# *Lecture 1: QCD Sum Rules in Quantum Mechanics*

A. P. Bakulev

Bogolyubov Lab. Theor. Phys., JINR (Dubna, Russia)

# *Contents of Lecture 1*

- Toy model: 2D Quantum Harmonic Oscillator

# *Contents of Lecture 1*

- Toy model: 2D Quantum Harmonic Oscillator
- Sum Rules: General Scheme, Green Function and Correlator

# *Contents of Lecture 1*

- Toy model: 2D Quantum Harmonic Oscillator
- Sum Rules: General Scheme, Green Function and Correlator
- Asymptotic Freedom for 2D QHO

# *Contents of Lecture 1*

- Toy model: 2D Quantum Harmonic Oscillator
- Sum Rules: General Scheme, Green Function and Correlator
- Asymptotic Freedom for 2D QHO
- Duality conception: method to describe excited states

# *Contents of Lecture 1*

- Toy model: 2D Quantum Harmonic Oscillator
- Sum Rules: General Scheme, Green Function and Correlator
- Asymptotic Freedom for 2D QHO
- Duality conception: method to describe excited states
- Numerical results and lessons

# *Contents of Lecture 1*

- Toy model: 2D Quantum Harmonic Oscillator
- Sum Rules: General Scheme, Green Function and Correlator
- Asymptotic Freedom for 2D QHO
- Duality conception: method to describe excited states
- Numerical results and lessons
- QCD: Quarks – inside, hadrons – outside! How to proceed?

# *Contents of Lecture 1*

- Toy model: 2D Quantum Harmonic Oscillator
- Sum Rules: General Scheme, Green Function and Correlator
- Asymptotic Freedom for 2D QHO
- Duality conception: method to describe excited states
- Numerical results and lessons
- QCD: Quarks – inside, hadrons – outside! How to proceed?
- QCD SRs: Way to study hadrons in np-QCD.

# *Contents of Lecture 1*

- Toy model: 2D Quantum Harmonic Oscillator
- Sum Rules: General Scheme, Green Function and Correlator
- Asymptotic Freedom for 2D QHO
- Duality conception: method to describe excited states
- Numerical results and lessons
- QCD: Quarks – inside, hadrons – outside! How to proceed?
- QCD SRs: Way to study hadrons in np-QCD.
- QCD: Currents, Correlators and Spectral Densities.

# Quantum-mechanical toy model:

## Two-Dimensional Harmonic Oscillator

# *Two-Dimensional Oscillator*

Simplest system with confinement – oscillator with potential $V(\vec{r}) = m\omega^2 r^2/2$. All formulas greatly simplify if $D = 2$.

# Two-Dimensional Oscillator

Simplest system with confinement – oscillator with potential $V(\vec{r}) = m\omega^2 r^2/2$. All formulas greatly simplify if $D = 2$. Then energy levels are

$$E_n = (2n + 1)\omega\,,$$

# Two-Dimensional Oscillator

Simplest system with confinement – oscillator with potential $V(\vec{r}) = m\omega^2 r^2/2$. All formulas greatly simplify if $D = 2$. Then energy levels are

$$E_n = (2n + 1)\omega\,,$$

and wave function values in the origin are

$$|\psi_n(0)|^2 = \frac{m\omega}{\pi}\,.$$

# *Two-Dimensional Oscillator*

Simplest system with confinement – oscillator with potential $V(\vec{r}) = m\omega^2 r^2/2$. All formulas greatly simplify if $D = 2$. Then energy levels are

$$E_n = (2n+1)\omega\,,$$

and wave function values in the origin are

$$|\psi_n(0)|^2 = \frac{m\omega}{\pi}\,.$$

We will consider the regular quasi-perturbative method of Sum Rules to determine energy $\boldsymbol{E_0}$ and $\boldsymbol{|\psi_0(0)|^2}$ of the ground state.

# *General scheme of Sum Rule method*

# *The general scheme of Sum Rule method*

- We study correlator $M(\mu)$, which has spectral expansion:

$$M^{\text{spec}}(\mu) = |\psi_0(0)|^2 \, e^{-E_0/\mu} + \text{“higher states”}$$

# The general scheme of Sum Rule method

- We study correlator $M(\mu)$, which has spectral expansion:

$$M^{\rm spec}(\mu) = |\psi_0(0)|^2 \, e^{-E_0/\mu} + \text{“higher states”}$$

- We construct perturbative expansion of this correlator:

$$M^{\rm pert}(\mu) = M_0(\mu) + \sum_{n\geq 1} C_{2n} \frac{\omega^{2n}}{\mu^{2n}} \,,$$

where $M_0(\mu)$ corresponds to free particle and has dispersion representation:

$$M_0(\mu) = \int_0^\infty \rho_0(E)\, e^{-E/\mu}\, dE \,.$$

# *The general scheme of Sum Rule method*

- We study correlator $M(\mu)$, which has expansion:

$$M^{\text{spec}}(\mu) = |\psi_0(0)|^2 \, e^{-E_0/\mu} + \text{“higher states”}$$

- We construct perturbative expansion of this correlator:

$$M^{\text{pert}}(\mu) = M_0(\mu) + \sum_{n\geq 1} C_{2n} \frac{\omega^{2n}}{\mu^{2n}}$$

- Sum Rule – it is simply

$$\boxed{M^{\text{spec}}(\mu) = M^{\text{pert}}(\mu)}$$

# *The general scheme of Sum Rule method*

- It appears that higher state contributions can be well approximated by

$$\text{"higher states"} = \text{"free states" outside interval } (0, s_0)$$

# *The general scheme of Sum Rule method*

- It appears that higher state contributions can be well approximated by

  "higher states" = "free states" outside interval $(0, s_0)$

- As a result we have Sum Rule (SR):

$$|\psi_0(0)|^2\, e^{-E_0/\mu} = \int_0^{S_0} \rho_0(E)\, e^{-E/\mu}\, dE + \text{“power corrections”}$$

# *The general scheme of Sum Rule method*

- It appears that higher state contributions can be well approximated by

  "higher states" = "free states" outside interval $(0, s_0)$

- As a result we have Sum Rule (SR):

$$|\psi_0(0)|^2 \, e^{-E_0/\mu} = \int_0^{S_0} \rho_0(s) \, e^{-s/\mu} \, ds + C_2 \frac{\omega^2}{\mu^2} + C_4 \frac{\omega^4}{\mu^4} + \dots$$

# *The general scheme of Sum Rule method*

- It appears that higher state contributions can be well approximated by

  "higher states" = "free states" outside interval $(0, s_0)$

- As a result we have Sum Rule (SR):

$$|\psi_0(0)|^2\, e^{-E_0/\mu} = \int_0^{S_0} \rho_0(s)\, e^{-s/\mu}\, ds + C_2 \frac{\omega^2}{\mu^2} + C_4 \frac{\omega^4}{\mu^4} + \dots$$

- Our aim: to determine $|\boldsymbol{\psi_0(0)}|^2$ and $\boldsymbol{E_0}$ from this SR by calculating spectral density $\boldsymbol{\rho_0(E)}$ and coefficients $\boldsymbol{C_{2n}}$ and by demanding stability of this SR in variable $\boldsymbol{\mu \in [\mu_L, \mu_U]}$.

# *Green functions and Correlators*

# $M(\mu)$ *and Green function* $G(\vec{x}, t)$

- Consider 2-time Green function

$$G(0, 0|\vec{x}, t) = \sum_{k \geq 0} \psi_k^*(\vec{x})\psi_k(0)e^{-iE_k t}\,.$$

$=$ probability amplitude for $(x = 0, t = 0) \to (\vec{x}, t)$.

# $M(\mu)$ and Green function $G(\vec{x},t)$

- Consider 2-time Green function

$$G(0,0|\vec{x},t) = \sum_{k\geq 0} \psi_k^*(\vec{x})\psi_k(0)e^{-iE_k t}\,.$$

$=$ probability amplitude for $(x=0,t=0) \to (\vec{x},t)$.

- To get $M(\mu)$ put $x=0$, $t=1/i\mu$:

$$M(\mu) = G(0,0|0,1/i\mu) = \sum_{k\geq 0} \left|\psi_k(0)\right|^2 e^{-E_k/\mu} = M^{\rm spec}(\mu)\,.$$

# $M(\mu)$ and Green function $G(\vec{x}, t)$

- Consider 2-time Green function

$$G(0, 0|\vec{x}, t) = \sum_{k \geq 0} \psi_k^*(\vec{x})\psi_k(0)e^{-iE_k t} \,.$$

= probability amplitude for $(x = 0, t = 0) \to (\vec{x}, t)$.

- To get $M(\mu)$ put $x = 0$, $t = 1/i\mu$:

$$M(\mu) = G(0, 0|0, 1/i\mu) = \sum_{k \geq 0} \left|\psi_k(0)\right|^2 e^{-E_k/\mu} = M^{\text{spec}}(\mu) \,.$$

In our case $\left|\psi_k(0)\right|^2 = m\omega/\pi$, so we have

$$M(\mu) = \quad ???$$

# $M(\mu)$ and Green function $G(\vec{x}, t)$

- Consider 2-time Green function

$$G(0,0|\vec{x},t) = \sum_{k\geq 0} \psi_k^*(\vec{x})\psi_k(0)e^{-iE_k t}\,.$$

$=$ probability amplitude for $(x = 0, t = 0) \to (\vec{x}, t)$.

- To get $M(\mu)$ put $x = 0$, $t = 1/i\mu$:

$$M(\mu) = G(0,0|0,1/i\mu) = \sum_{k\geq 0} \left|\psi_k(0)\right|^2 e^{-E_k/\mu} = M^{\text{spec}}(\mu)\,.$$

In our case $\left|\psi_k(0)\right|^2 = m\omega/\pi$, so we have

$$M(\mu) = \frac{m\omega}{2\pi\,\sinh\left(\omega/\mu\right)}\,.$$

# *Spectral expansion for $M(\mu)$*

- Exact correlator:

$$M(\mu) = \frac{m\omega}{2\pi \sinh(\omega/\mu)} .$$

# *Spectral expansion for $M(\mu)$*

- Exact correlator:

$$M(\mu) = \frac{m\omega}{2\pi\,\sinh(\omega/\mu)}\,.$$

- Spectral representation = expansion in powers of $e^{-2\omega/\mu}$

$$M^{\rm spec}(\mu) = \frac{m\omega}{\pi}\left(e^{-\omega/\mu} + e^{-3\omega/\mu} + e^{-5\omega/\mu} + e^{-7\omega/\mu} + \dots\right)\,.$$

# *Spectral expansion for $M(\mu)$*

- Exact correlator:

$$M(\omega) = \frac{m\omega}{2\pi} \cdot (0.851) \ .$$

- Spectral representation = expansion in powers of $e^{-2\omega/\mu}$

$$M^{\text{spec}}(\mu) = \frac{m\omega}{\pi} \left( e^{-\omega/\mu} + e^{-3\omega/\mu} + e^{-5\omega/\mu} + e^{-7\omega/\mu} + \dots \right) \ .$$

Numerically at $\mu = \omega$:

$$M^{\text{spec}}(\omega) = \frac{m\omega}{2\pi} \left(0.736 + 0.100 + 0.013 + 0.002 + \dots\right) \ .$$

# *Spectral expansion for $M(\mu)$*

- Exact correlator:

$$M(\omega) = \frac{m\omega}{2\pi} \cdot (0.851) \; .$$

- Spectral representation = expansion in powers of $e^{-2\omega/\mu}$

$$M^{\rm spec}(\mu) = \frac{m\omega}{\pi} \left( e^{-\omega/\mu} + e^{-3\omega/\mu} + e^{-5\omega/\mu} + e^{-7\omega/\mu} + \ldots \right) \; .$$

Numerically at $\mu = \omega$:

$$M^{\rm spec}(\omega) = \frac{m\omega}{2\pi} \left(0.736 + 0.100 + 0.013 + 0.002 + \ldots\right) \; .$$

Ground state contributes $86\%$, first excitation – $12\%$, while the second – $1.5\%$.

