Jan Kotański

Supersymmetric Yang-Mills Quantum Mechanics in $D = 4$ dimensions for $SU(2)$ gauge group

- Energy spectrum
- Hamiltonian eigenfunctions
- Virial theorem

P. van Baal, World Scientific, Singapore, 556 (2002), hep-th/0112072

J. K., J. Wosiek, Nucl. Phys. Proc. Suppl. **119** 932 (2003)

M. Campostrini, J. Wosiek, Nucl. Phys. **B703** 454 (2004)

Supersymmetric Y-M Quantum Mechanics

zero-volume Hamiltonian – in one point

$$H = \underbrace{\underbrace{-\frac{1}{2}\left(\frac{\partial}{\partial \hat{c}_i^a}\right)^2}_{H_T} + \underbrace{\frac{1}{2}\left(\hat{B}_i^a\right)^2}_{H_V}}_{H_B} \underbrace{- i\varepsilon_{abd}\bar{\lambda}^a\bar{\sigma}^i\lambda^b\hat{c}_i^d}_{H_F}$$

$\sigma^j = \tau^j$ - Pauli matices

λ_a^β, $\bar{\lambda}_a^{\dot{\beta}}$ - Weyl spinors

$\hat{c}_i^a$ - bosonic variables

$$\hat{B}_i^a = -\tfrac{1}{2}\varepsilon_{ijk}\varepsilon_{abd}\hat{c}_j^b\hat{c}_k^d$$

anticommutation relations

$$\{\lambda^{a\alpha}, \bar{\lambda}^{b\dot{\beta}}\} = \bar{\sigma}_0^{\dot{\beta}\alpha}\delta^{ab}$$

$$\{\lambda^{a\alpha}, \lambda^{b\beta}\} = 0$$

$$\{\bar{\lambda}^{a\dot{\alpha}}, \bar{\lambda}^{b\dot{\beta}}\} = 0$$

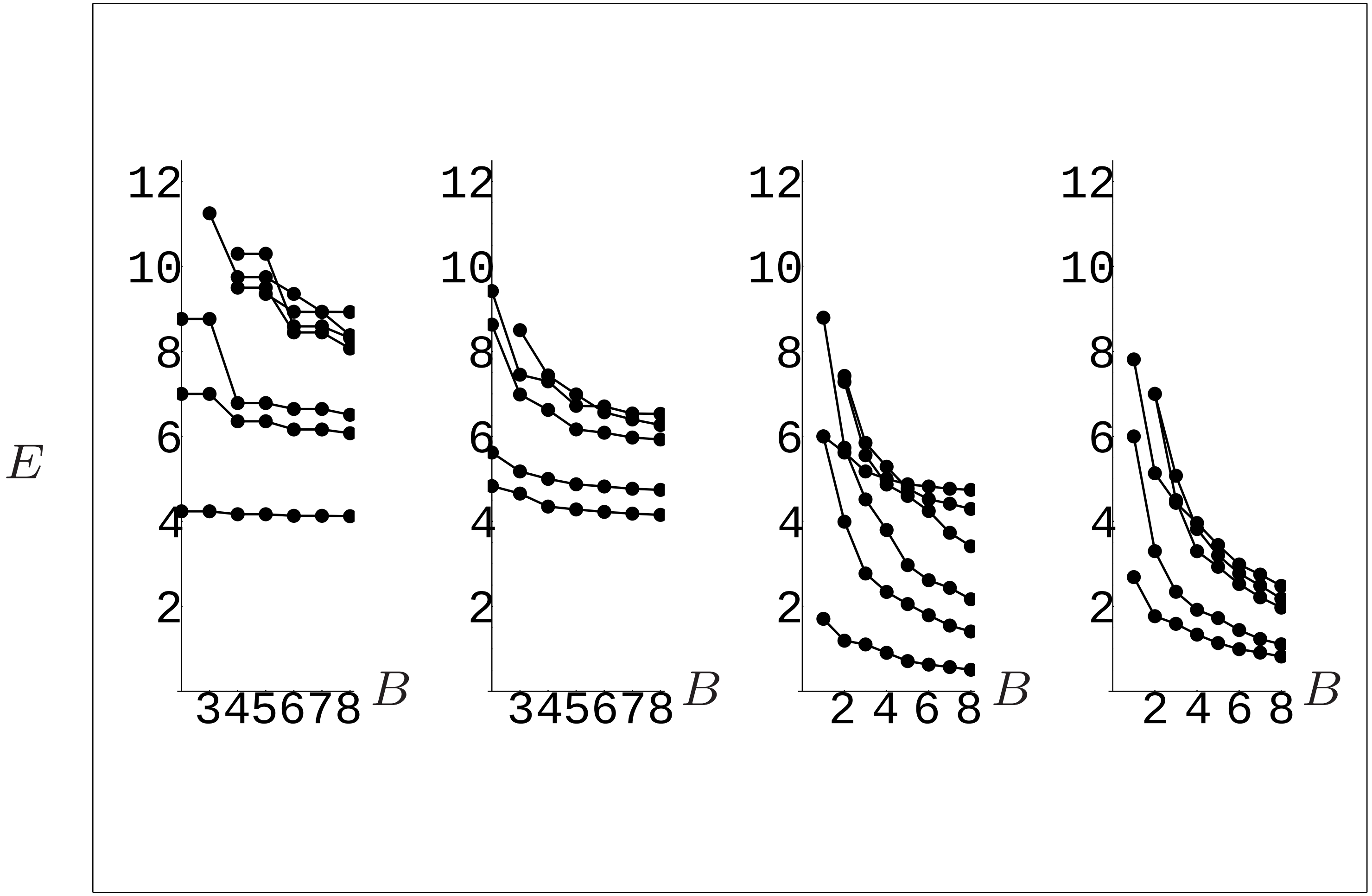
colour indices - $a, b, d \in \{1, 2, 3\}$

spatial indices - $i, j \in \{1, 2, 3\}$

spinor indices - $\alpha, \beta \in \{1, 2\}$

Old results

Energy dependence on cut-off $B \geq n_B$



n_F =0 n_F =1 n_F =2 n_F =3

n_B - # of bosons

n_F - # of fermions

particle-hole sym.:

