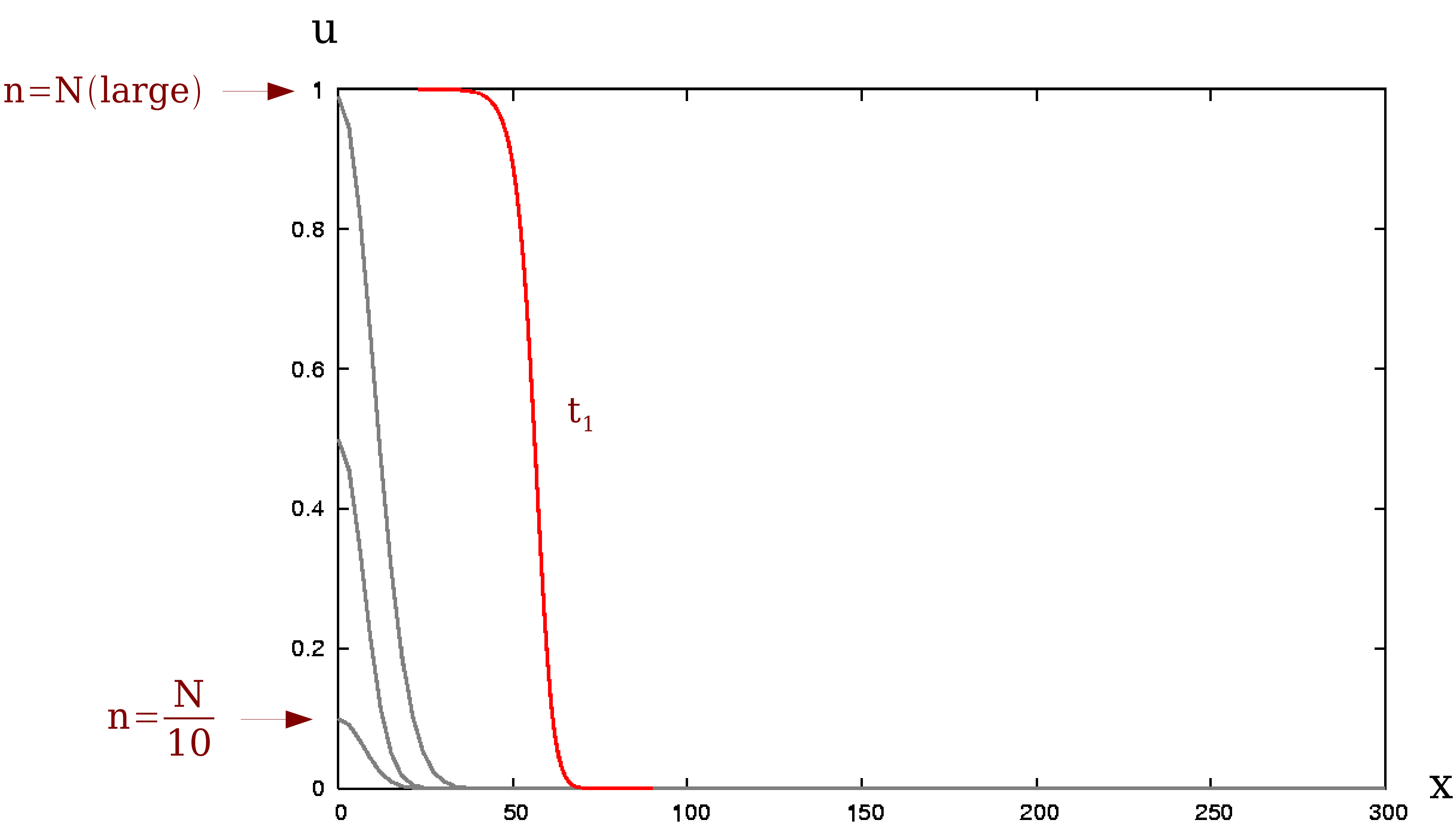
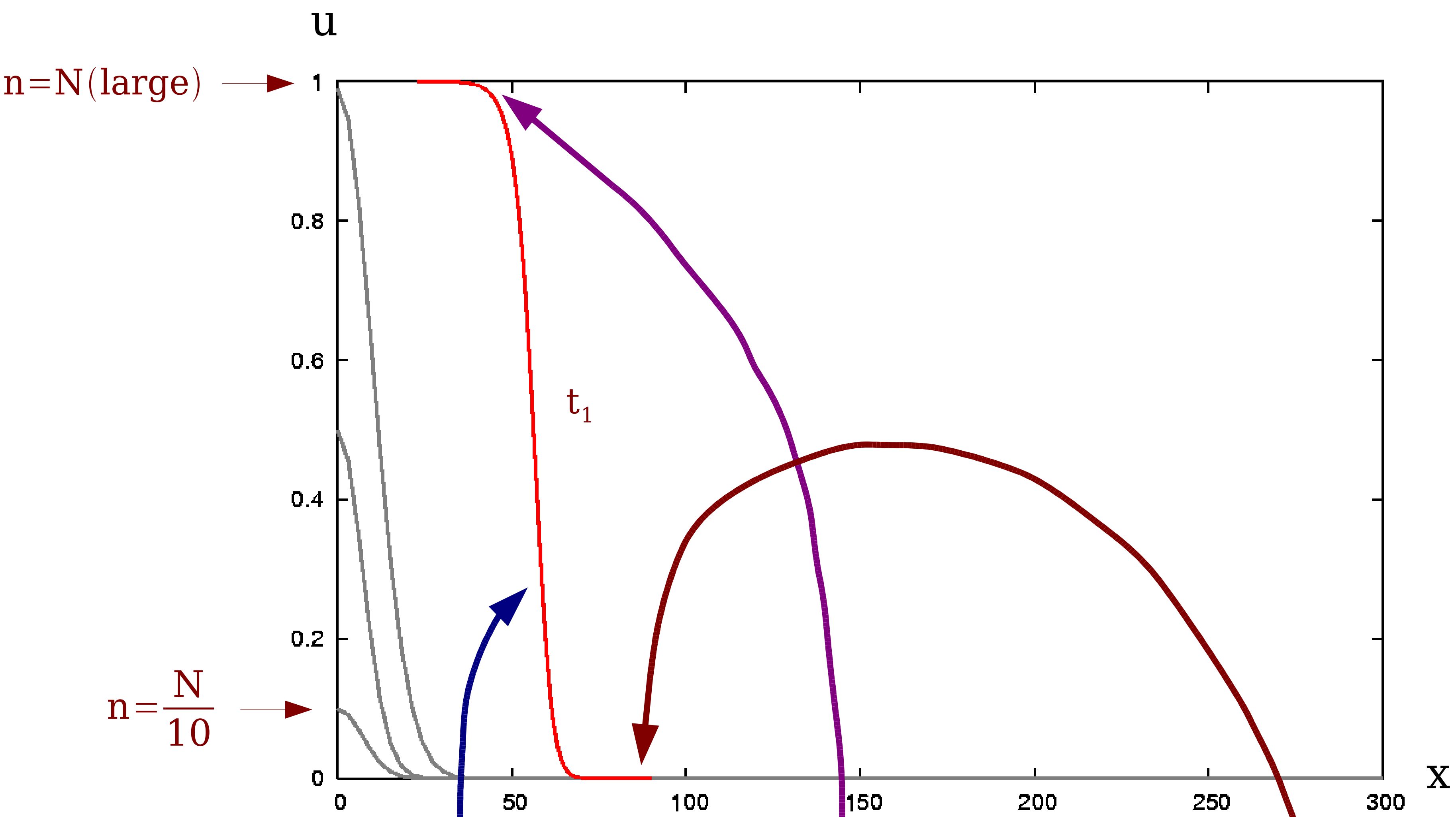
Traveling wave equations: solutions