# *Perturbative expansion for $M(\mu)$*

- Exact correlator:

$$M(\mu) = \frac{m\omega}{2\pi \sinh(\omega/\mu)} \, .$$

# Perturbative expansion for $M(\mu)$

- Exact correlator:

$$M(\omega) = \frac{m\omega}{2\pi} \cdot (0.851) \ .$$

- Perturbative expansion in powers $(\omega/\mu)^n$

$$M^{\text{pert}}(\mu) = \frac{m\mu}{2\pi}\left(1 - \frac{\boldsymbol{\omega^2}}{6\mu^2} + \frac{7}{360}\frac{\boldsymbol{\omega^4}}{\mu^4} - \frac{31}{15120}\frac{\boldsymbol{\omega^6}}{\mu^6} + \dots\right) ,$$

# Perturbative expansion for $M(\mu)$

- Exact correlator:

$$M(\omega) = \frac{m\omega}{2\pi} \cdot (0.851) \; .$$

- Perturbative expansion in powers $(\omega/\mu)^n$

$$M^{\rm pert}(\mu) = \frac{m\mu}{2\pi} \left( 1 - \frac{\boldsymbol{\omega^2}}{6\mu^2} + \frac{7}{360} \frac{\boldsymbol{\omega^4}}{\mu^4} - \frac{31}{15120} \frac{\boldsymbol{\omega^6}}{\mu^6} + \ldots \right) ,$$

Here $m\mu/2\pi$ corresponds to Green function of free particle:

$$M^{\rm free}(\mu) = \frac{m\mu}{2\pi} \, ,$$

# *Perturbative expansion for $M(\mu)$*

- Exact correlator:

$$M(\omega) = \frac{m\omega}{2\pi} \cdot (0.851) \ .$$

- Perturbative expansion in powers $(\omega/\mu)^n$

$$M^{\rm pert}(\mu) = \frac{m\mu}{2\pi} \left( 1 - \frac{\boldsymbol{\omega^2}}{6\mu^2} + \frac{7}{360}\frac{\boldsymbol{\omega^4}}{\mu^4} - \frac{31}{15120}\frac{\boldsymbol{\omega^6}}{\mu^6} + \ldots \right) ,$$

Numerically at $\mu = \omega$:

$$M^{\rm pert}(\omega) = \frac{m\omega}{2\pi} \left(1 - 0.167 + 0.019 - 0.002 + \ldots\right)$$

First correction specifies free result by $17\%$, while the second – by $3\%$

# *Asymptotic Freedom for HO Correlator*

# *Asymptotic Freedom for $M(\mu)$*

Perturbative expansion can be rewritten

$$\frac{M(\mu)-M_0(\mu)}{M_0(\mu)} = -\frac{\omega^2}{6\mu^2} + \frac{7}{360}\frac{\omega^4}{\mu^4} - \frac{31}{15120}\frac{\omega^6}{\mu^6} + \ldots$$

# *Asymptotic Freedom for $M(\mu)$*

Perturbative expansion can be rewritten

$$\frac{M(\mu) - M_0(\mu)}{M_0(\mu)} = -\frac{\omega^2}{6\mu^2} + \frac{7}{360}\frac{\omega^4}{\mu^4} - \frac{31}{15120}\frac{\omega^6}{\mu^6} + \dots$$

That means **Asymptotic Freedom**:

$M(\mu)$ behaves like $M_0(\mu)$ at large $\mu \gg \omega$!

# Asymptotic Freedom for $M(\mu)$

Perturbative expansion can be rewritten

$$\frac{M(\mu) - M_0(\mu)}{M_0(\mu)} = -\frac{\omega^2}{6\mu^2} + \frac{7}{360}\frac{\omega^4}{\mu^4} - \frac{31}{15120}\frac{\omega^6}{\mu^6} + \dots$$

That means **Asymptotic Freedom**:

$M(\mu)$ behaves like $M_0(\mu)$ at large $\mu \gg \omega$!

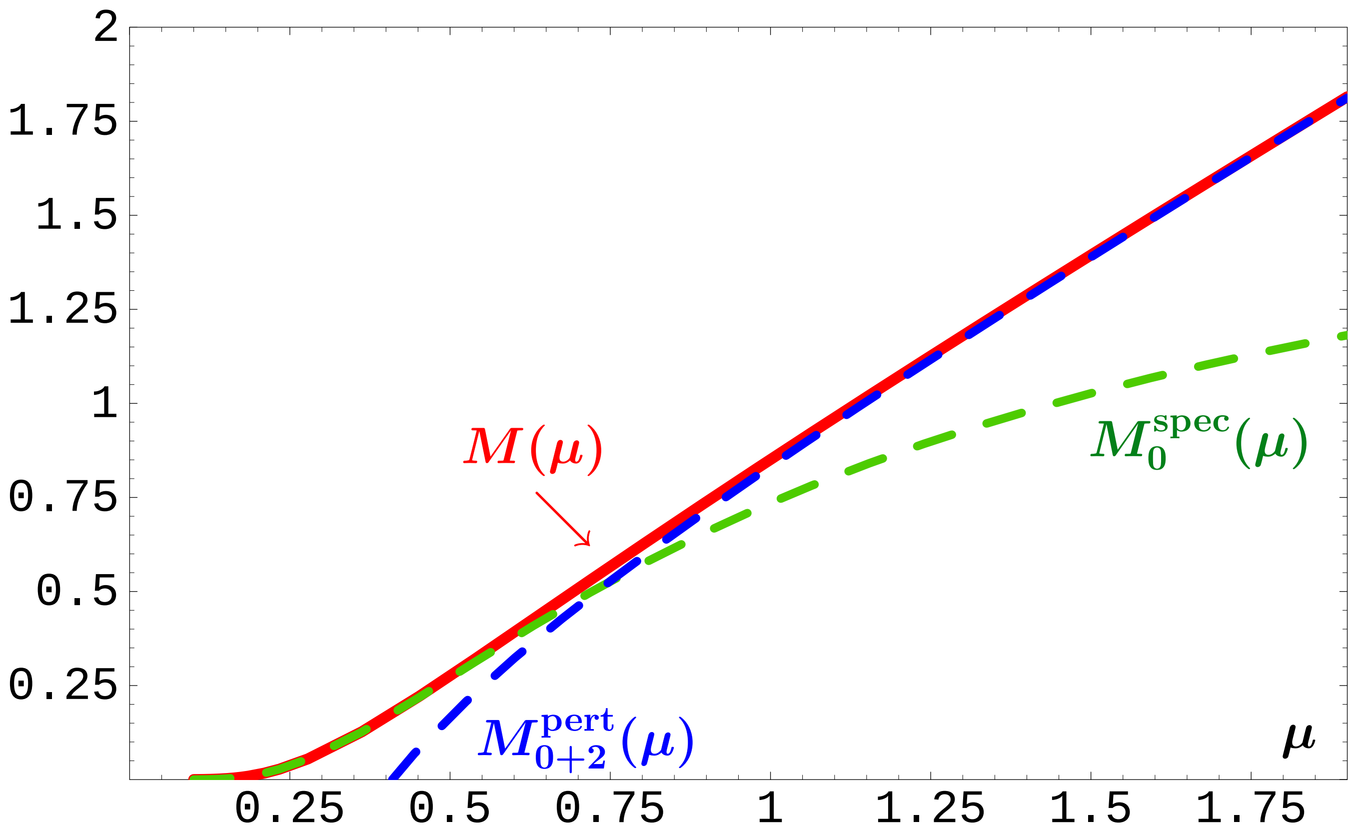
**Asymptotic Freedom** in Quantum Mechanics