$n_F \longleftrightarrow 6 - n_F$

$[\hat{n}_F, H] = 0$

$[\hat{n}_B, H] \neq 0$

New invariant bosonic variables

angular momentum $j = 0$

and gauge invariance

$(\hat{r}, u, v)$

$\hat{r}^2 = (\hat{c}_j^a)^2$

$u = \hat{r}^{-4}(\hat{B}_j^a)^2$

$v = \hat{r}^{-3}\,\det \hat{c}$

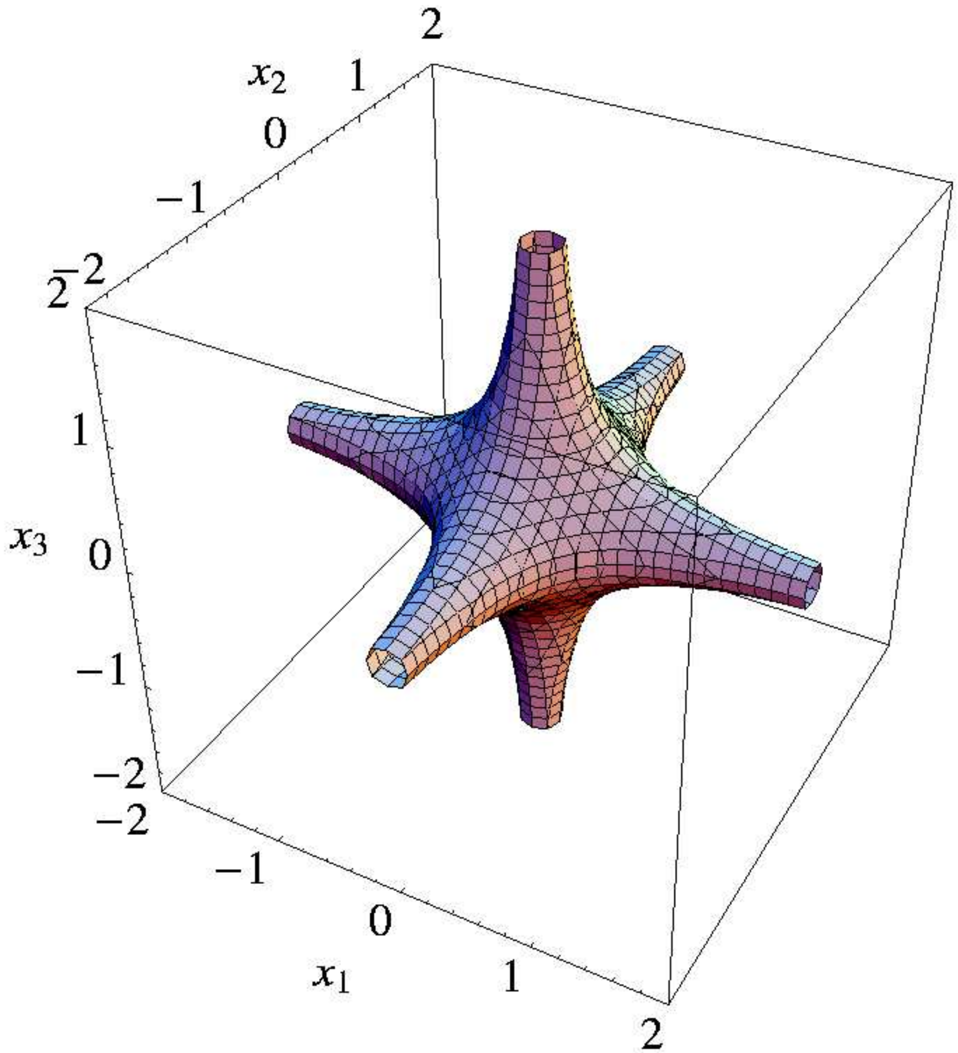
where $\hat{B}_i^a = -\frac{1}{2}\varepsilon_{ijk}\varepsilon_{abd}\hat{c}_j^b\hat{c}_k^d$

(x_1, x_2, x_3)

$\hat{r}^2 = \sum_j x_j^2$

$u = \hat{r}^{-4}\sum_{i>j} x_i^2 x_j^2$

$v = \hat{r}^{-3}\prod_j x_j$



equipotential surface

$2H_V = u\,\hat{r}^4 = x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2 = 0.1$

Cut Fock space for $n_F = 2$

Acting $\hat{c}^b_j$ and $\bar{\lambda}^a_{\dot{\alpha}}$ on empty state:

$$|n\rangle = \sum_{\substack{\text{convolutions} \\ \{a_1,\ldots,a_r\}}} \hat{c}^{a_1}_{k_1} \ldots \hat{c}^{a_m}_{k_m} \bar{\lambda}^{a_{m+1}}_{\dot{\alpha}} \ldots \bar{\lambda}^{a_r}_{\dot{\beta}} |0\rangle$$

Combinations of fermions

$$\mathcal{I}_j{}^a = -2i\varepsilon_{cba} \bar{\lambda}^c_{\dot{\alpha}} (\bar{\sigma}^{j0})^{\dot{\alpha}}{}_{\dot{\beta}} \bar{\lambda}^{b\dot{\beta}} |0\rangle \qquad \mathcal{J}^{ab} = -\bar{\lambda}^a_{\dot{\alpha}} \bar{\lambda}^b_{\dot{\beta}} \epsilon^{\dot{\alpha}\dot{\beta}} |0\rangle$$

$$\bar{\sigma}^{j0} = \tfrac{1}{2}\tau_j \qquad \epsilon_{\alpha\beta} = \epsilon_{\dot{\alpha}\dot{\beta}} = -i\tau_2$$

Contractions between bosons and fermions

$$|e_1(u,v)\rangle = \hat{c}^a_j / \hat{r} \mathcal{I}^j{}_a \qquad |e_4(u,v)\rangle = \delta^{ab} \mathcal{J}_{ab}$$

$$|e_2(u,v)\rangle = \hat{B}^a_j / \hat{r}^2 \mathcal{I}^j{}_a \qquad |e_5(u,v)\rangle = \hat{c}^a_j \hat{c}^b_j / \hat{r}^2 \mathcal{J}_{ab}$$