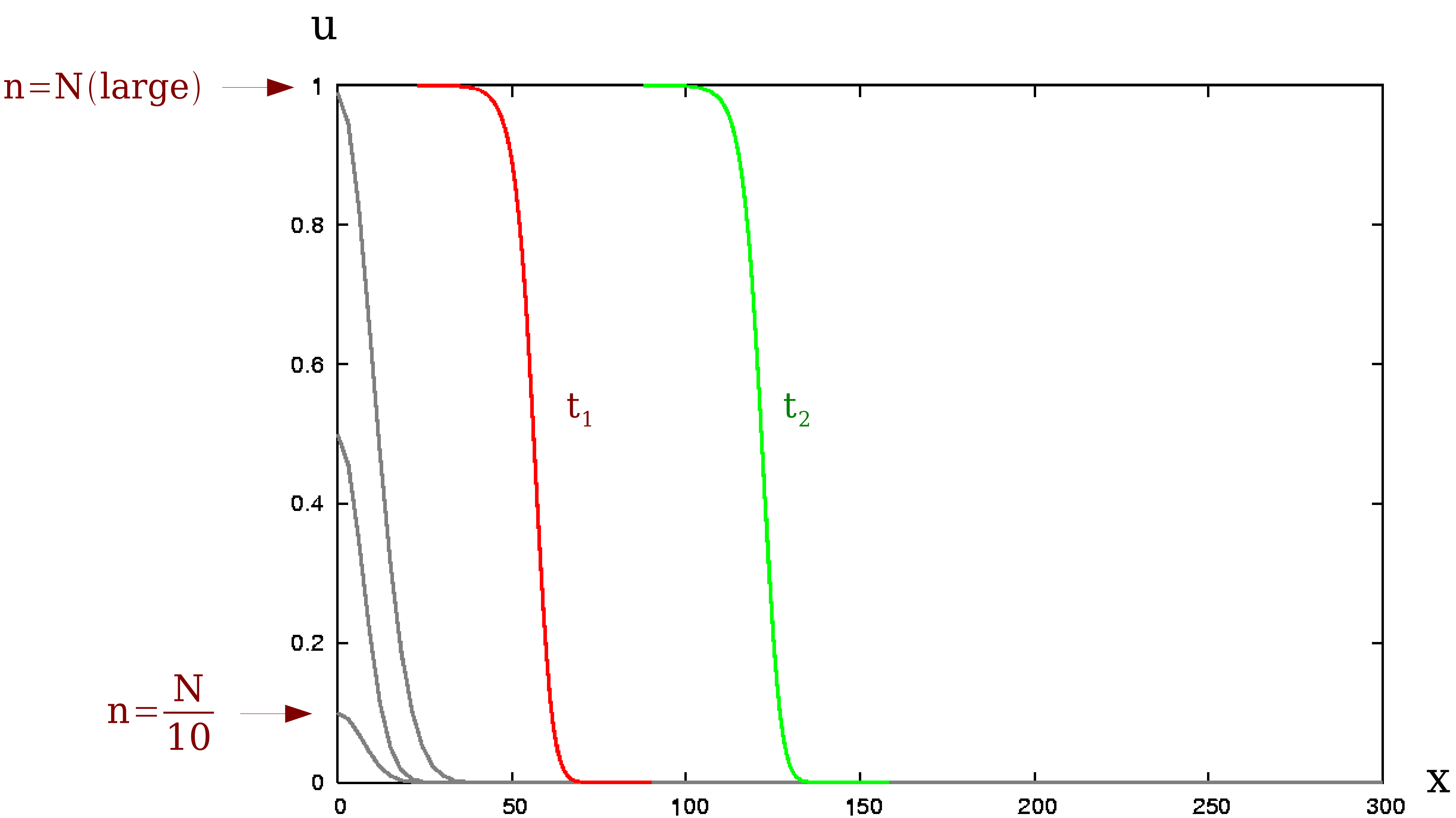
$$\partial_t u = \left[\begin{array}{c}\chi(-\partial_x)u \\ \text{encodes diffusive growth of u}\end{array}\right] - \left[\begin{array}{c}\text{nonlinear function of u} \\ \text{compensates the growth of u near 1}\end{array}\right] + \left[\text{noise of order } \sqrt{\frac{u}{N}}\right]$$

Traveling wave equations: solutions



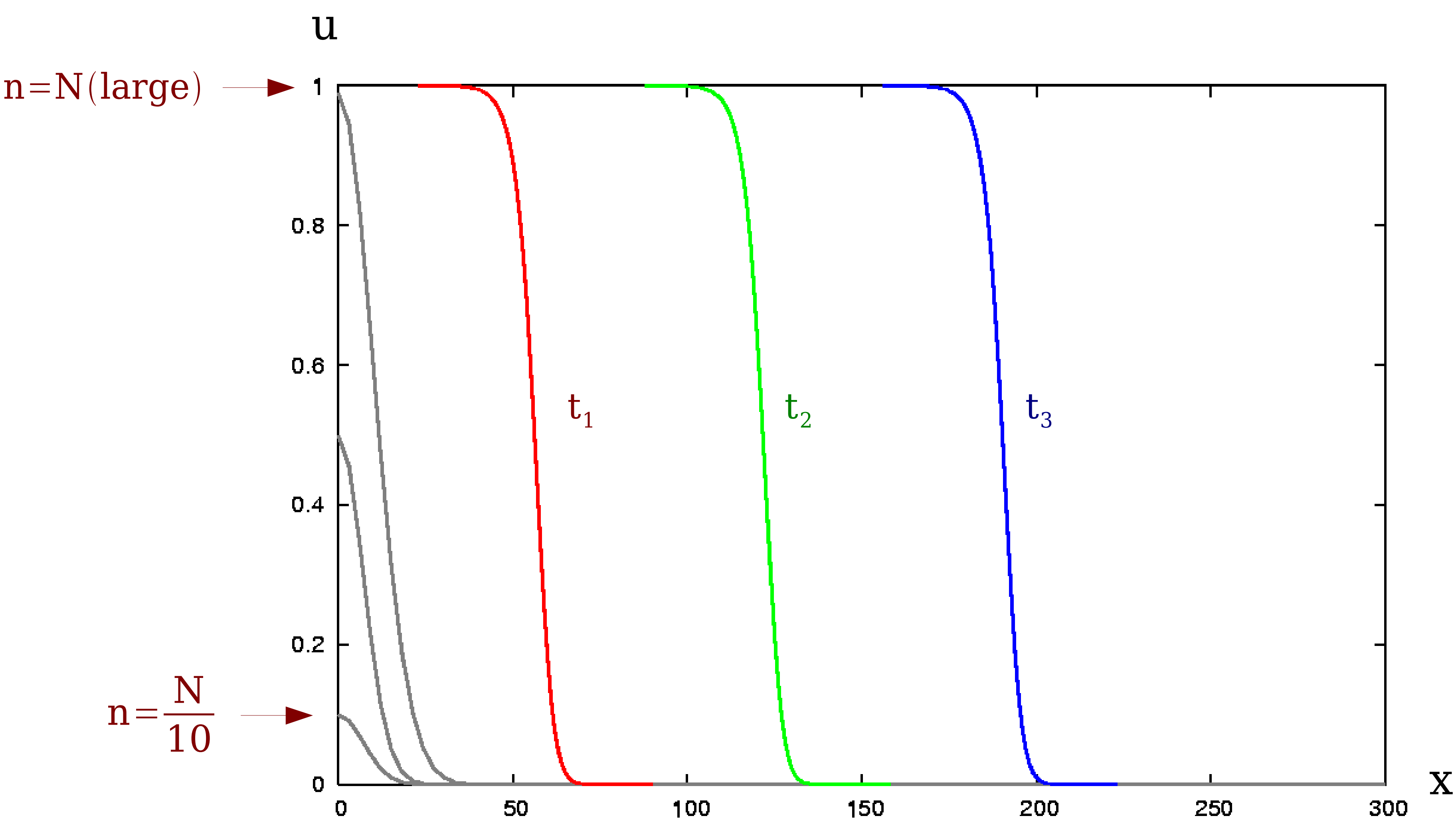
$$\partial_t u = \left[\underset{\text{encodes diffusive growth of u}}{\chi(-\partial_x)u}\right] - \left[\underset{\text{compensates the growth of u near 1}}{\text{nonlinear function of u}}\right] + \left[\text{noise of order } \sqrt{\frac{u}{N}}\right]$$

Traveling wave equations: solutions



$$\partial_t u = \left[\underset{\text{encodes diffusive growth of u}}{\chi(-\partial_x) u} \right] - \left[\underset{\text{compensates the growth of u near 1}}{\text{nonlinear function of u}} \right] + \left[\text{noise of order } \sqrt{\frac{u}{N}} \right]$$