is violated by **Power Corrections**

of the type $\omega^2/\mu^2$

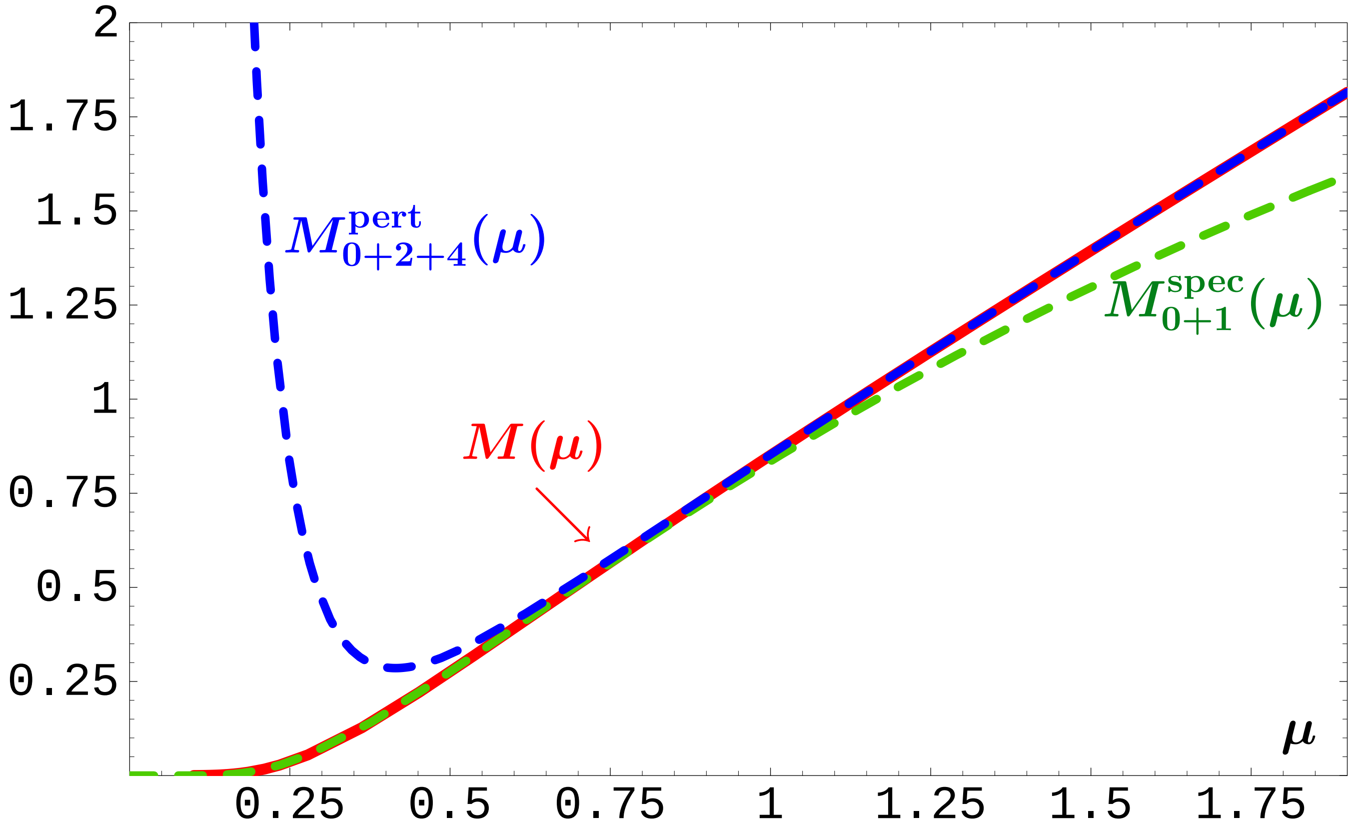
# *Graphics for $M(\mu)$*

**Exact $M(\mu)$**; **Ground state only**; $\boldsymbol{M_0(\mu) + O(\omega^2/\mu^2)}$.



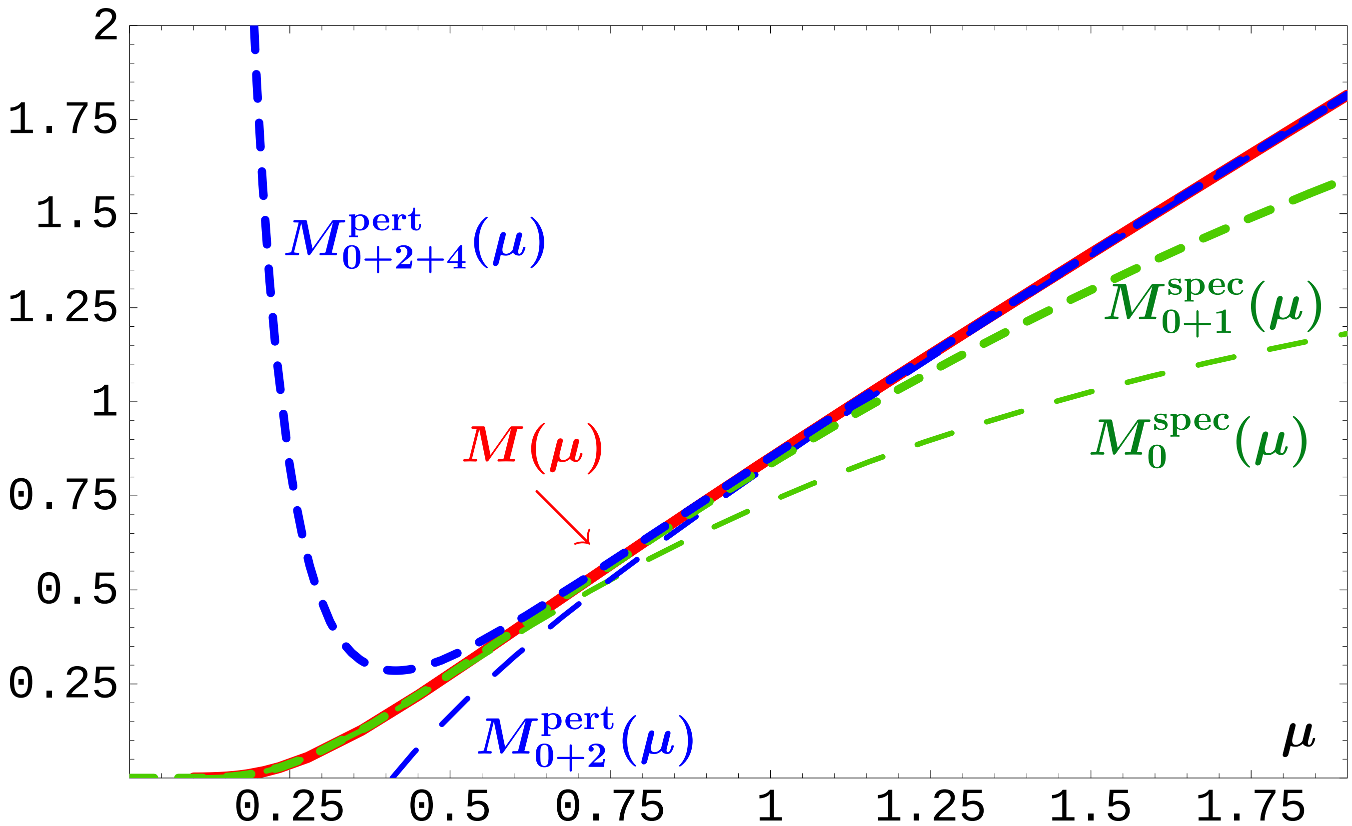
# Graphics for $M(\mu)$

**Exact $M(\mu)$**; **$0+1$ states only**; **$M_0(\mu)+O(\omega^4/\mu^4)$**.



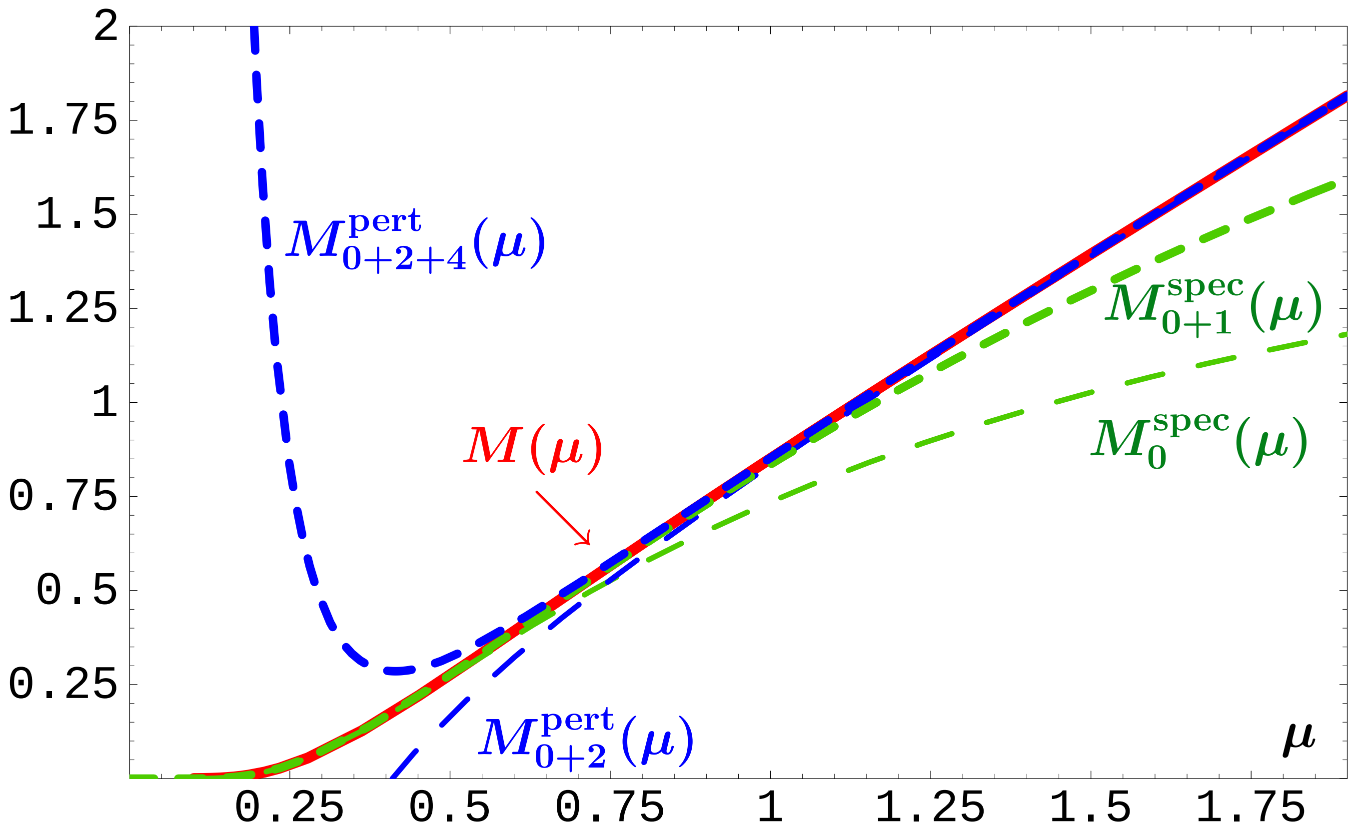
# *Graphics for $M(\mu)$*

For small $\mu$ in spectral part survives only ground state $|\psi_0|^2 e^{-E_0/\mu}$. **But**: PT breaks down.



# Graphics for $M(\mu)$

For large $\mu$ **AF** works well: $M(\mu) \simeq M_0(\mu)$. **But**: We need more and more resonances to saturate $M(\mu)$.



# *Global and Local Dualities*

# *Global Duality: Free $\Leftrightarrow$ Confined*

We need to model higher resonances in spectral repr. of our correlator $M(\mu)$:

$$M^{\rm spec}(\mu) = \sum_{k\geq 0} \frac{m\omega}{\pi}\, e^{-E_k/\mu} \equiv \int_0^\infty \rho^{\rm osc}(E)\, e^{-E/\mu}\, dE$$

# *Global Duality: Free $\Leftrightarrow$ Confined*

We need to model higher resonances in spectral repr. of our correlator $M(\mu)$:

$$M^{\text{spec}}(\mu) = \sum_{k\geq 0} \frac{m\omega}{\pi}\, e^{-E_k/\mu} \equiv \int_0^\infty \rho^{\text{osc}}(E)\, e^{-E/\mu}\, dE$$

Here spectral density is just sum of $\delta$-functions:

$$\rho^{\text{osc}}(E) = \sum_{k\geq 0} \frac{m\omega}{\pi}\, \delta(E - E_k)\,.$$

# *Global Duality: Free $\Leftrightarrow$ Confined*

We need to model higher resonances in spectral repr. of our correlator $M(\mu)$:

$$M^{\rm spec}(\mu) = \sum_{k\geq 0} \frac{m\omega}{\pi}\, e^{-E_k/\mu} \equiv \int_0^\infty \rho^{\rm osc}(E)\, e^{-E/\mu}\, dE$$

Analogously we have integral representation for free correlator:

$$M_0(\mu) = \frac{m\mu}{2\pi} \equiv \int_0^\infty \rho_0(E)\, e^{-E/\mu}\, dE\,.$$

**Who knows what is $\rho_0(E)$ ?**

# *Global Duality: Free $\Leftrightarrow$ Confined*

We need to model higher resonances in spectral repr. of our correlator $M(\mu)$:

$$M^{\text{spec}}(\mu) = \sum_{k\geq 0} \frac{m\omega}{\pi}\, e^{-E_k/\mu} \equiv \int_0^\infty \rho^{\text{osc}}(E)\, e^{-E/\mu}\, dE$$

Analogously we have integral representation for free correlator:

$$M_0(\mu) = \frac{m\mu}{2\pi} \equiv \int_0^\infty \rho_0(E)\, e^{-E/\mu}\, dE\,.$$

**Who knows what is $\rho_0(E)$? Answer**: $\rho_0(E) = \dfrac{m}{2\pi}$.

# *Global Duality: Free ⇔ Confined*

We need to model higher resonances in spectral repr. of our correlator $M(\mu)$:

$$M^{\text{spec}}(\mu) = \int_0^\infty \rho^{\text{osc}}(E)\, e^{-E/\mu}\, dE\,; \;\; M_0(\mu) = \int_0^\infty \rho_0(E)\, e^{-E/\mu}\, dE\,.$$

**Asymptotic Freedom**:

$$M(\mu \to \infty) = M_0(\mu \to \infty)$$

# *Global Duality: Free ⇔ Confined*

We need to model higher resonances in spectral repr. of our correlator $M(\mu)$:

$$M^{\rm spec}(\mu) = \int_0^\infty \rho^{\rm osc}(E)\, e^{-E/\mu}\, dE\,; \;\; M_0(\mu) = \int_0^\infty \rho_0(E)\, e^{-E/\mu}\, dE\,.$$