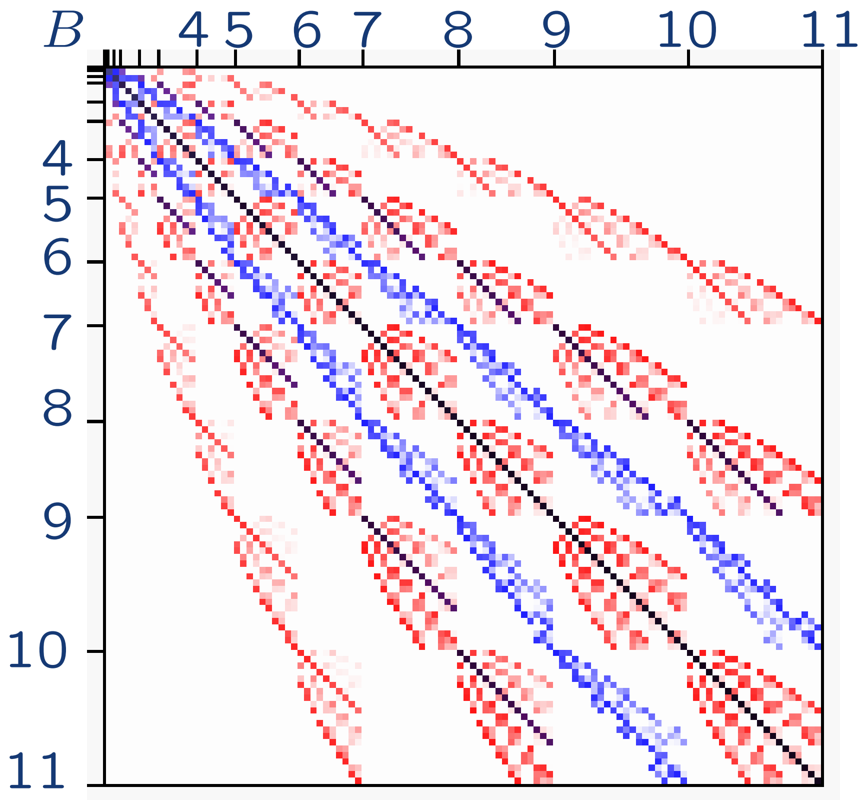
$$|e_3(u,v)\rangle = \hat{c}^b_j \hat{c}^b_k \hat{c}^a_k / \hat{r}^3 \mathcal{I}^j{}_a \qquad |e_6(u,v)\rangle = \hat{c}^b_j \hat{c}^d_j \hat{c}^d_k \hat{c}^b_k / \hat{r}^4 \mathcal{J}_{ab}$$

Basis vectors

$$|n\rangle = \textstyle\sum_{m=1}^{6} h^n_m(\hat{r},u,v) |e_m(u,v)\rangle$$

Hamiltonianu Diagonalization $H^{n'n}$

Eigenequation: $\sum_n H^{n'n} v_k^n = E_k v_k^{n'}$ where $H^{n'n} = \langle n'|H|n\rangle$



Eigenstates of Hamiltonian:

$$|\Phi_k(\hat{r}, u, v)\rangle = \sum_n v_k^n |n\rangle$$
$$= \sum_n v_k^n \sum_{m=1}^{6} h_m^n(\hat{r}, u, v)|e_m\rangle$$

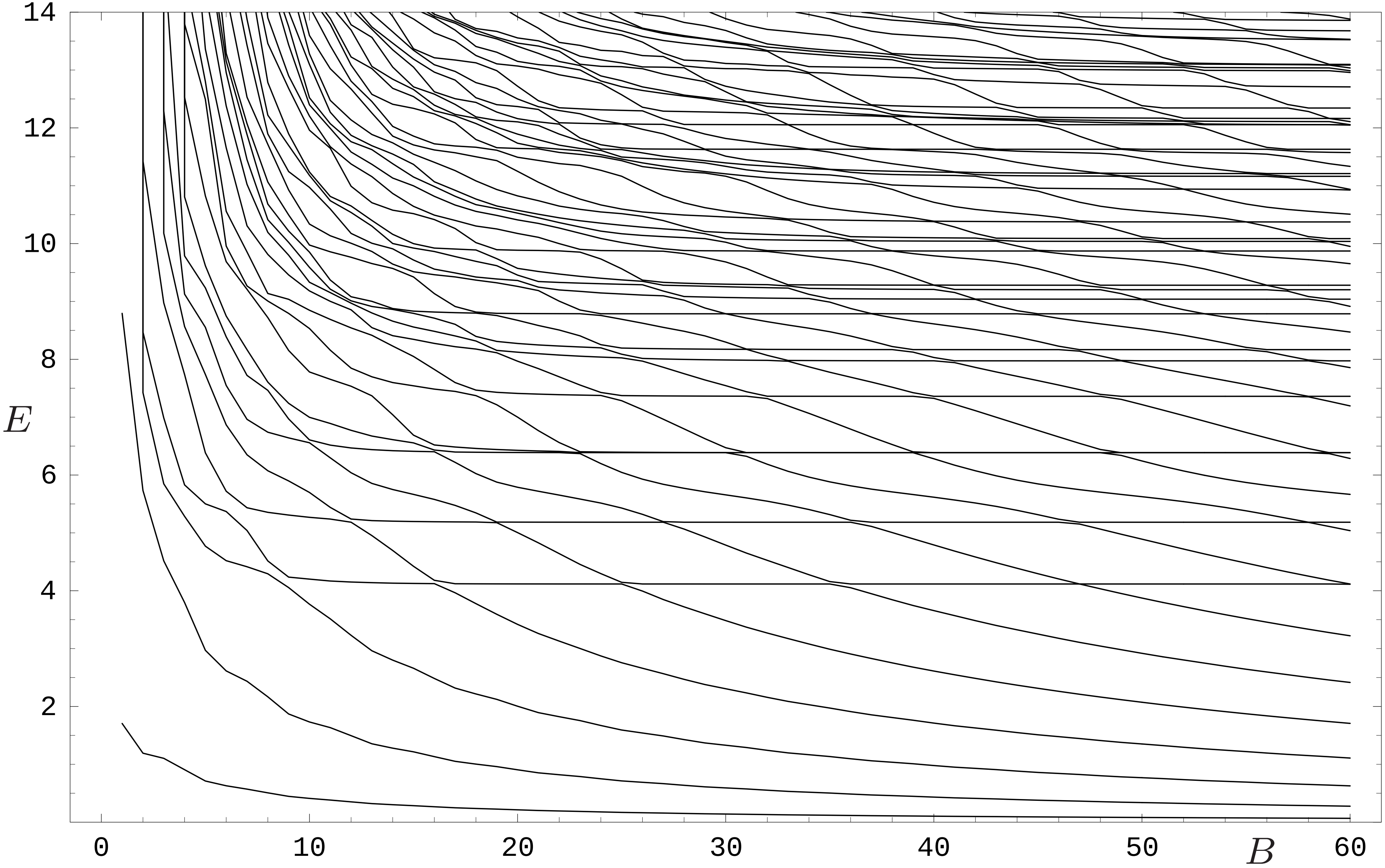
Observables:

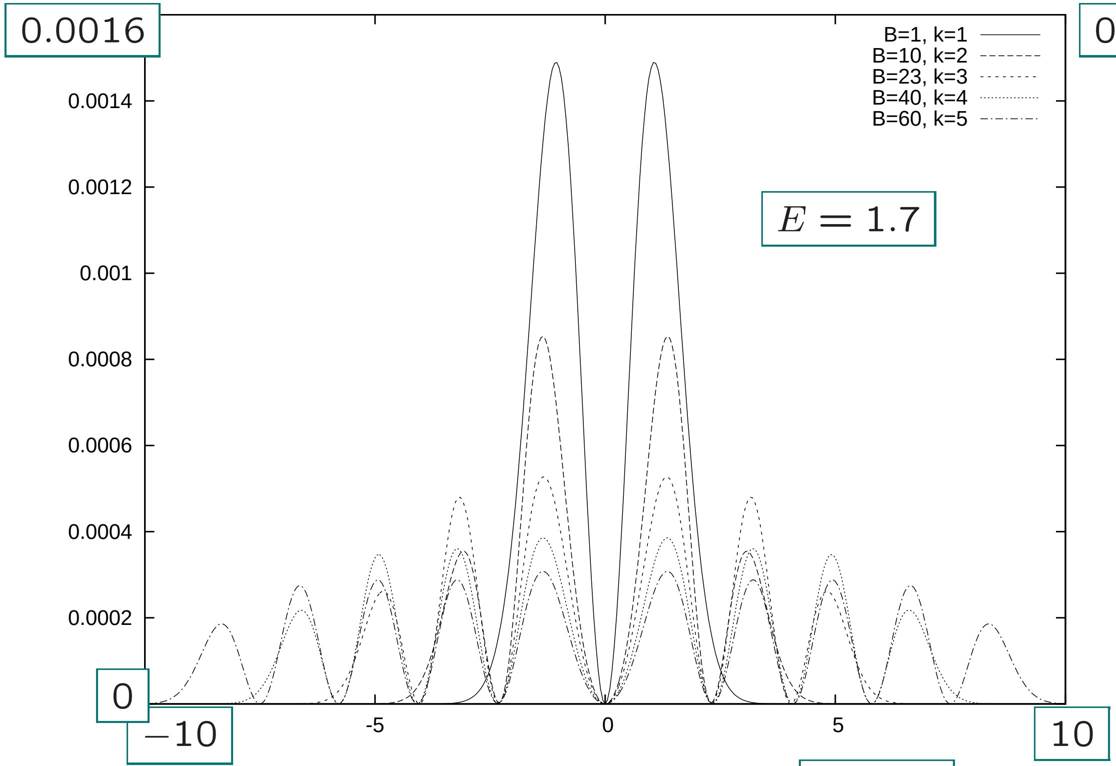
$$\langle \mathcal{O}\rangle_k = \langle \Phi_k(\hat{r}, u, v)\rangle|\mathcal{O}|\Phi_k(\hat{r}, u, v)\rangle$$