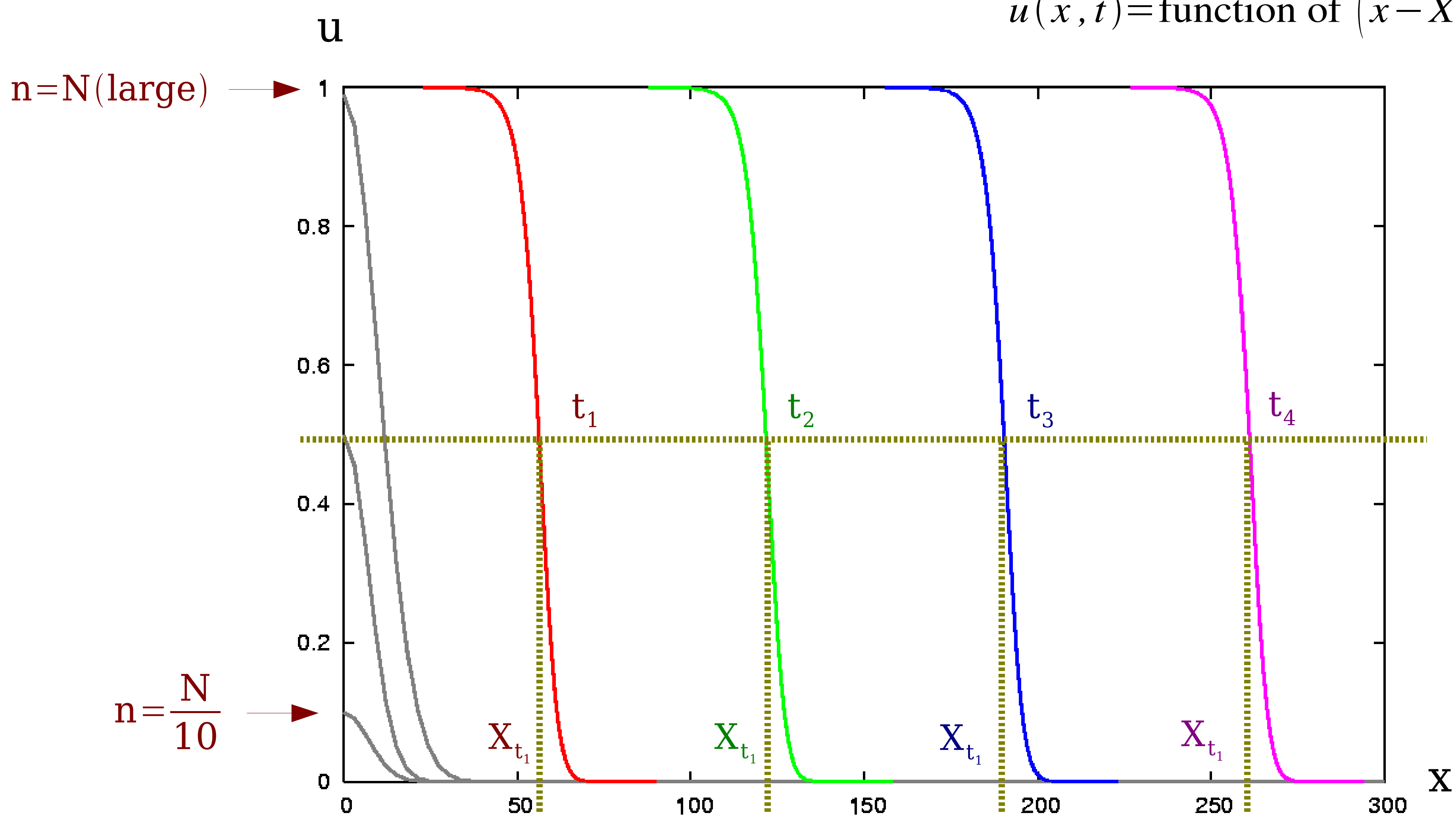
Traveling wave equations: solutions



$$\partial_t u = \left[\begin{array}{c}\chi(-\partial_x)u \\ \text{encodes diffusive growth of u}\end{array}\right] - \left[\begin{array}{c}\text{nonlinear function of u} \\ \text{compensates the growth of u near 1}\end{array}\right] + \left[\text{noise of order } \sqrt{\frac{u}{N}}\right]$$

Traveling wave equations: solutions

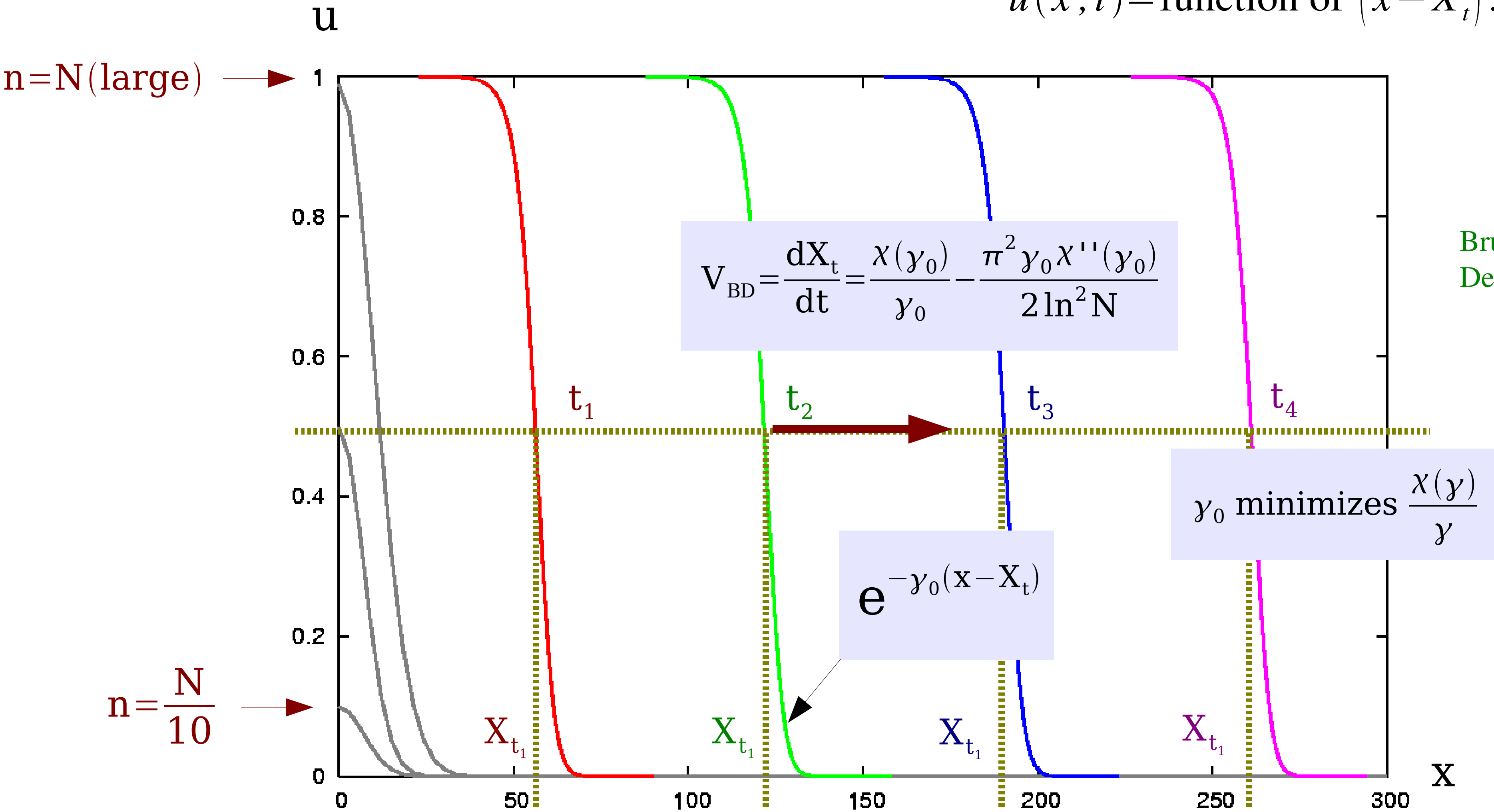
$u(x,t) = \text{function of } (x - X_t)\,?$



$$\partial_t u = \left[\underset{\text{encodes diffusive growth of u}}{\chi(-\partial_x) u} \right] - \left[\underset{\text{compensates the growth of u near 1}}{\text{nonlinear function of u}} \right] + \left[\text{noise of order } \sqrt{\frac{u}{N}} \right]$$

Traveling wave equations: solutions

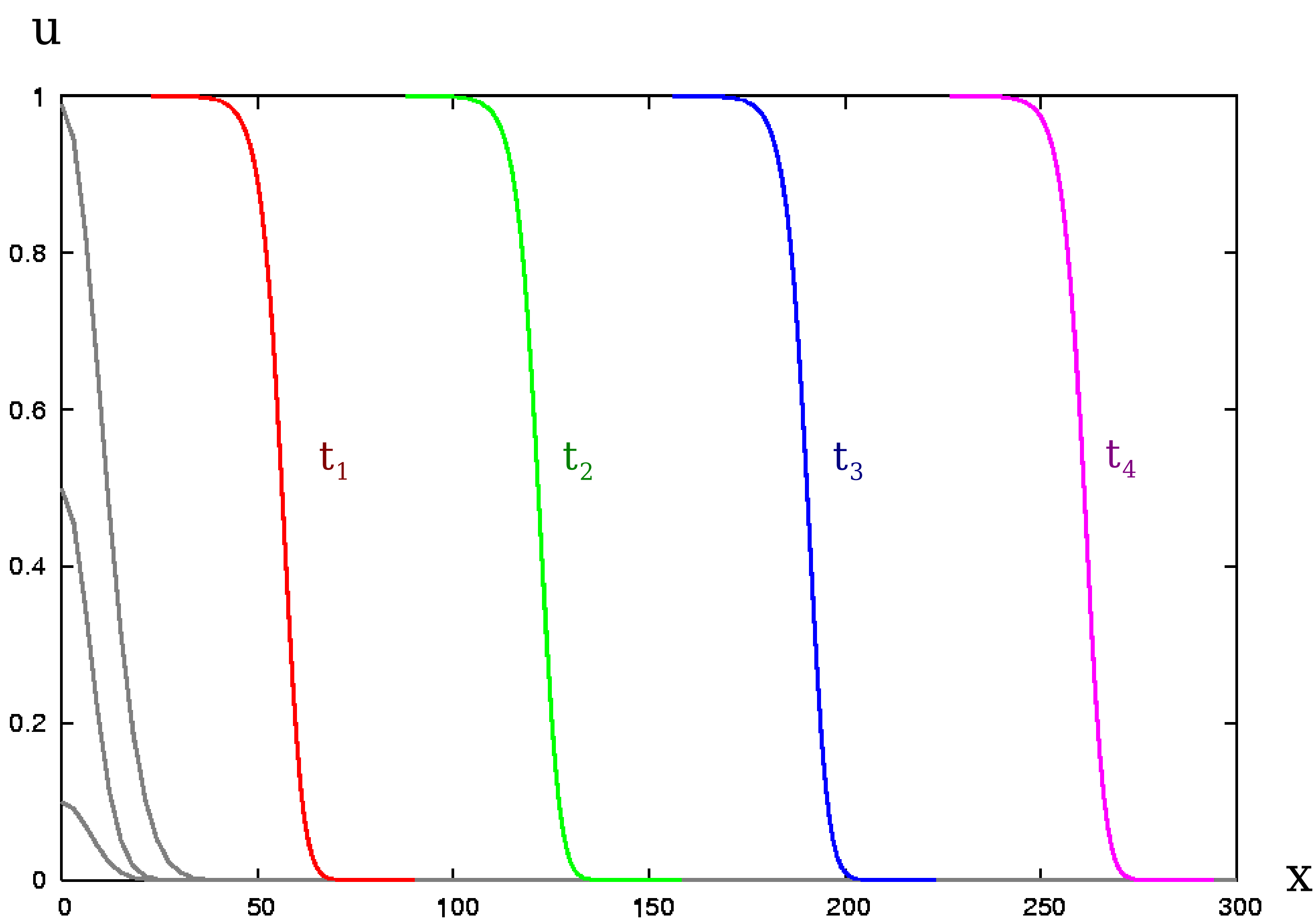
$$u(x,t) = \text{function of } \left(x - X_t\right)?$$