**Asymptotic Freedom**:

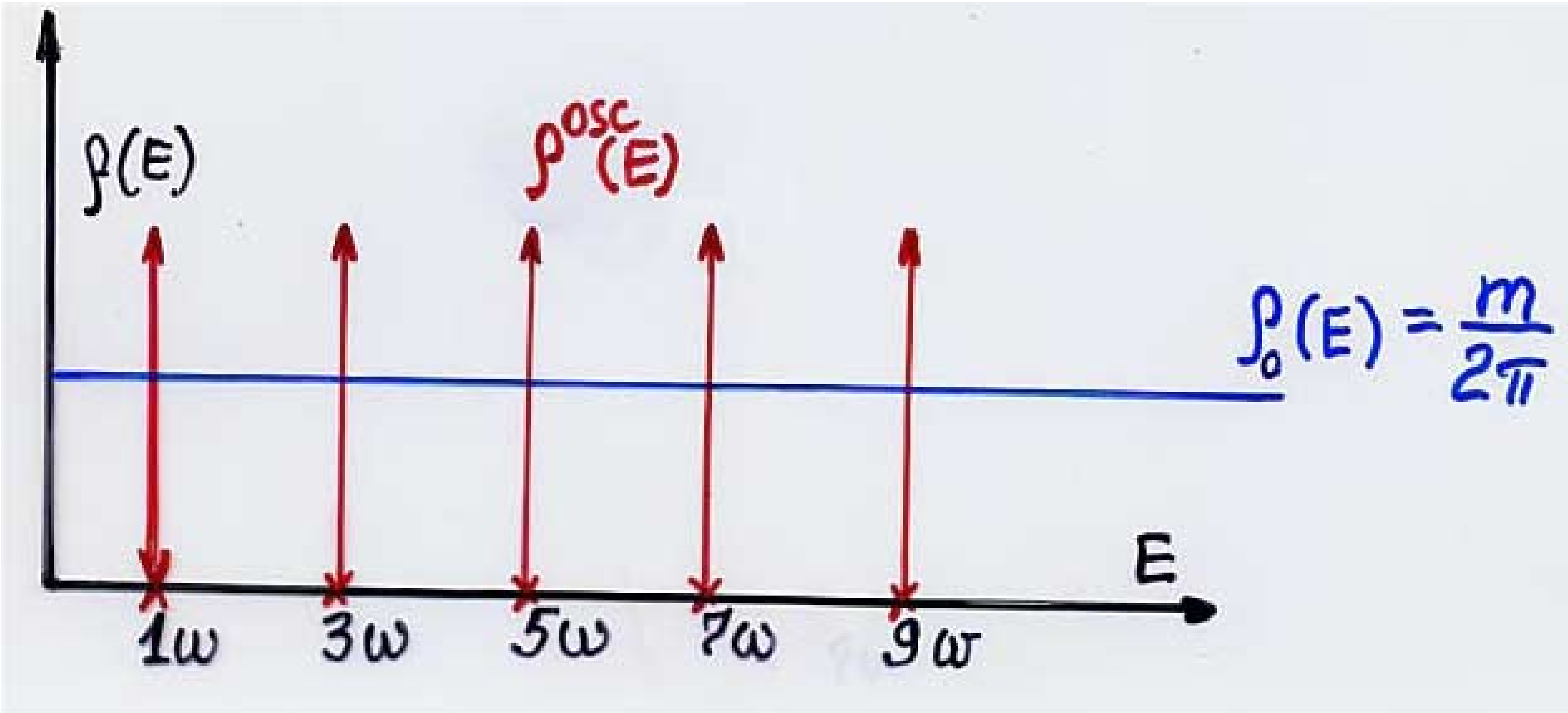
$$M(\mu \to \infty) = M_0(\mu \to \infty)$$

dictates **Global Duality** for these two densities

$$\int_0^\infty \rho^{\rm osc}(E)\, dE = \int_0^\infty \rho_0(E)\, dE$$

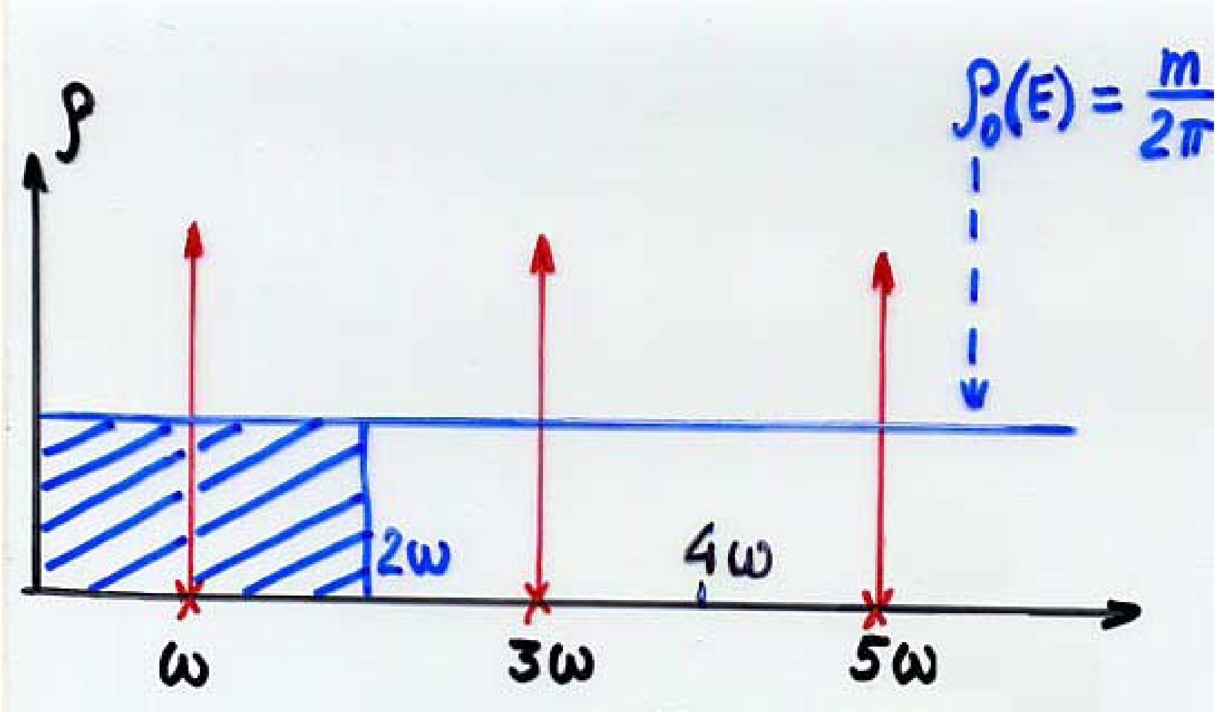
# *Graphics of dual spectral densities*

At first glance they have completely different behaviour:



# *Graphics of dual spectral densities*

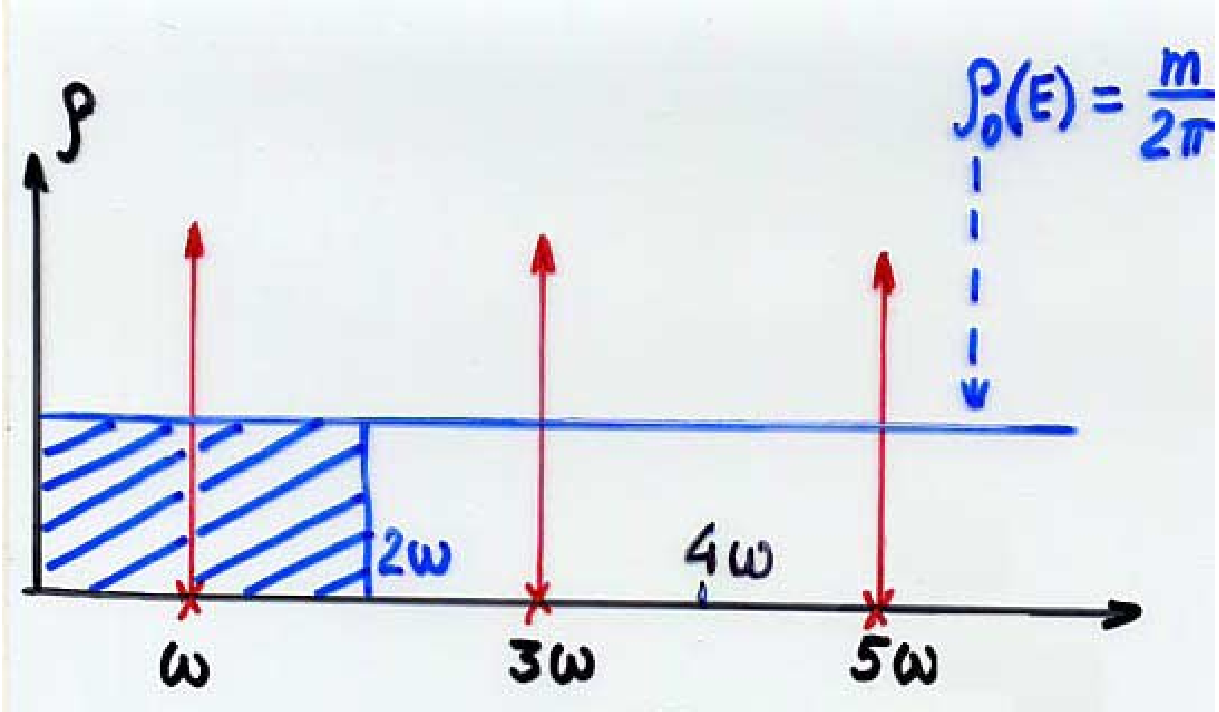
But we have very interesting relations between $2k\omega$-partial integral moments of this dual densities, namely, $< E^N >_{2k\omega} = \int_{2k\omega}^{2k\omega+2\omega} E^N \rho(E) dE$. For $N = 0$:




$$\int_{2k\omega}^{2(k+1)\omega} \rho^{\rm osc}(E)\, dE = \frac{m\omega}{\pi} = \int_{2k\omega}^{2(k+1)\omega} \rho_0(E)\, dE$$

# Graphics of dual spectral densities

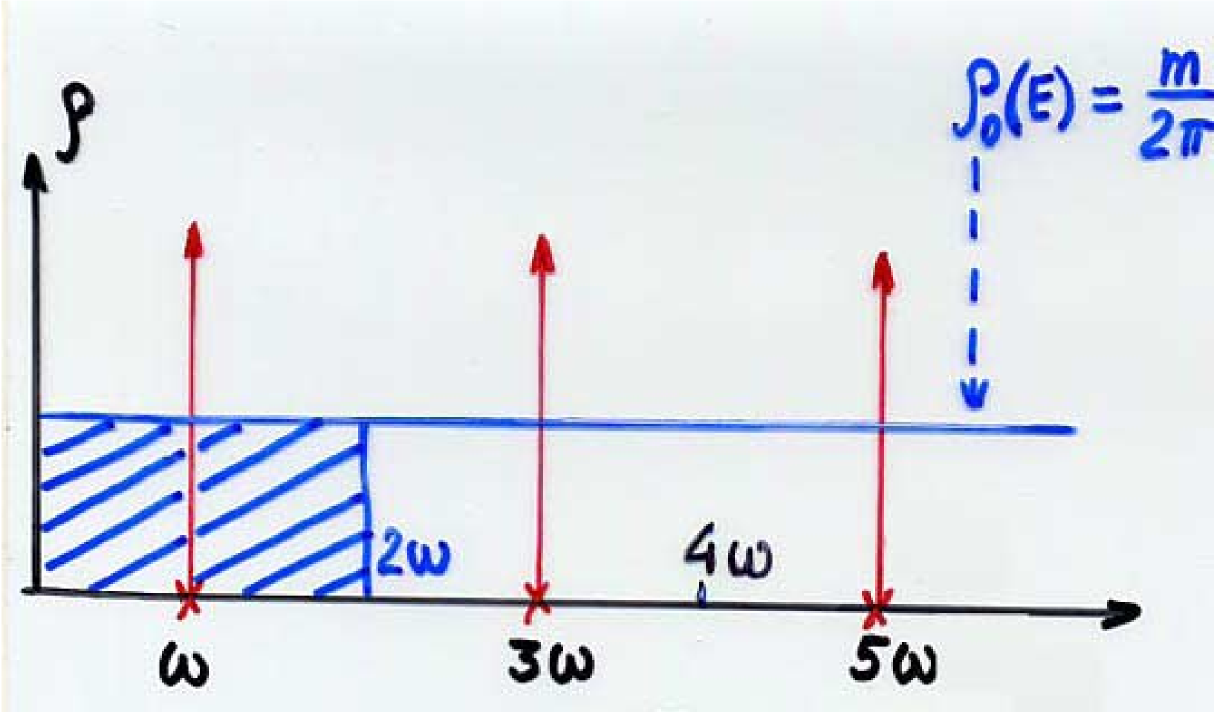
But we have very interesting relations between $2k\omega$-partial integral moments of this dual densities, namely, $< E^N >_{2k\omega} = \int_{2k\omega}^{2k\omega+2\omega} E^N \rho(E) dE$. For $N = 1$:




$$\int_{2k\omega}^{2(k+1)\omega} E\,\rho^{\rm osc}(E)\,dE = \frac{m\omega^2(2k+1)}{\pi} = \int_{2k\omega}^{2(k+1)\omega} E\,\rho_0(E)\,dE$$