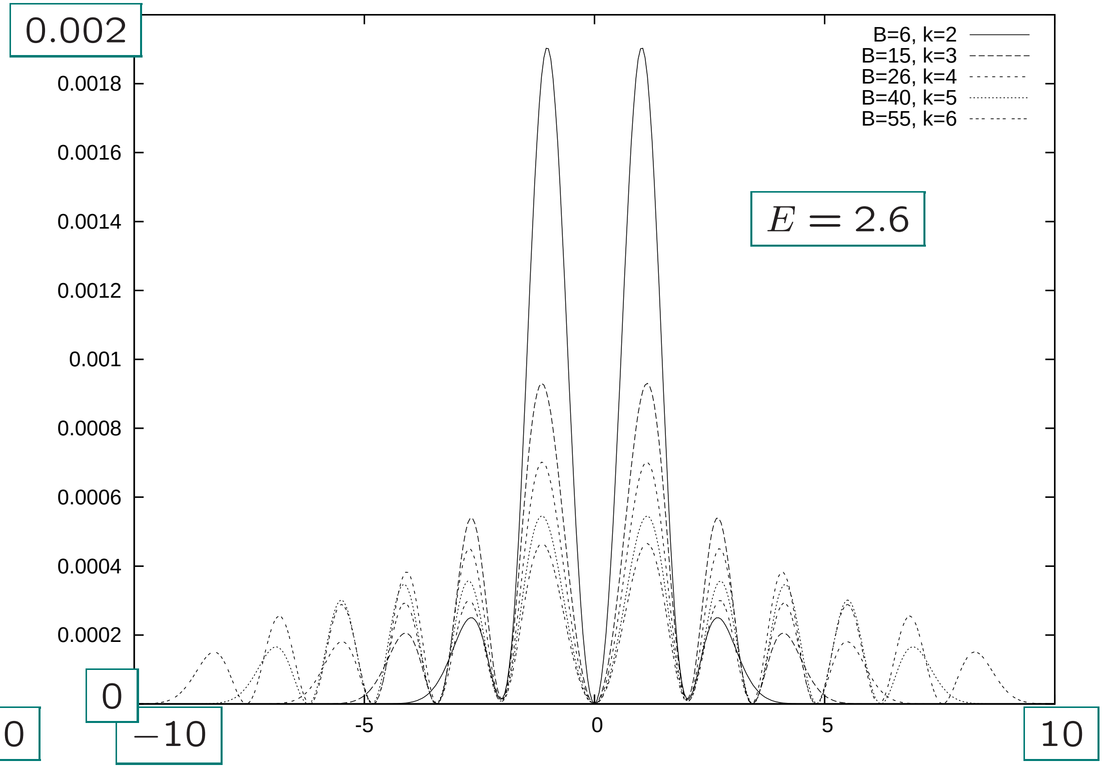
112×112 matrix for $B = 11$

10416×10416 matrix for $B = 60$

Energy spectrum



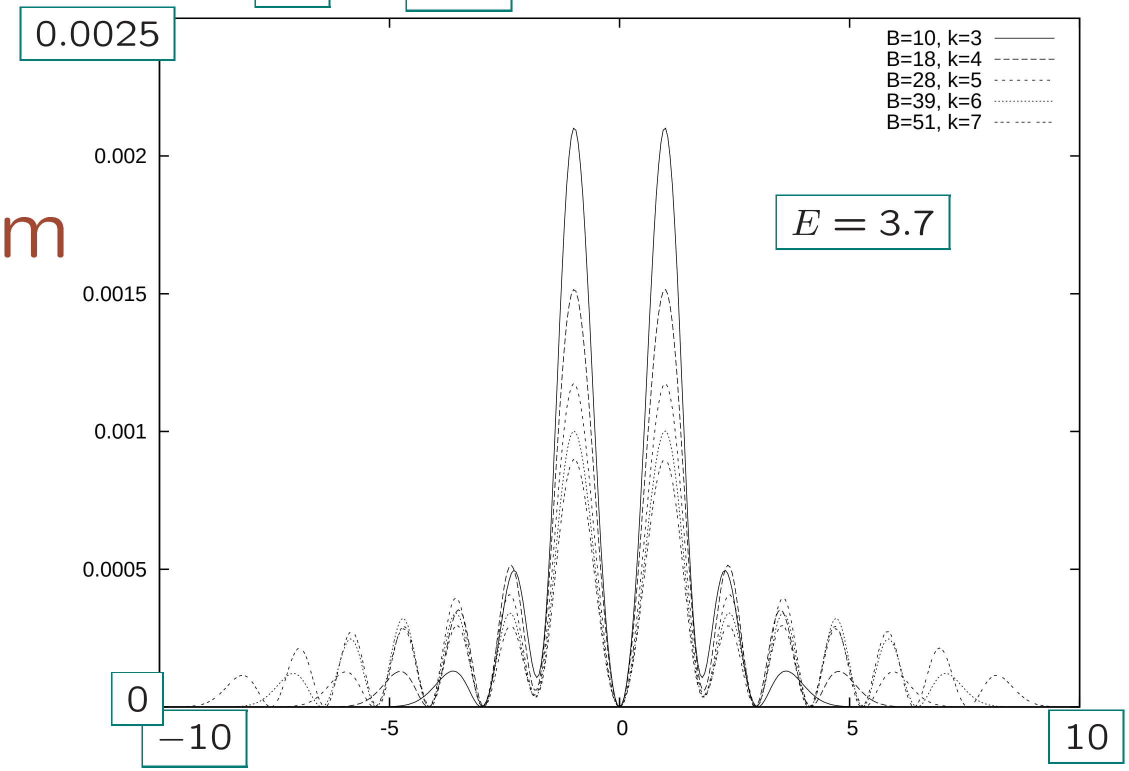




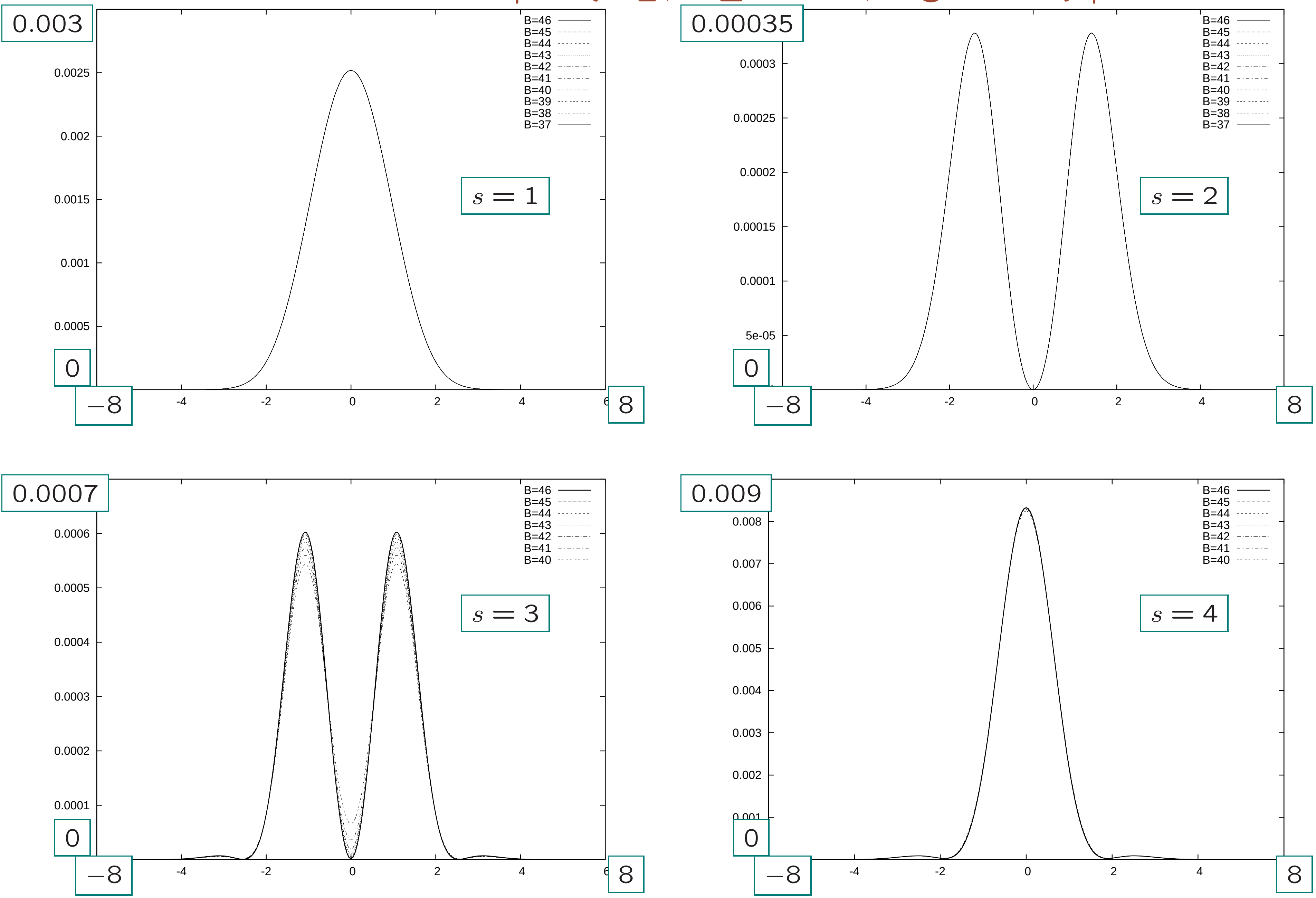
Continuous spectrum for fixed energy

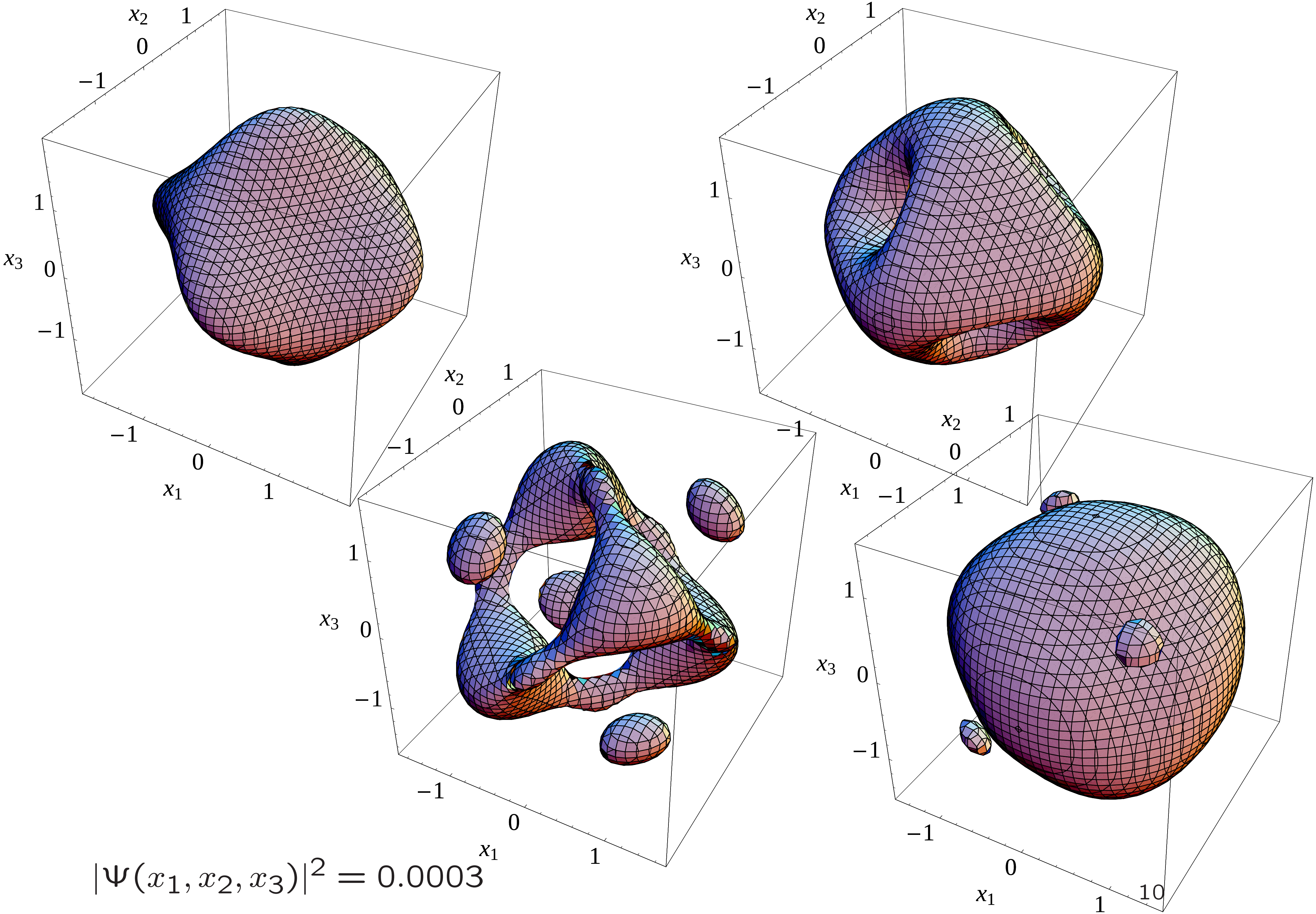
plot: $x^2|\Psi(x)|^2$

$x_2 = x_3 = 0,\ x_1 = x$



Bound states $|\Psi(x_1, x_2 = 0, x_3 = 0)|^2$





$|\Psi(x_1, x_2, x_3)|^2 = 0.0003$

Quantum virial theorem

Heisenberg equation

$$\frac{d\widehat{F}}{dt} = \frac{\partial \widehat{F}}{\partial t} + \frac{1}{i\hbar}[\widehat{F}, \widehat{H}]$$

For motion encolsed in bound area: $\frac{d}{dt}\langle \vec{x} \cdot \vec{p} \rangle = 0$

$$0 = \frac{d}{dt}\langle \vec{x} \cdot \vec{p} \rangle = \frac{1}{i\hbar}\langle [\vec{x} \cdot \vec{p}, \widehat{H}] \rangle$$