Brunet, Derrida (1997)

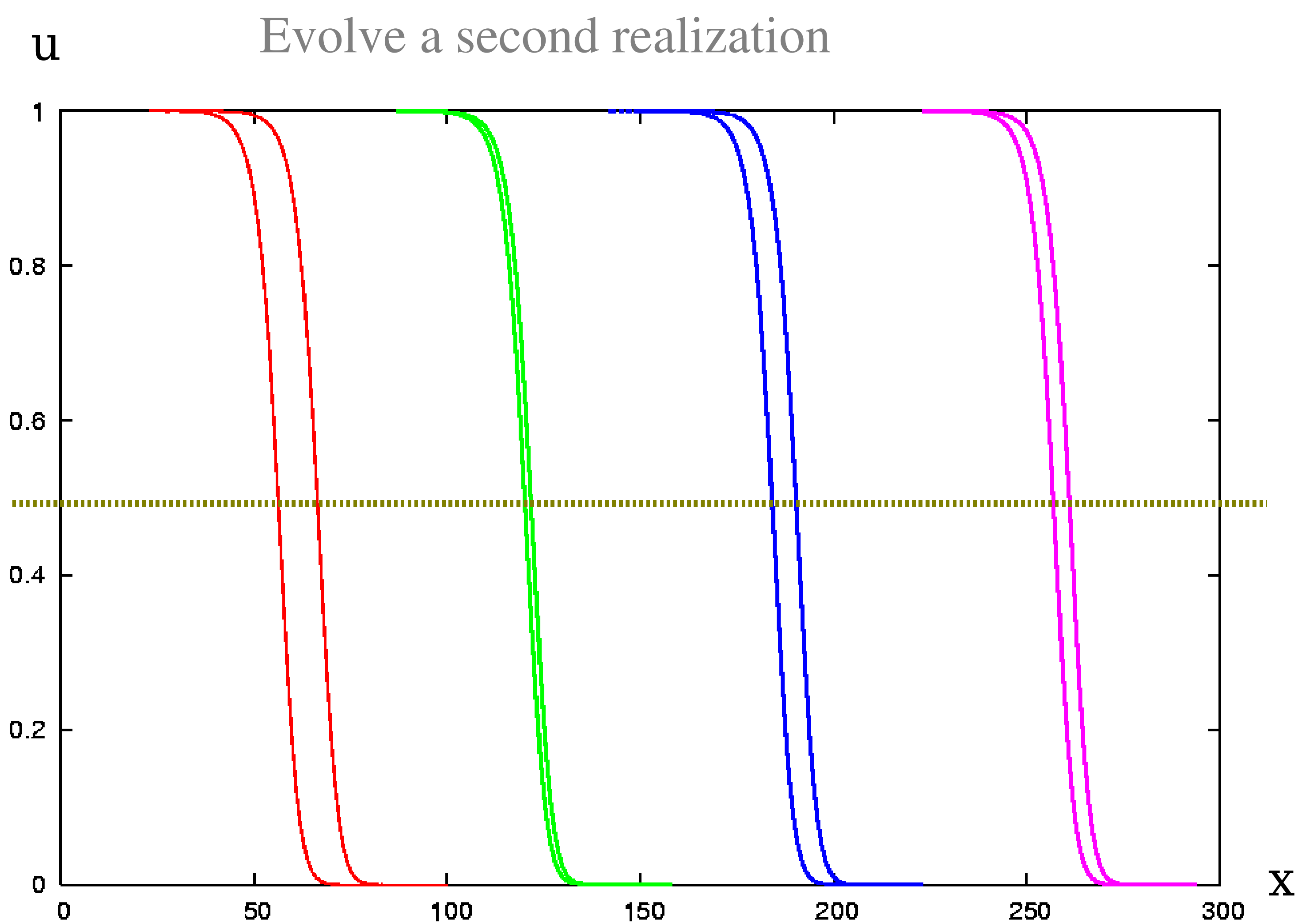
$$\partial_t u = \left[\begin{array}{c} \chi(-\partial_x)\,u \\ \text{encodes diffusive growth of u} \end{array}\right] - \left[\begin{array}{c} \text{nonlinear function of u} \\ \text{compensates the growth of u near 1} \end{array}\right] + \left[\text{noise of order } \sqrt{\frac{u}{N}}\right]$$

Traveling wave equations: solutions



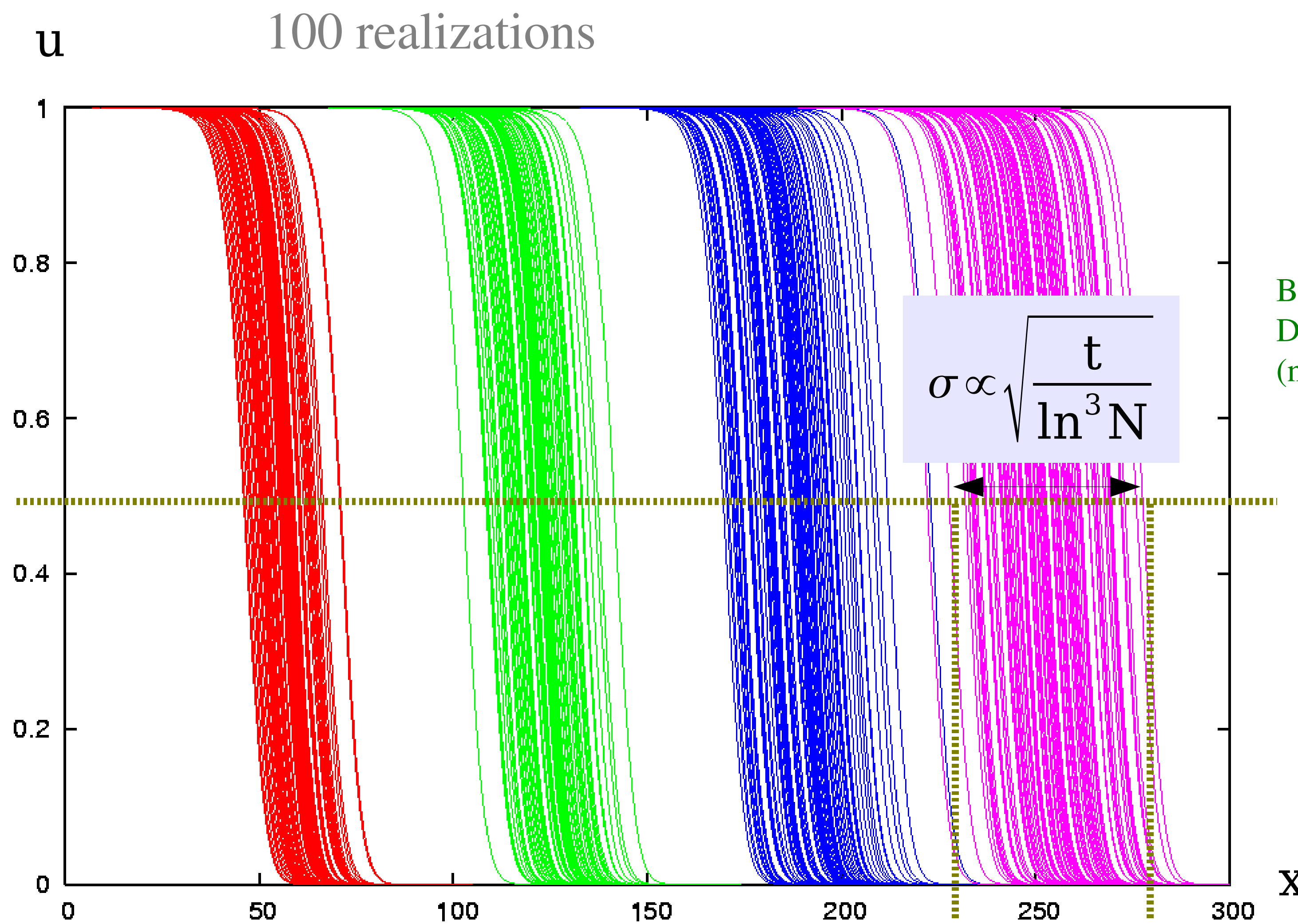
$$\partial_t u = \left[\begin{array}{c}\chi(-\partial_x)u \\ \text{encodes diffusive growth of u}\end{array}\right] - \left[\begin{array}{c}\text{nonlinear function of u} \\ \text{compensates the growth of u near 1}\end{array}\right] + \left[\text{noise of order } \sqrt{\frac{u}{N}}\right]$$

Traveling wave equations: solutions



$$\partial_t u = \left[\begin{matrix}\chi(-\partial_x)u \\ \text{encodes diffusive growth of u}\end{matrix}\right] - \left[\begin{matrix}\text{nonlinear function of u} \\ \text{compensates the growth of u near 1}\end{matrix}\right] + \left[\text{noise of order } \sqrt{\frac{u}{N}}\right]$$