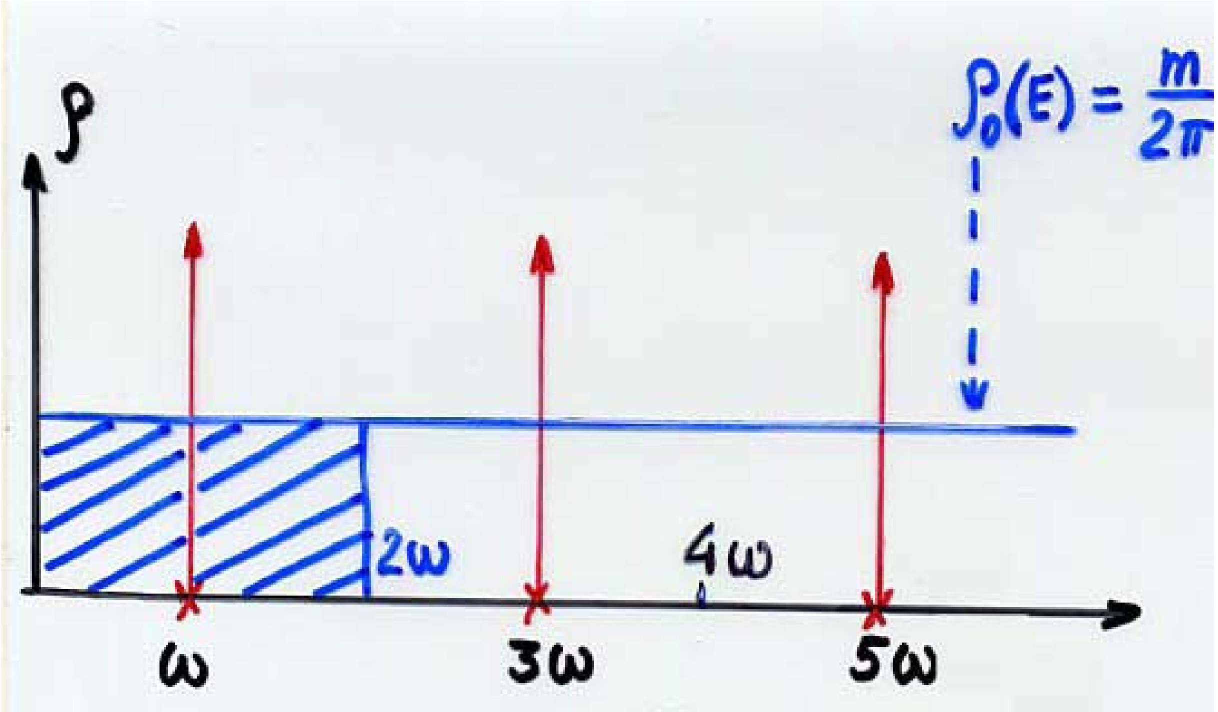
# Graphics of dual spectral densities

But we have very interesting relations between $2k\omega$-partial integral moments of this dual densities, namely, $< E^N >_{2k\omega} = \int_{2k\omega}^{2k\omega+2\omega} E^N \rho(E) dE$. For $N \geq 2$:




$$\int_{2k\omega}^{2(k+1)\omega} E^N \rho^{\rm osc}(E)\, dE = \int_{2k\omega}^{2(k+1)\omega} E^N \rho_0(E)\, dE \left[1 + O\left(\frac{N^2}{k^2}\right)\right]$$

# *Graphics of dual spectral densities*




**We have duality between each excited resonance in oscillator and free particle in some spectral domain ⇒ "Local Duality"**

# QM Sum Rules for Harmonic Oscillator

# *QM Sum Rules*

We can model higher state contributions by

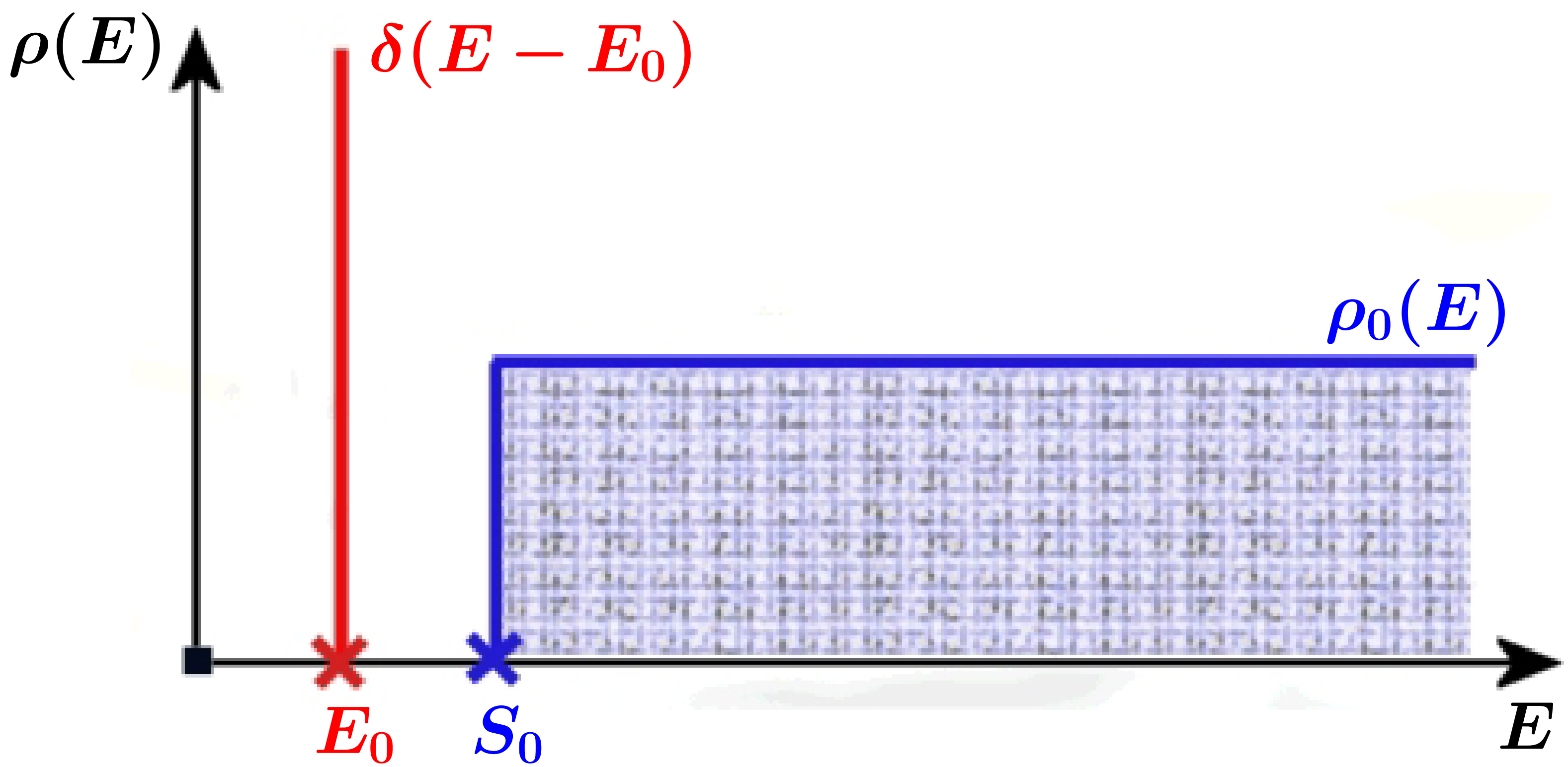
"higher states" = "free states" outside interval $(0, S_0)$

# QM Sum Rules

We can model higher state contributions by

"higher states" = "free states" outside interval $(0, S_0)$

or: $\rho^{\text{mod}}(E) = |\psi_0(0)|^2\,\delta\,(E - E_0) + \rho_0(E)\,\theta\,(E - S_0)$



# QM Sum Rules

Our model for HSs gives

$$M^{\mathrm{mod}}(\mu) = |\psi_0(0)|^2\, e^{-E_0/\mu} + \int_{S_0}^{\infty} \rho_0(s)\, e^{-E/\mu}\, dE\,.$$

# *QM Sum Rules*

Our model for HSs gives

$$M^{\rm mod}(\mu) = |\psi_0(0)|^2\, e^{-E_0/\mu} + \int_{S_0}^{\infty} \rho_0(s)\, e^{-E/\mu}\, dE\,.$$

After all we have Sum Rule:

$$\boldsymbol{|\psi_0(0)|^2} e^{-\boldsymbol{E_0}/\mu} = \int_0^{S_0} \rho_0(E)\, e^{-E/\mu}\, dE + \textbf{power corrections}$$

# *QM Sum Rules*

Our model for HSs gives

$$M^{\rm mod}(\mu) = |\psi_0(0)|^2 \, e^{-E_0/\mu} + \int_{S_0}^{\infty} \rho_0(s) \, e^{-E/\mu} \, dE \, .$$

or equivalent SR (with $\Psi_0(0) \equiv \psi_0(0)\sqrt{\pi/\omega}$):

$$|\boldsymbol{\Psi_0(0)}|^2 e^{-\boldsymbol{E_0}/\mu} = \frac{\mu}{2\omega} \left\{ 1 - e^{-S_0/\mu} - \frac{\omega^2}{6\mu^2} + \dots \right\}$$

# QM Sum Rules

Our model for HSs gives

$$M^{\rm mod}(\mu) = |\psi_0(0)|^2\, e^{-E_0/\mu} + \int_{S_0}^{\infty} \rho_0(s)\, e^{-E/\mu}\, dE\,.$$

or equivalent SR (with $\Psi_0(0) \equiv \psi_0(0)\sqrt{\pi/\omega}$):

$$|\boldsymbol{\Psi_0(0)}|^{\mathbf{2}} e^{-\boldsymbol{E_0}/\mu} = \frac{\mu}{2\omega}\left\{1 - e^{-S_0/\mu} - \frac{\omega^2}{6\mu^2} + \ldots\right\}$$

Daughter SR — by $\dfrac{-\partial\ldots}{\partial\mu^{-1}}$:

$$|\boldsymbol{\Psi_0(0)}|^{\mathbf{2}} \boldsymbol{E_0}\, e^{-\boldsymbol{E_0}/\mu} = \frac{\mu^2}{2\omega}\left\{1 - \left(1 + \frac{S_0}{\mu}\right) e^{-S_0/\mu} + \frac{\omega^2}{6\mu^2} + \ldots\right\}$$