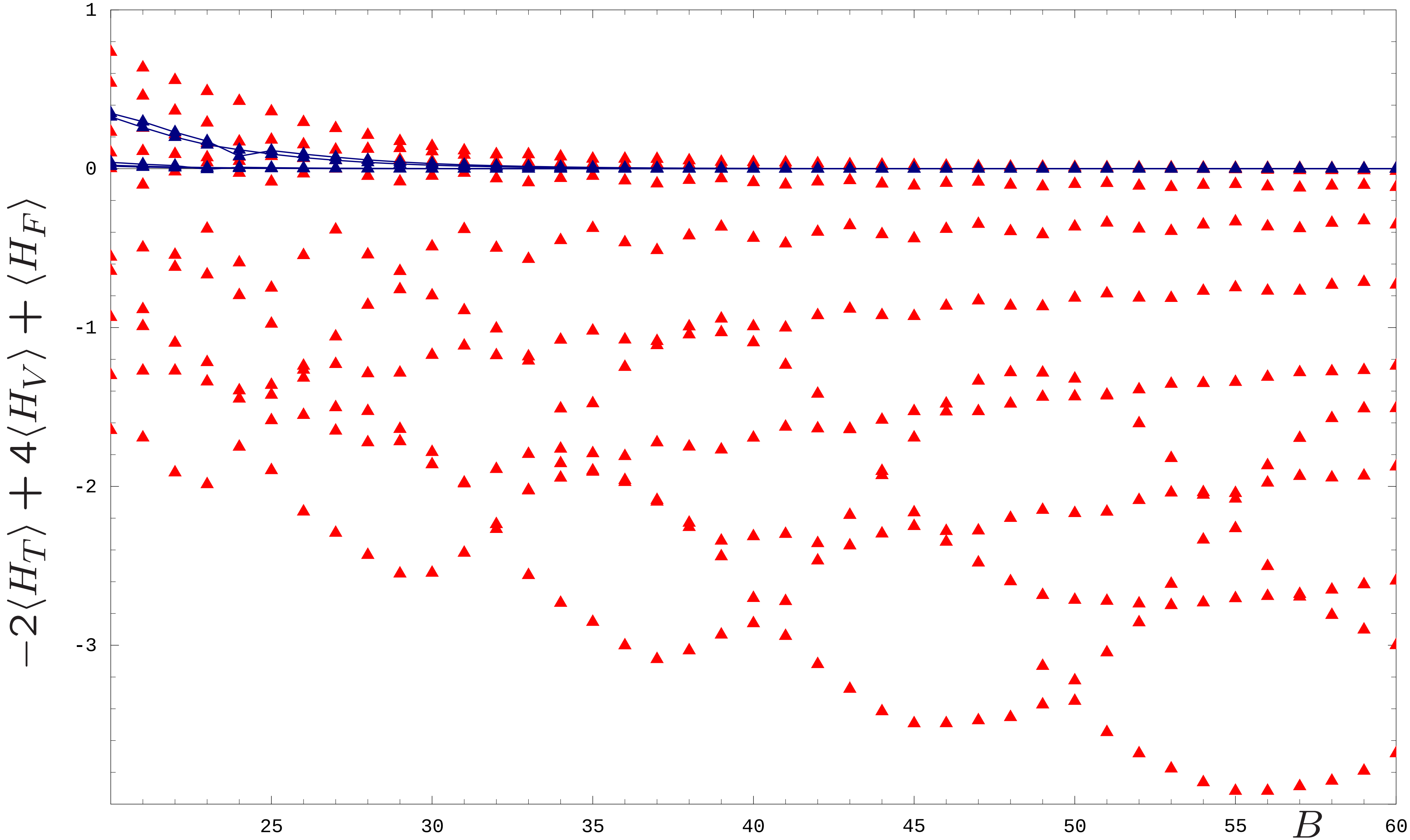
When $V(\alpha\vec{x}) = \alpha^n V(\vec{x})$ then $2\langle T \rangle = n\langle V \rangle$

Here, virial - $\vec{x} \cdot \vec{p} \sim r\,\frac{\partial}{\partial r}$

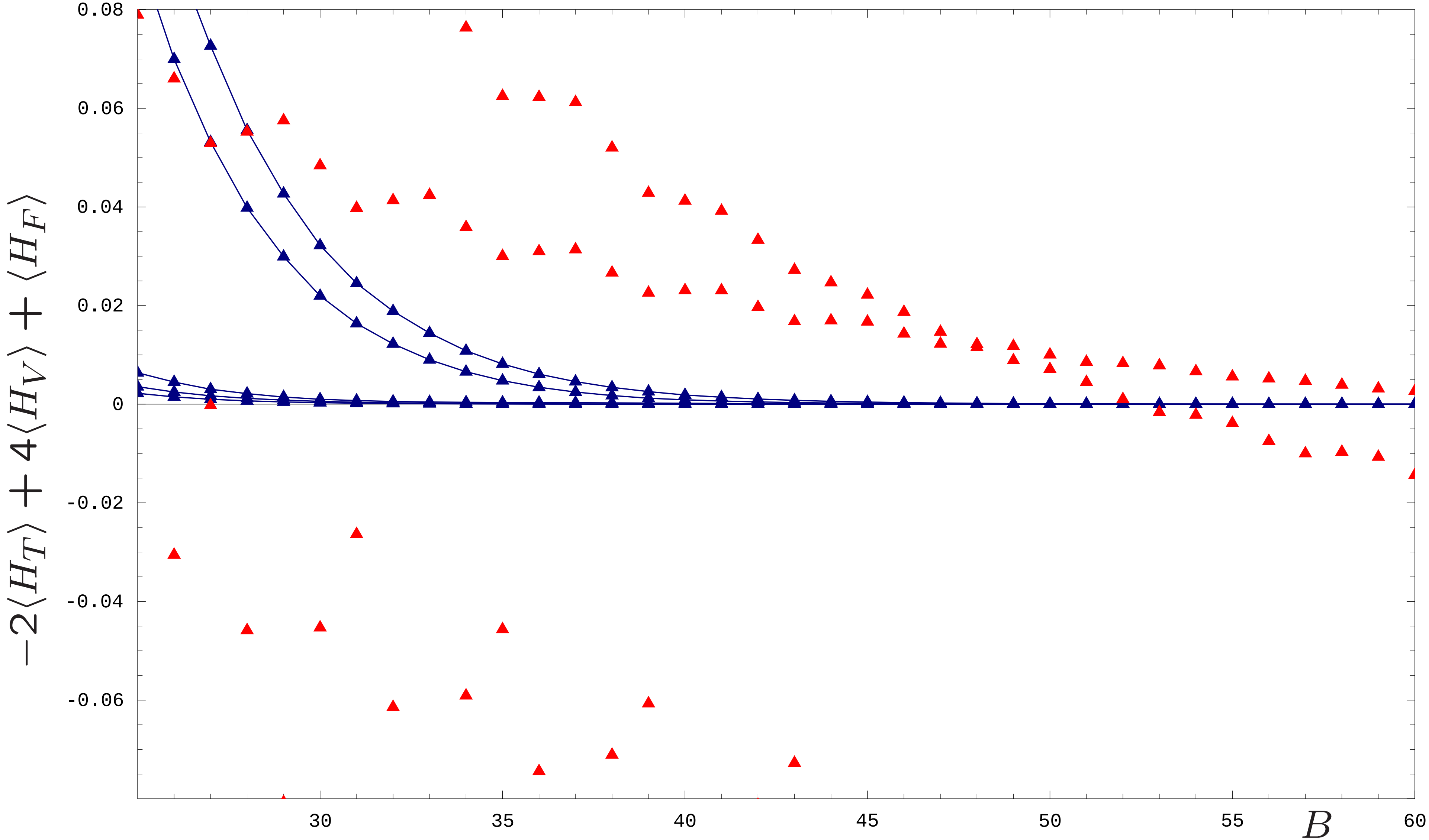
$$H_T(\alpha r) = \alpha^{-2} H_T(r) \quad H_V(\alpha r) = \alpha^4 H_V(r) \quad H_F(\alpha r) = \alpha H_F(r)$$