Traveling wave equations: solutions

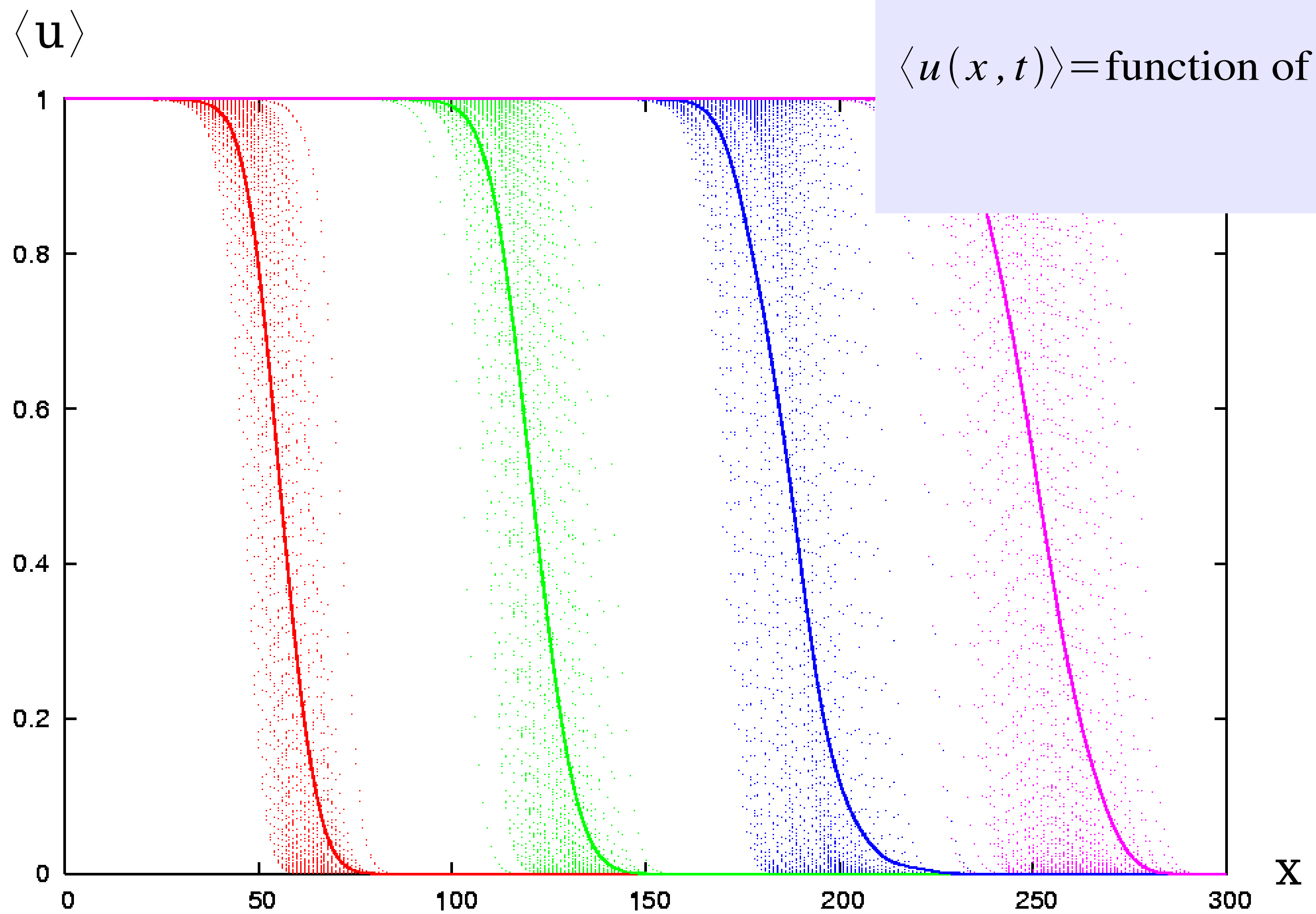


Brunet, Derrida (1999) (numerical result)

$$\partial_t u = \left[\begin{array}{c} \chi(-\partial_x)u \\ \text{encodes diffusive growth of u} \end{array} \right] - \left[\begin{array}{c} \text{nonlinear function of u} \\ \text{compensates the growth of u near 1} \end{array} \right] + \left[\text{noise of order } \sqrt{\frac{u}{N}} \right]$$

Traveling wave equations: solutions

$$\langle u(x,t)\rangle = \text{function of}\left(\frac{x-X_t}{\sqrt{\frac{t}{\ln^3 N}}}\right)$$



$$\partial_t u = \left[\begin{matrix}\chi(-\partial_x)u\\ \text{encodes diffusive growth of u}\end{matrix}\right] - \left[\begin{matrix}\text{nonlinear function of u}\\ \text{compensates the growth of u near 1}\end{matrix}\right] + \left[\text{noise of order } \sqrt{\frac{u}{N}}\right]$$

Summary of the part on stochastic processes

We have considered models that evolve according to nonlinear stochastic partial differential equations of the form

$$\partial_t u = \left[\begin{matrix}\chi(-\partial_x)u \\ \text{encodes diffusive growth of u}\end{matrix}\right] - \left[\begin{matrix}\text{nonlinear function of u} \\ \text{compensates the growth of u near 1}\end{matrix}\right] + \left[\text{noise of order } \sqrt{\frac{u}{N}}\right]$$

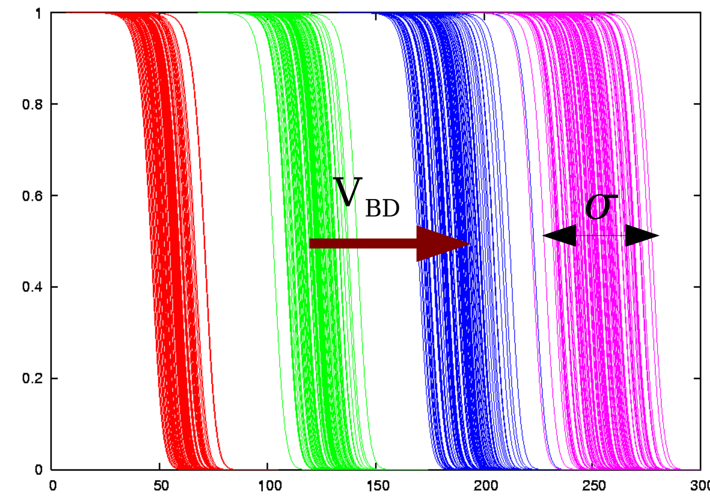
which is in the universality class of the sF-KPP equation

$$\partial_t u = \partial_x^2 u + u - u^2 + \sqrt{\frac{u}{N}(1-u)}\,\nu$$

$$\chi(\gamma) = \gamma^2 + 1$$
$$\text{nonlinear function}: u^2$$

These equations admit ***traveling wave solutions***, with ***universal features at large N and t***

average velocity	$V_{BD} = \frac{\chi(\gamma_0)}{\gamma_0} - \frac{\pi^2 \gamma_0 \chi''(\gamma_0)}{2\ln^2 N}$
shape	$u \sim e^{-\gamma_0(x - V_{BD} t)}$
dispersion in the position	$\sigma \propto \sqrt{\frac{t}{\ln^3 N}}$



Outline

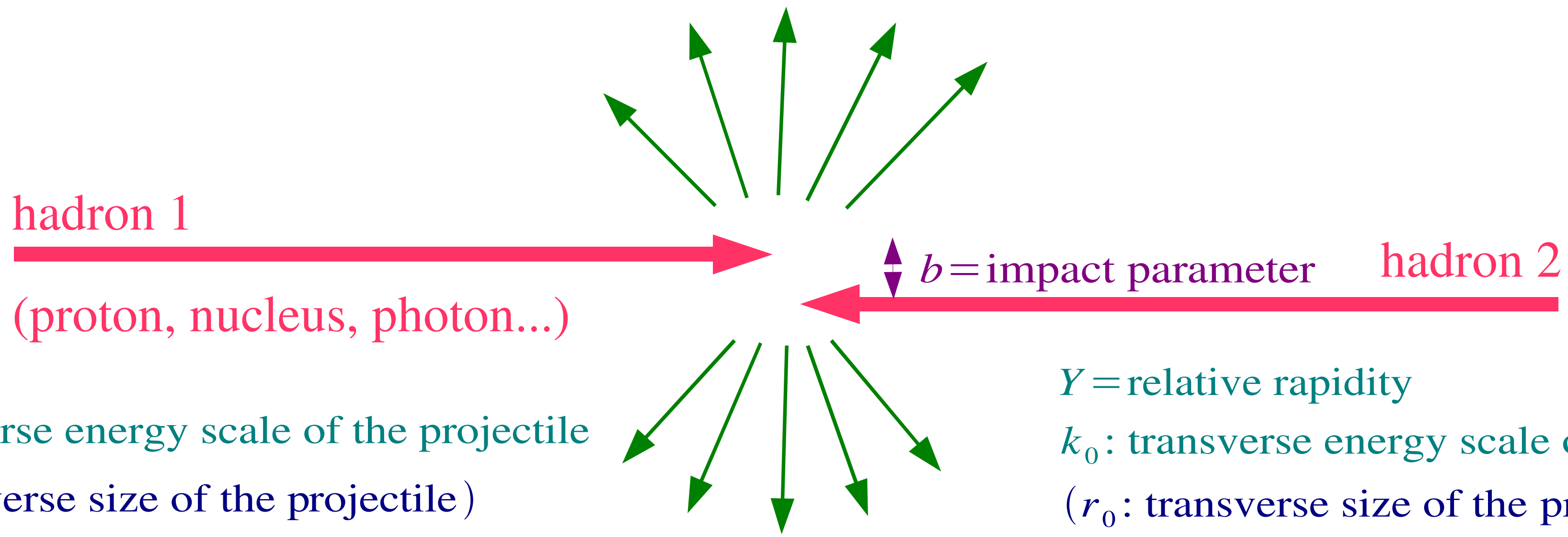
Lecture 1

- ✯ Universality: lessons from condensed matter
- ✯ Stochastic processes: simple examples
- ✯ Reaction-diffusion and traveling wave equations
- ✯ High energy scattering as a reaction-diffusion process

Lecture 2

- ✯ Results on noisy traveling waves
- ✯ Genealogies in selective evolution models
- ✯ A connection to the Parisi theory of spin glasses?