# QM Sum Rules: The Scheme

Main SR:

$$|\boldsymbol{\Psi_0(0)}|^2 \approx \boldsymbol{\Psi_0^2}(\boldsymbol{E_0}, S_0, \boldsymbol{\mu}) = \frac{\mu\, e^{\boldsymbol{E_0}/\mu}}{2\omega}\left\{1 - e^{-S_0/\mu} - \frac{\omega^2}{6\mu^2} + \ldots\right\}$$

# QM Sum Rules: The Scheme

Main SR:

$$|\boldsymbol{\Psi_0(0)}|^2 \approx \boldsymbol{\Psi_0^2}(\boldsymbol{E_0}, S_0, \boldsymbol{\mu}) = \frac{\mu\, e^{\boldsymbol{E_0}/\mu}}{2\omega} \left\{ 1 - e^{-S_0/\mu} - \frac{\omega^2}{6\mu^2} + \dots \right\}$$

Daughter SR:

$$\boldsymbol{E_0} \approx \boldsymbol{E_0}(S_0, \boldsymbol{\mu}) = \mu \frac{1 - \left(1 + \frac{S_0}{\mu}\right) e^{-S_0/\mu} + \frac{\omega^2}{6\mu^2} + \dots}{1 - e^{-S_0/\mu} - \frac{\omega^2}{6\mu^2} + \dots}$$

# *QM Sum Rules: The Scheme*

Main SR:

$$|\Psi_0(0)|^2 \approx \Psi_0^2(E_0, S_0, \mu) = \frac{\mu\, e^{E_0/\mu}}{2\omega}\left\{1 - e^{-S_0/\mu} - \frac{\omega^2}{6\mu^2} + \dots\right\}$$

Daughter SR:

$$E_0 \approx E_0(S_0, \mu) = \mu\,\frac{1 - \left(1 + \frac{S_0}{\mu}\right) e^{-S_0/\mu} + \frac{\omega^2}{6\mu^2} + \dots}{1 - e^{-S_0/\mu} - \frac{\omega^2}{6\mu^2} + \dots}$$

Strategy of processing SRs:

- Determine $E_0 \approx E_0(S_0, \mu)$ by minimal sensitivity to variation of $\mu \in [\mu_L; \mu_U]$ at appropriate $S_0$;

# QM Sum Rules: The Scheme

Main SR:

$$|\Psi_0(0)|^2 \approx \Psi_0^2(E_0, S_0, \mu) = \frac{\mu\, e^{E_0/\mu}}{2\omega}\left\{1 - e^{-S_0/\mu} - \frac{\omega^2}{6\mu^2} + \dots\right\}$$

Daughter SR:

$$E_0 \approx E_0(S_0, \mu) = \mu\,\frac{1 - \left(1 + \frac{S_0}{\mu}\right) e^{-S_0/\mu} + \frac{\omega^2}{6\mu^2} + \dots}{1 - e^{-S_0/\mu} - \frac{\omega^2}{6\mu^2} + \dots}$$

Strategy of processing SRs:

- Determine $E_0 \approx E_0(S_0, \mu)$ by minimal sensitivity to variation of $\mu \in [\mu_L; \mu_U]$ at appropriate $S_0$;
- Determine $|\Psi_0(0)|^2 \approx \Psi_0^2(S_0, E_0, \mu)$ by minimal sensitivity to variation of $\mu$ at appropriate $S_0$.

# QM Sum Rules: Fidelity Window

- Power corrections are of the type $(\omega/\mu)^{2n}$ and they are huge at $\mu \ll \omega$. Demand:

$$\Delta_{\rm pert}(\mu) \equiv \sum_{n\geq 1} \frac{C_{2n}(\omega/\mu)^{2n}}{M_0(\mu)} \leq 0.33 \quad \text{for all} \quad \mu \geq \mu_{\rm L}$$

# *QM Sum Rules: Fidelity Window*

- Power corrections are of the type $(\omega/\mu)^{2n}$ and they are huge at $\mu \ll \omega$. Demand:

$$\Delta_{\text{pert}}(\mu) \equiv \sum_{n\geq 1} \frac{C_{2n}(\omega/\mu)^{2n}}{M_0(\mu)} \leq 0.33 \quad \text{for all} \quad \mu \geq \mu_{\text{L}}$$

- Higher states at large $\mu \gg \omega$ are not suppressed by $e^{-E_k/\mu} \approx 1$. Demand:

$$\Delta_{\text{pert}}(\mu) \equiv \int_{S_0}^{\infty} \frac{\rho_0(E)}{M_0(\mu)} e^{-E/\mu} dE \leq 0.33 \quad \text{for all} \quad \mu \leq \mu_{\text{U}}$$

# *QM Sum Rules: Fidelity Window*

- Power corrections are of the type $(\omega/\mu)^{2n}$ and they are huge at $\mu \ll \omega$. Demand:

$$\Delta_{\text{pert}}(\mu) \equiv \sum_{n\geq 1} \frac{C_{2n}(\omega/\mu)^{2n}}{M_0(\mu)} \leq 0.33 \quad \text{for all} \quad \mu \geq \mu_{\text{L}}$$

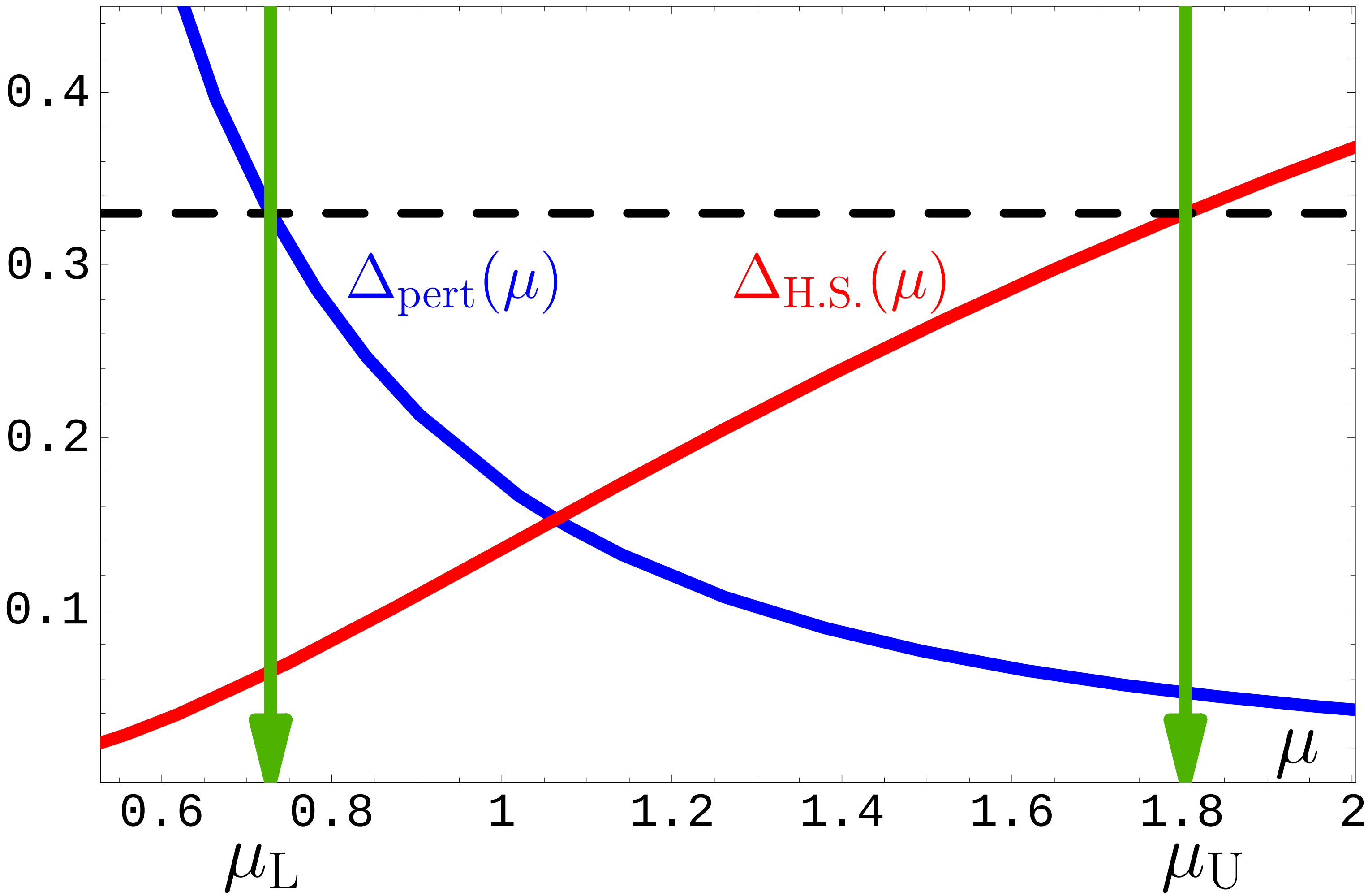
- Higher states at large $\mu \gg \omega$ are not suppressed by $e^{-E_k/\mu} \approx 1$. Demand:

$$\Delta_{\text{pert}}(\mu) \equiv \int_{S_0}^{\infty} \frac{\rho_0(E)}{M_0(\mu)} e^{-E/\mu} dE \leq 0.33 \quad \text{for all} \quad \mu \leq \mu_{\text{U}}$$

- Fidelity window: $\mu_{\text{L}} \leq \mu \leq \mu_{\text{U}}$. Only for $\mu$ inside it is reasonable to demand **minimal sensitivity** of SRs to variations in $\mu$!