$$-2\langle H_T \rangle + 4\langle H_V \rangle + \langle H_F \rangle = 0$$

Virial theorem



Virial theorem (zoom in)



Conclusions

1. Solution of Schrödinger equation for $n_F = 2$ and $j = 0$ in $D = 4$

2. Energy spectum and eigenstates

3. Virial theorem for SYMQM: bound states and non-localized states

4. Future: other observables and laws

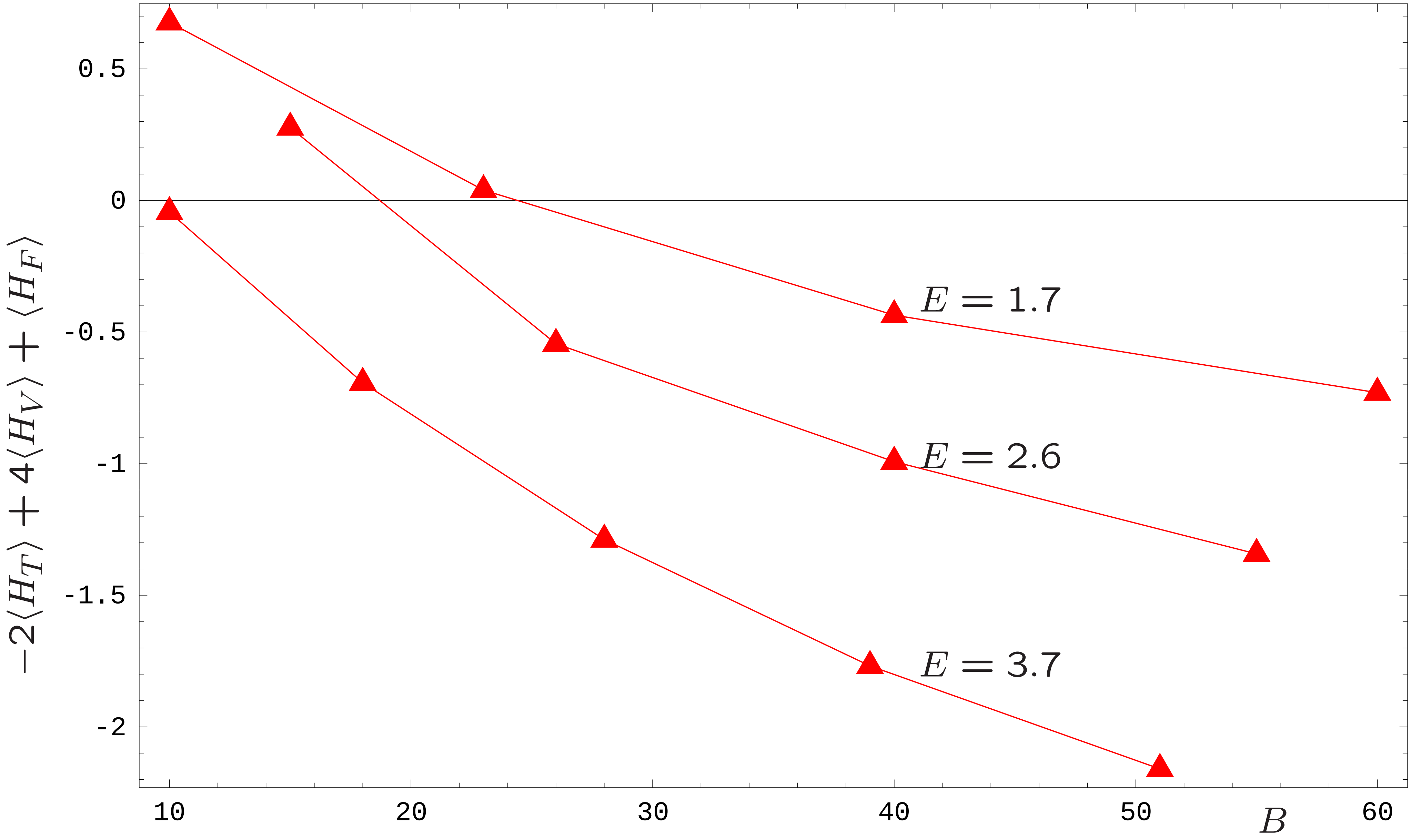
   systems with $D = 10$

   $SU(2) \to SU(3)$

   $D-$brane scattering

5. Difficulty: large number of states (i.e. $D = 10$ and $j \neq 0$)

Virial theorem: Continuous Spectrum



Disperstion relation: $E \sim p^2$