High energy QCD



Y = relative rapidity

k : transverse energy scale of the projectile

$(r$: transverse size of the projectile$)$

k_0 : transverse energy scale of the target

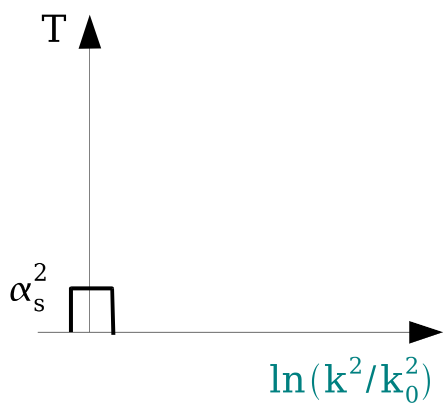
$(r_0$: transverse size of the projectile$)$

$$A(Y,\vec{k}) = \int d^2b\, A(b,Y,\vec{k}) = \text{elastic amplitude}$$

$$A(b,Y,\vec{k}) = \text{fixed impact parameter amplitude} \;\; \boxed{\leq 1}$$

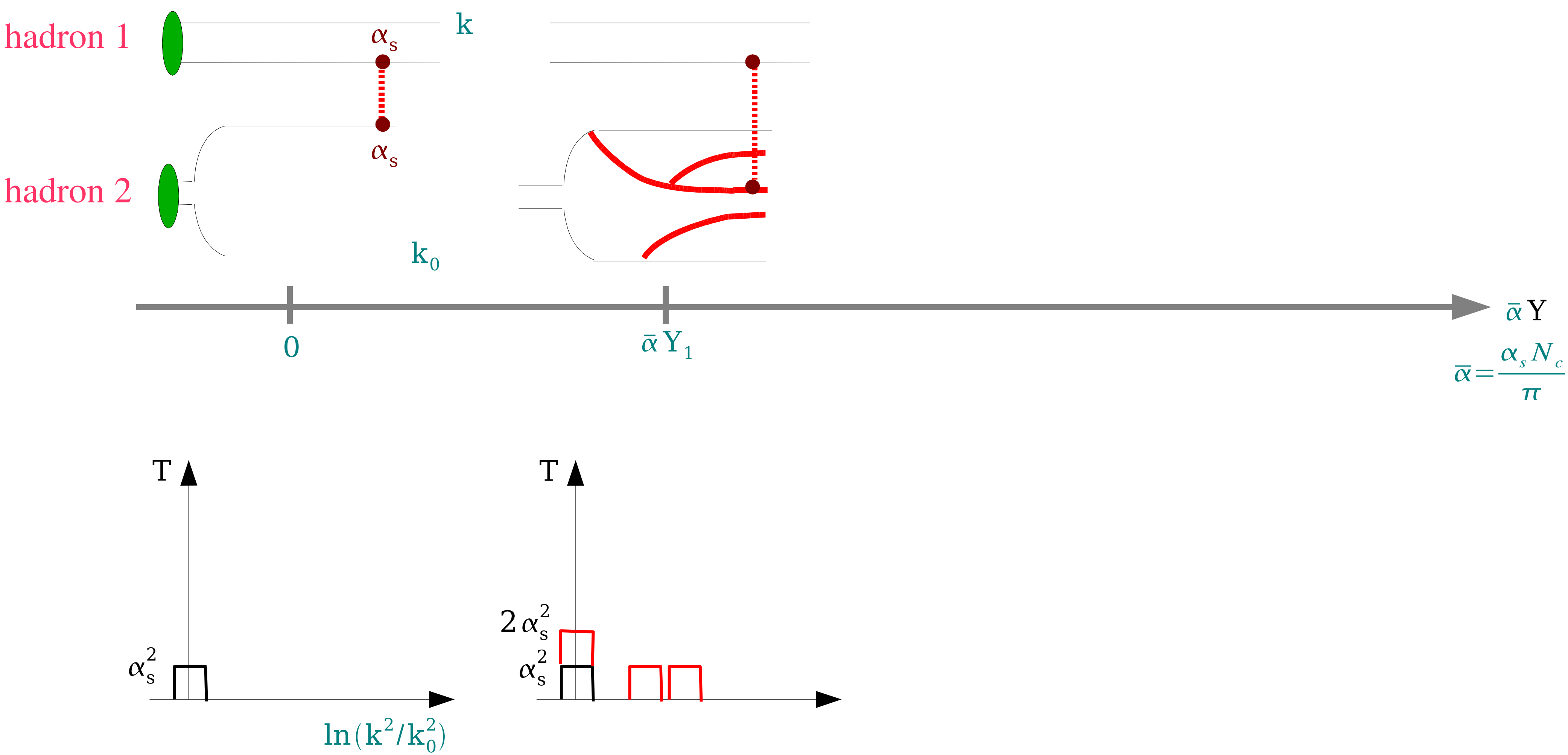
(High) energy dependence of QCD amplitudes?

High energy QCD = reaction-diffusion



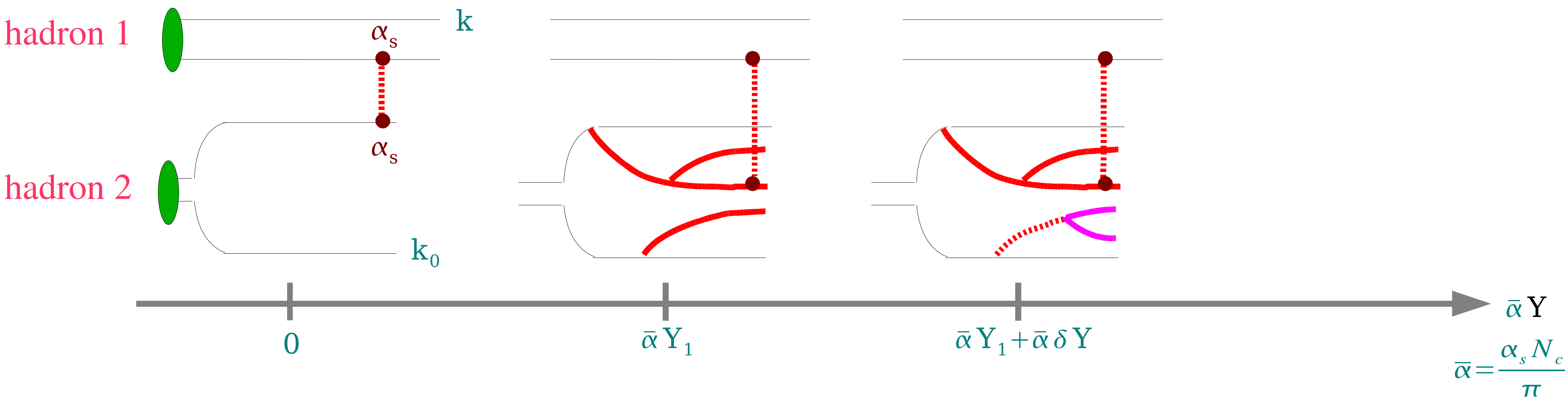
$$T(k) = \alpha_s^2 \times \delta(\ln k^2 - \ln k_0^2)$$