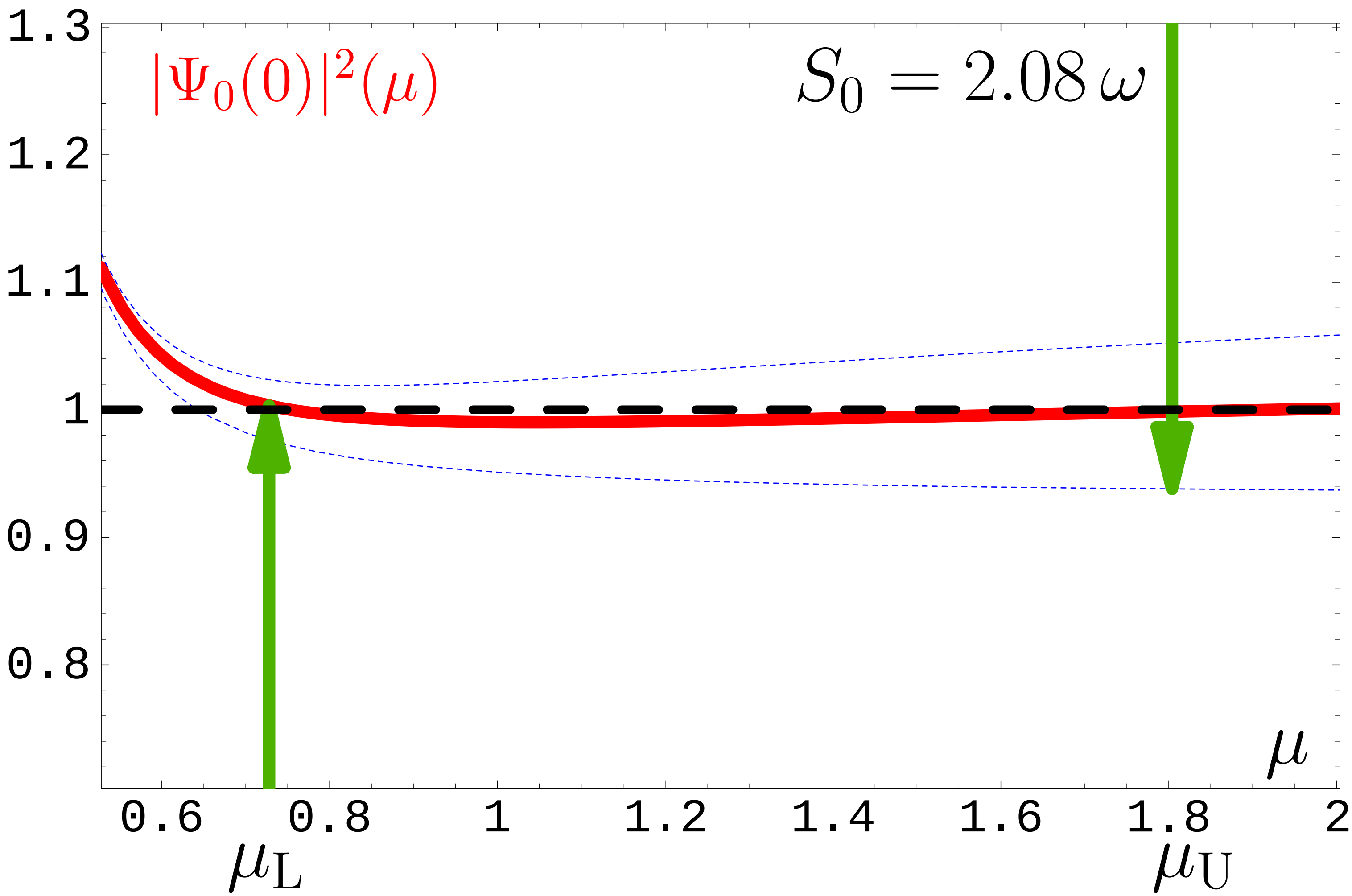
# *QM SRs: Setup with fixed $E_0 = 1$*

We fix energy to the exact value $E_0 = 1$ and obtain fidelity window: $\mu_L = 0.73\,\omega$ and $\mu_U = 1.80\,\omega$



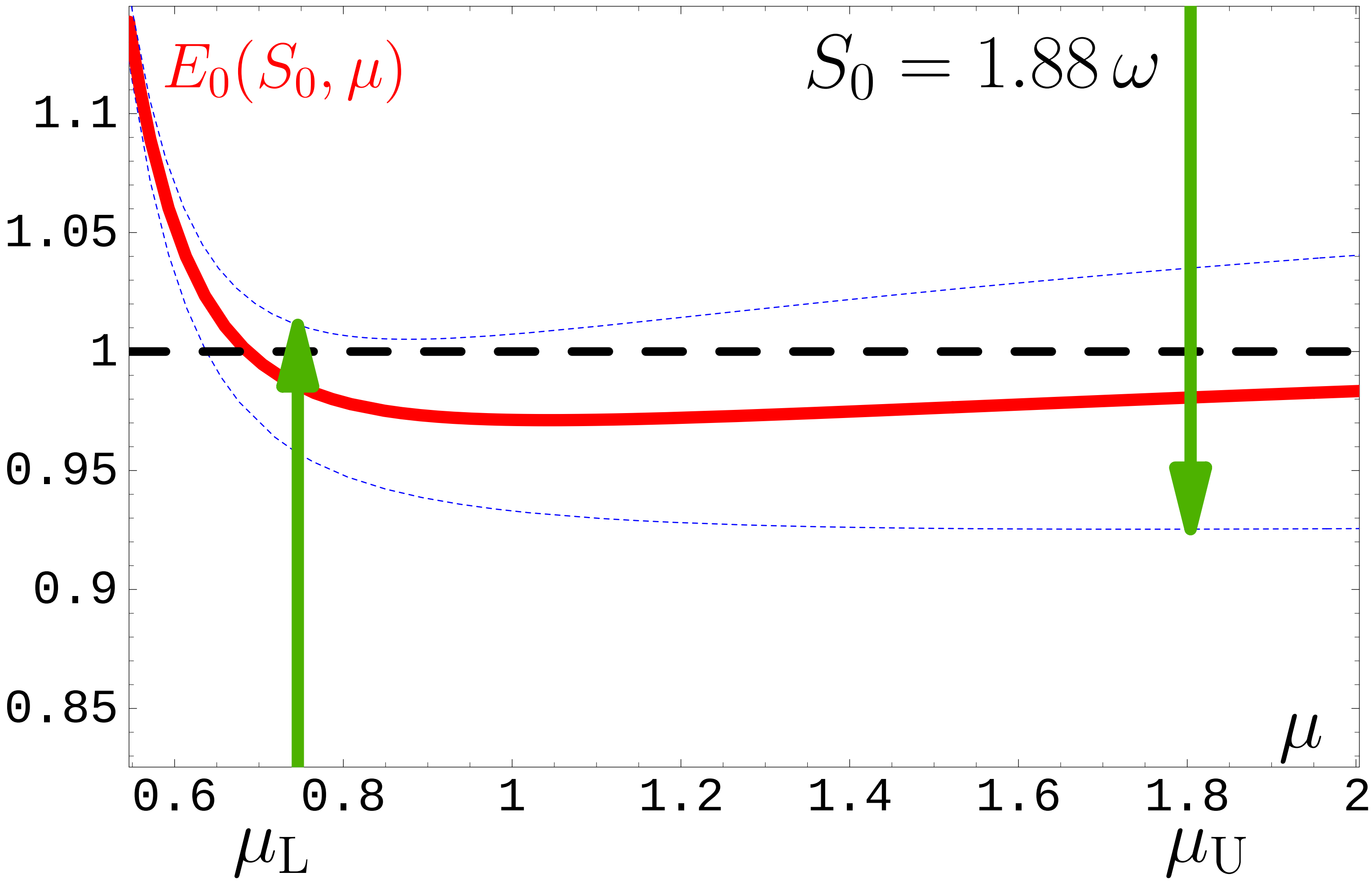
# QM SRs: Setup with fixed $E_0 = 1$

We fix energy to the exact value $E_0 = 1$ and obtain $|\Psi_0(0)|^2 = 0.99$ with only 2 pow.corrs. (exact $|\Psi_0(0)|^2 = 1$)



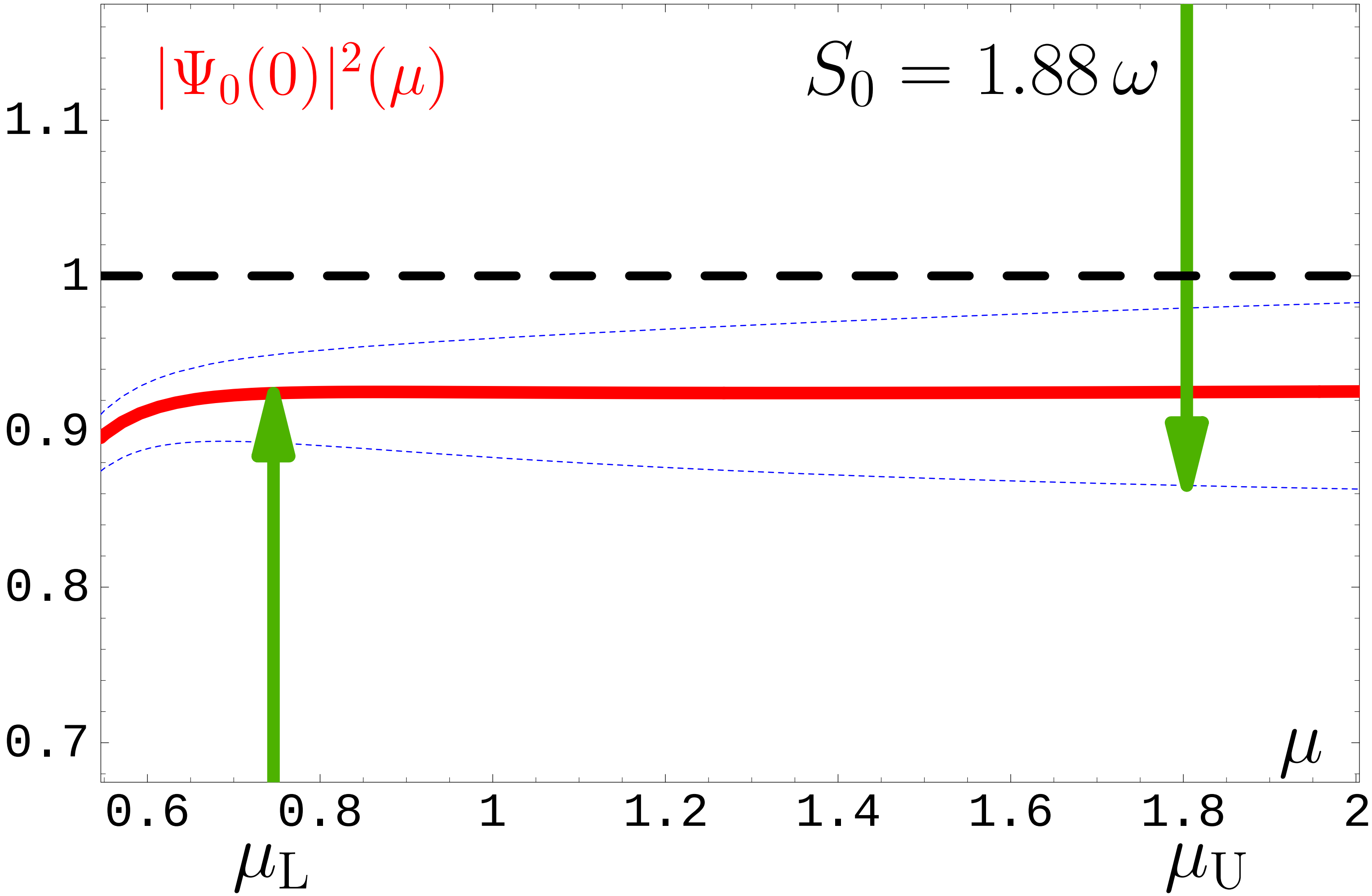
# QM SRs: Complete Setup

We take into account 3 power corrs. and obtain fidelity window $[0.74\,\omega; 1.8\,\omega]$ and $E_0 = 0.98\,\omega$ for $S_0 = 1.88\,\omega$ :



# *QM SRs: Complete Setup*

We take into account 3 power corrs. and obtain  and
$|\Psi_0(0)|^2 = 0.92$



# *QM Sum Rules:*

# *Conclusions*

# *QM SRs: Conclusions*

- SRs give $E_0$ and $|\psi_0(0)|^2$ with accuracy **not worser 10%** ;

# *QM SRs: Conclusions*

- SRs give $E_0$ and $|\psi_0(0)|^2$ with accuracy **not worser 10%** ;
- Main source of the error – **crude model** for spectral density of higher states: even taking into account 10 power corrections we obtain $E_0 = 0.95\,\omega$, $S_0 = 1.79\,\omega$, and $|\psi_0(0)|^2 = 0.89$ ;

# *QM SRs: Conclusions*

- SRs give $E_0$ and $|\psi_0(0)|^2$ with accuracy **not worser 10%**;

- Main source of the error – **crude model** for spectral density of higher states: even taking into account 10 power corrections we obtain $E_0 = 0.95\,\omega$, $S_0 = 1.79\,\omega$, and $|\psi_0(0)|^2 = 0.89$;

- **But**: If we know $E_0 = 1$ exactly (say, from Particle Data Group), then accuracy can be twice higher: with taking into account 2 power corrections we obtain $S_0 = 2.08\,\omega$ and $|\psi_0(0)|^2 = 0.99$!