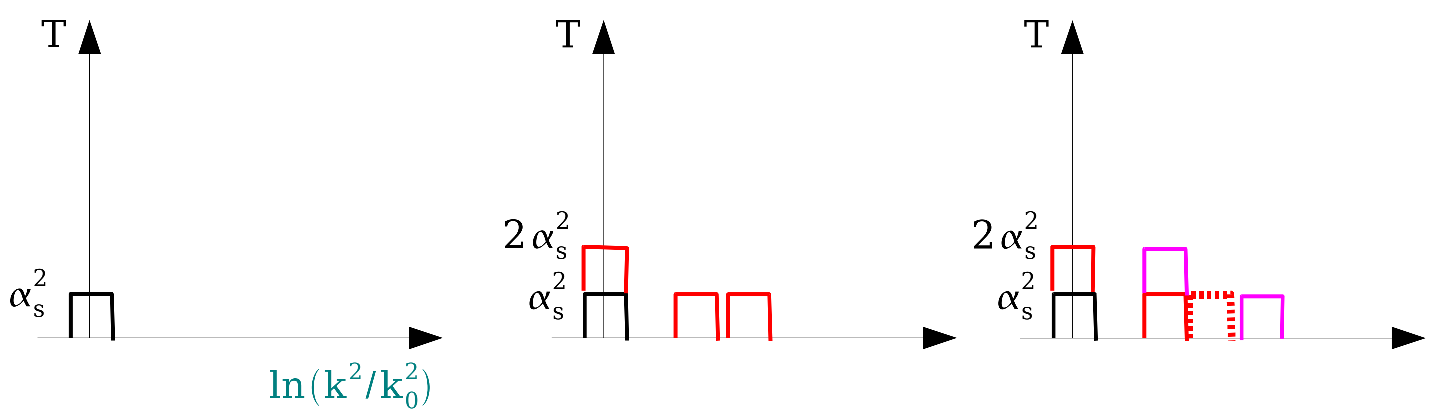
High energy QCD = reaction-diffusion



$$T(k) = \alpha_s^2 \times n(k)$$

High energy QCD = reaction-diffusion



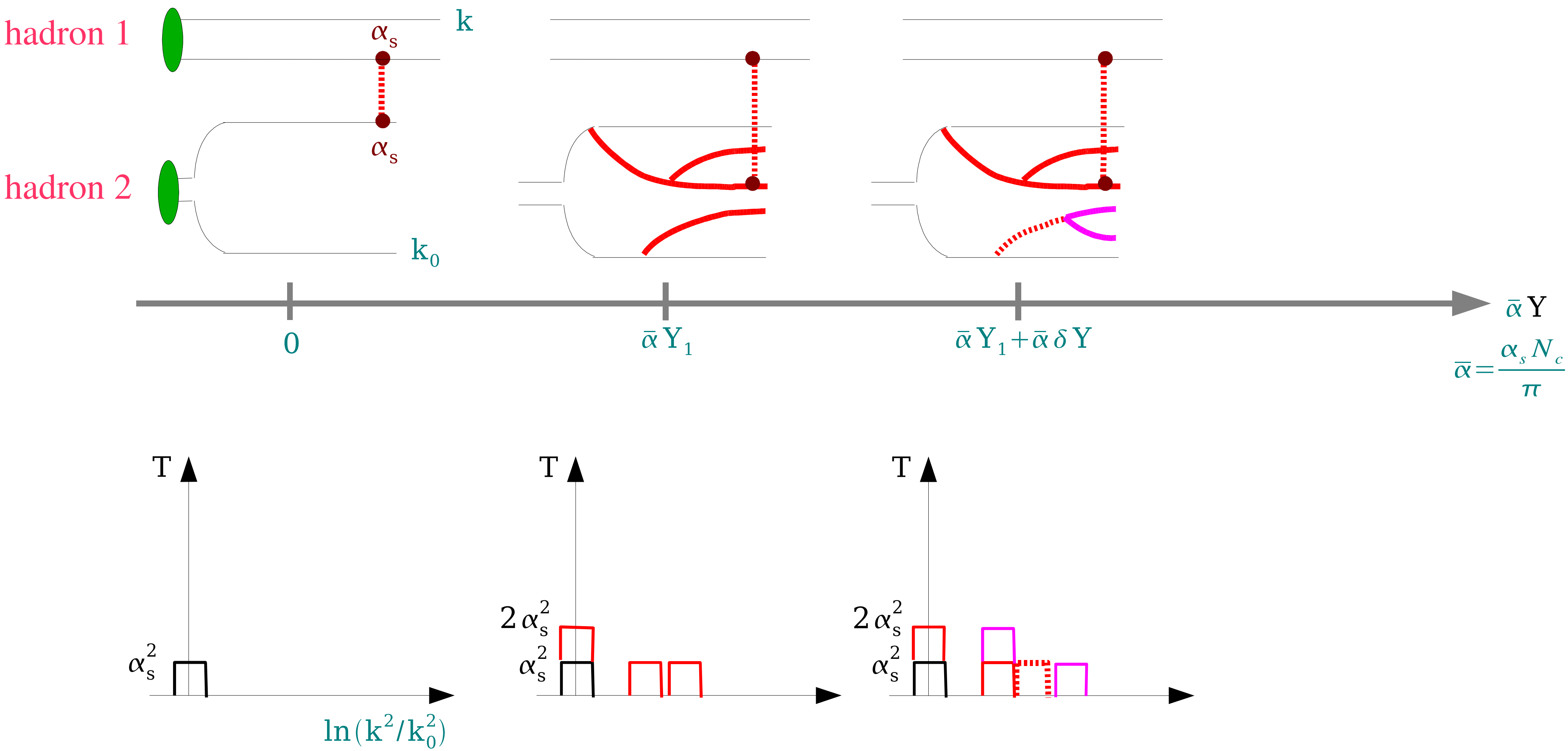


$T(k)=\alpha_s^2\times n(k)$

$$\partial_{\bar{\alpha}Y} n=\chi(-\partial_{\ln(k^2/k_0^2)})n$$

BFKL kernel

High energy QCD = reaction-diffusion



$T(k) = \alpha_s^2 \times n(k)$

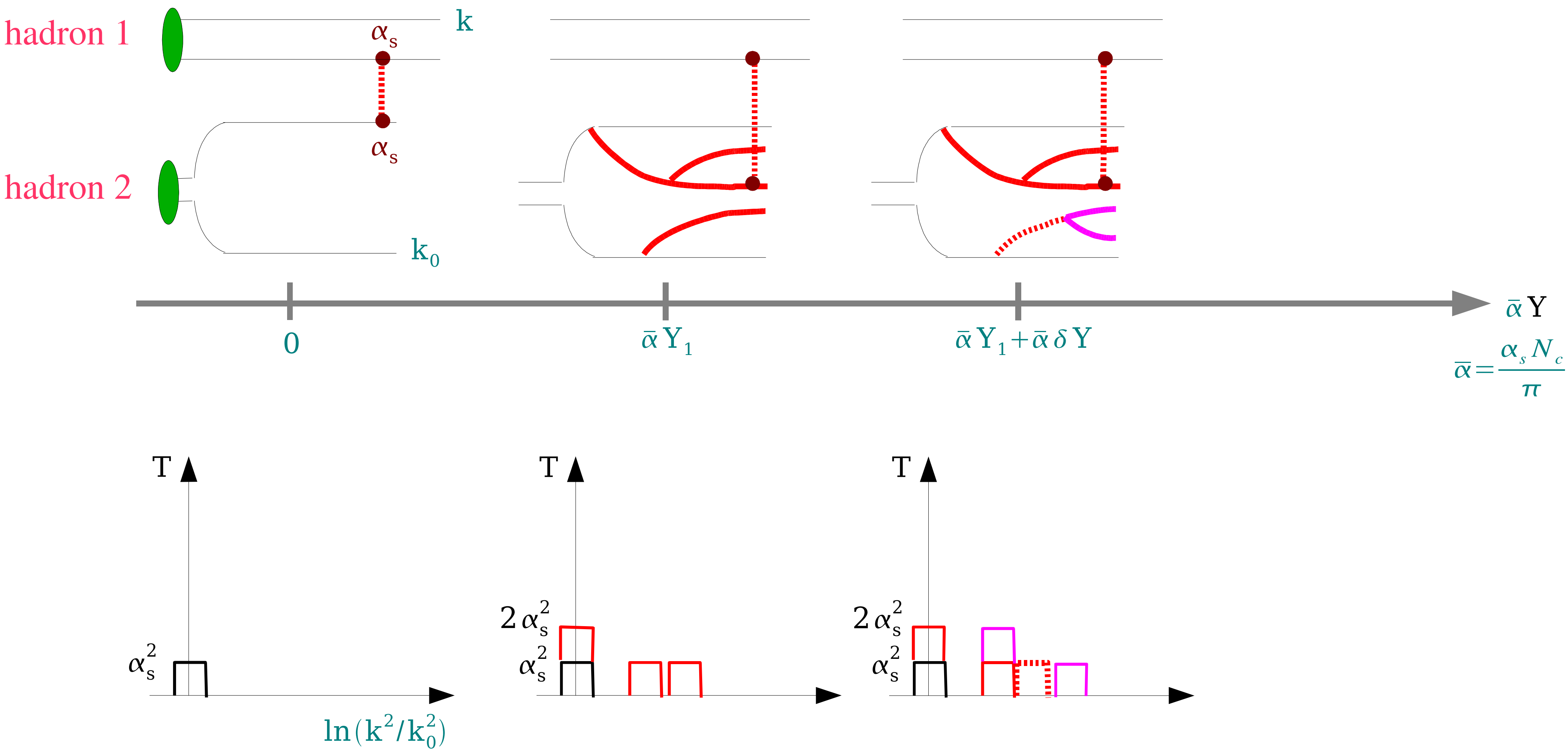
$$\partial_{\bar{\alpha} Y} n = \chi\left(-\partial_{\ln(k^2/k_0^2)}\right) n \quad + \sqrt{n}\,\nu$$

analogous to t

BFKL kernel

analogous to x

High energy QCD = reaction-diffusion



$$T(k) = \alpha_s^2 \times n(k)$$

$$\partial_{\bar\alpha Y} T = \chi(-\partial_{\ln(k^2/k_0^2)})T \qquad + \sqrt{\alpha_s^2 T}\,\nu$$

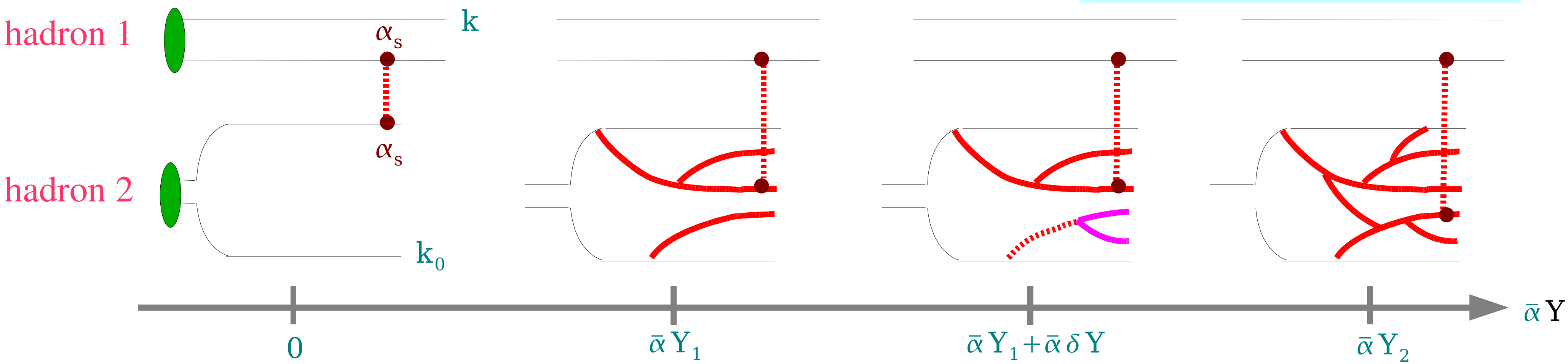
analogous to t

BFKL kernel

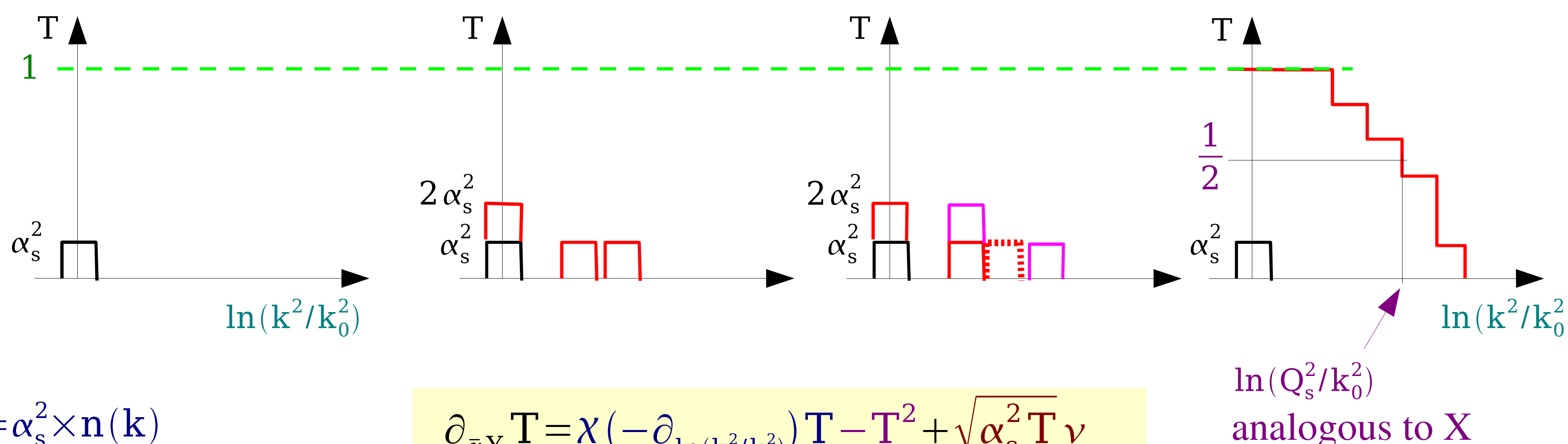
analogous to x

High energy QCD = reaction-diffusion

parton density saturation



unitarity limit: $T < 1$ because $2T - T^2 =$ interaction proba.



$T(k) = \alpha_s^2 \times n(k)$

$$\partial_{\bar\alpha Y} T = \chi\left(-\partial_{\ln(k^2/k_0^2)}\right) T - T^2 + \sqrt{\alpha_s^2 T}\,\nu$$

analogous to t

BFKL kernel

analogous to x

analogous to u

analogous to 1/N

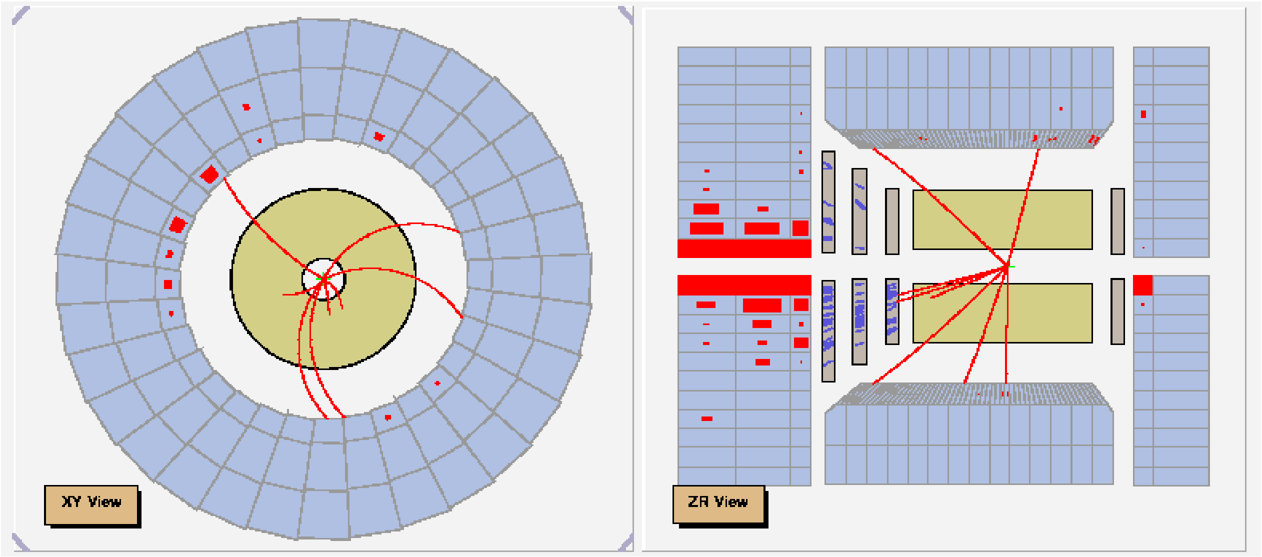
High energy QCD = reaction-diffusion

$$\partial_{\bar{\alpha} Y} T = \chi\left(-\partial_{\ln(k^2/k_0^2)}\right) T - T^2 + \sqrt{\alpha_s^2 T}\, \nu$$

describes the evolution of a particular Fock state

T would be the scattering amplitude of the probe off one ***random*** Fock state

1 Fock state realization corresponds to 1 event



Experimentalists need many events to measure a cross section!

physical amplitude: $A = \langle T \rangle$

analogous to $\langle u \rangle$

Dictionary and predictions

Position x		$\ln(k^2/k_0^2)$
Time t		$\bar{\alpha}\,Y$
Particle density/fraction u	$\longleftrightarrow$	Partonic amplitude T
Maximum/equilibrium number of particles N		$\dfrac{1}{\alpha_s^2}$
Position of the wave front X		Saturation scale $\ln(Q_s^2/k_0^2)$

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Predictions from the correspondence

Shape of the *partonic* amplitude: $T \sim \left(r^2 Q_s^2(Y)\right)^{\gamma_0}$

Saturation scale: $$\langle \ln Q_s^2 \rangle_Y = \bar{\alpha}\,Y \left(\frac{\chi(\gamma_0)}{\gamma_0} - \frac{\pi^2 \gamma_0 \chi''(\gamma_0)}{2\ln^2(1/\alpha_s^2)} \right)$$

$$\sigma^2 = \langle \ln^2 Q_s^2 \rangle_Y - \langle \ln Q_s^2 \rangle_Y^2 \propto \frac{\bar{\alpha}\,Y}{\ln^3(1/\alpha_s^2)}$$

$$\Rightarrow A \sim A\left(\frac{r^2 Q_s^2(Y)}{\sqrt{\frac{\bar{\alpha}\,Y}{\ln^3(1/\alpha_s^2)}}} \right)$$

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Validity

A priori, $Y \gg 1, \ln(1/\alpha_s^2) \gg 1$

In practice: analytical results reliable for $\alpha_s \ll 10^{-5} (!!!)$

But we believe the picture itself for $\alpha_s < 0.1$

Fixed impact parameter

Summary

Instead of solving the full QCD evolution equations, we have looked for properties of the amplitude that would not depend on the details of the evolution, and in particular, on its exact form near the unitarity limit (which is still not fully understood in QCD).

We have conjectured that high energy QCD is in the universality class of reaction-diffusion processes from the physics of the parton model. Its solutions at small coupling and large rapidity are traveling waves.

The properties of these QCD traveling waves (shape and position, i.e. form of the amplitude and rapidity dependence of the saturation scale) may be obtained directly by solving simpler equations in the universality class of the sFKPP equation.

S.M., Nucl. Phys. A (2005)
Iancu, Mueller, S.M., Phys. Lett. B (2005)
Enberg, Golec-Biernat, S.M., Phys. Rev. D (2005)

More on these equations in the next lecture!