# *QM SRs: Conclusions*

- SRs give $E_0$ and $|\psi_0(0)|^2$ with accuracy **not worser 10%**;
- Main source of the error – **crude model** for spectral density of higher states: even taking into account 10 power corrections we obtain $E_0 = 0.95\,\omega$, $S_0 = 1.79\,\omega$, and $|\psi_0(0)|^2 = 0.89$;
- **But**: If we know $E_0 = 1$ exactly (say, from Particle Data Group), then accuracy can be twice higher: with taking into account 2 power corrections we obtain $S_0 = 2.08\,\omega$ and $|\psi_0(0)|^2 = 0.99$!
- In QCD spectral density more close to perturbative!

***Quarks inside,***

***Hadrons outside!***

***How to proceed?***

# *QCD: Lagrangian, quarks and gluons*

Gauge-invariant Lagrangian of QCD

$$\mathcal{L}_{QCD} = -\frac{1}{4} G^a_{\mu\nu} G^{a\mu\nu} + \sum_{q=u,d,s,\dots} \bar{\psi}_q (i\hat{D} - m_q) \psi_q$$

contains only gluon ($G^a_{\mu\nu}(x)$) and quark ($\psi_q(x)$) fields.

# QCD: Lagrangian, quarks and gluons

Gauge-invariant Lagrangian of QCD

$$\mathcal{L}_{QCD} = -\frac{1}{4} G^a_{\mu\nu} G^{a\mu\nu} + \sum_{q=u,d,s,\ldots} \bar{\psi}_q (i\hat{D} - m_q)\psi_q$$

contains only gluon ($G^a_{\mu\nu}(x)$) and quark ($\psi_q(x)$) fields. These fields has color degrees of freedom: 3 for quarks $\psi^A_q(x)$ ($A = \mathbf{1}, \mathbf{2}, \mathbf{3}$) and 8 for gluons $G^a_{\mu\nu}(x)$ ($a = \mathbf{1}, \ldots, \mathbf{8}$).

# *QCD: Lagrangian, quarks and gluons*

Gauge-invariant Lagrangian of QCD

$$\mathcal{L}_{QCD} = -\frac{1}{4} G^a_{\mu\nu} G^{a\mu\nu} + \sum_{q=u,d,s,\ldots} \bar{\psi}_q (i\hat{D} - m_q) \psi_q$$

contains only gluon ($G^a_{\mu\nu}(x)$) and quark ($\psi_q(x)$) fields. These fields has color degrees of freedom: 3 for quarks $\psi^A_q(x)$ ($A = \mathbf{1}, \mathbf{2}, \mathbf{3}$) and 8 for gluons $G^a_{\mu\nu}(x)$ ($a = \mathbf{1}, \ldots, \mathbf{8}$).
Interaction is hidden in $G^a_{\mu\nu}$ and covariant derivative $D^{AB}_\mu$:

$$\begin{aligned} G^a_{\mu\nu} &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g_s f^{abc} A^b_\mu A^c_\nu \\ D^{AB}_\mu &= \partial_\mu - i g_s \left(t^a\right)^{AB} A^a_\mu \end{aligned}$$

# QCD: Lagrangian, quarks and gluons

Gauge-invariant Lagrangian of QCD

$$\mathcal{L}_{QCD} = -\frac{1}{4} G^a_{\mu\nu} G^{a\mu\nu} + \sum_{q=u,d,s,\ldots} \bar{\psi}_q (i\hat{D} - m_q) \psi_q$$

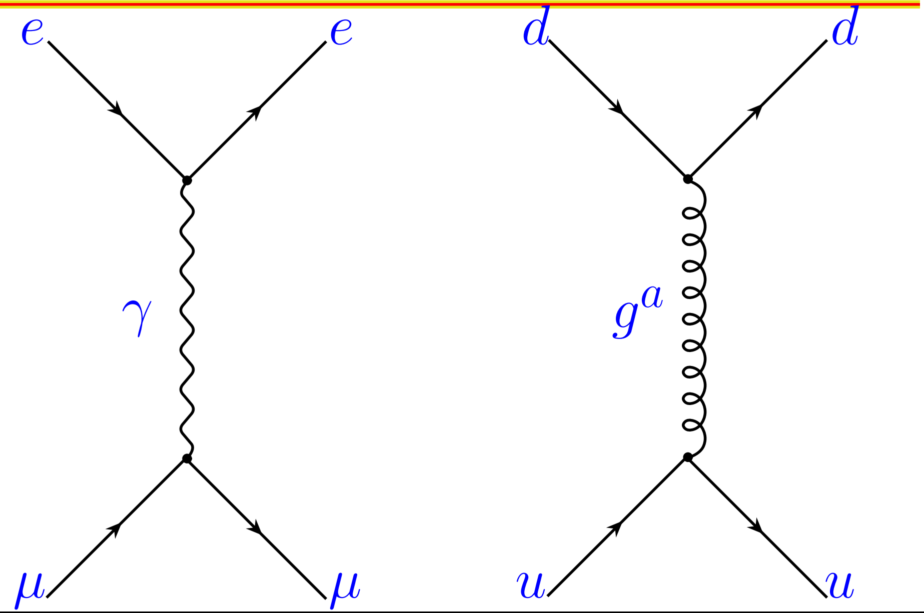
contains only gluon ($G^a_{\mu\nu}(x)$) and quark ($\psi_q(x)$) fields. These fields has color degrees of freedom: 3 for quarks $\psi_q^A(x)$ ($A = \mathbf{1}, \mathbf{2}, \mathbf{3}$) and 8 for gluons $G^a_{\mu\nu}(x)$ ($a = \mathbf{1}, \ldots, \mathbf{8}$).
Interaction is hidden in $G^a_{\mu\nu}$ and covariant derivative $D^{AB}_\mu$:

$$\begin{aligned} G^a_{\mu\nu} &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g_s f^{abc} A^b_\mu A^c_\nu \\ D^{AB}_\mu &= \partial_\mu - i g_s \left(t^a\right)^{AB} A^a_\mu \end{aligned}$$

It is nonlinear due to **Non-Abelian** character ($f^{abc} \neq 0$).

# QCD: Coloured gluons ⇒ Confinement

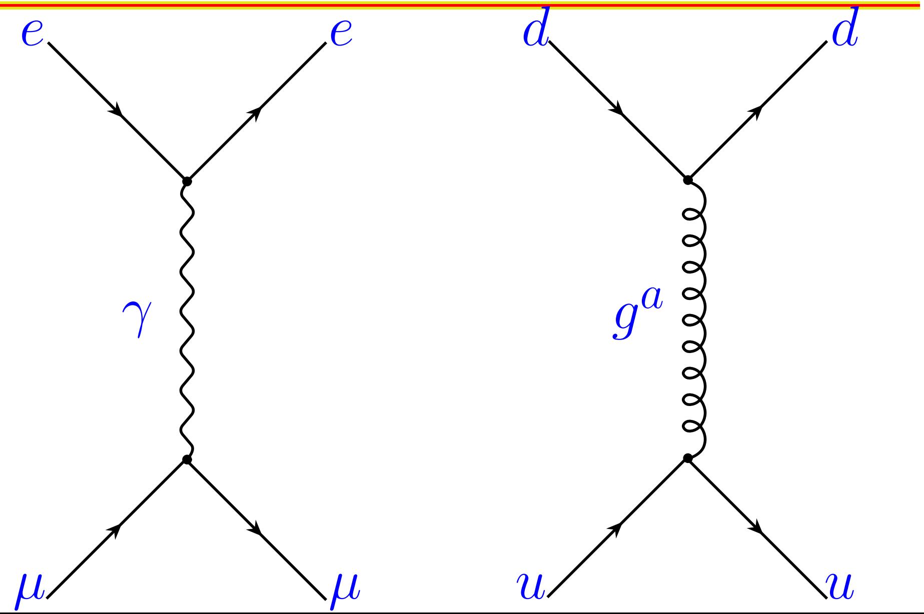
Consider $e\mu$- and $qq$-scattering (for $d$- and $u$-flavors): wavy line denotes photon and curved line – gluon.




Comparison:

| Parameter | Photon | Gluon |
|---|---|---|
| Mass | 0 | 0 |
| Spin | 1 | 1 |
| Vertex | $e\gamma_\mu$ | $g_s\gamma_\mu(t^a)_{ij}$ |
| Charge | 0 | $\neq 0$ |

# QCD: Coloured gluons ⇒ Confinement




Consider $e\mu$- and $qq$-scattering (for $d$- and $u$-flavors): wavy line denotes photon and curved line – gluon.

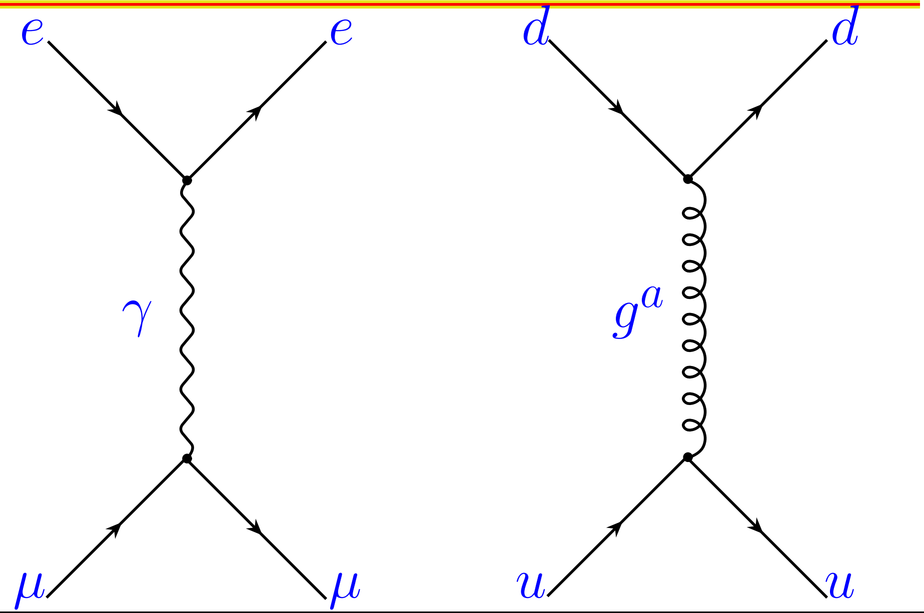
Comparison:

| Parameter | Photon | Gluon |
|---|---|---|
| Mass | 0 | 0 |
| Spin | 1 | 1 |
| Vertex | $e\gamma_\mu$ | $g_s\gamma_\mu(t^a)_{ij}$ |
| Charge | 0 | $\neq 0$ |

**Non-Abelian** character of QCD ⇒ charged **gluons**.

# *QCD: Coloured gluons ⇒ Confinement*

Consider $e\mu$- and $qq$-scattering (for $d$- and $u$-flavors): wavy line denotes photon and curved line – gluon.




Comparison:

| Parameter | Photon | Gluon |
|---|---|---|
| Mass | 0 | 0 |
| Spin | 1 | 1 |
| Vertex | $e\gamma_\mu$ | $g_s\gamma_\mu(t^a)_{ij}$ |
| Charge | 0 | $\neq 0$ |

Coloured **gluons** ⇒ **confinement!**

# *Massless QCD: What are Hadrons?*

| PS- and V-mesons composed of $u$- and $d$-quarks | | |
|---|---|---|
| meson type | PS | V |
| composition | $\pi^0[\bar{u}u - \bar{d}d]$, $\pi^\pm[\bar{u}d, \bar{d}u]$ | $\rho^0(\omega)[\bar{u}u - \bar{d}d]$, $\rho^\pm[\bar{u}d, \bar{d}u]$ |
| mass | 140 MeV | 770(780) MeV |

# *Massless QCD: What are Hadrons?*

| PS- and V-mesons composed of $u$- and $d$-quarks | | |
|---|---|---|
| meson type | PS | V |
| composition | $\pi^0[\bar{u}u - \bar{d}d]$, $\pi^\pm[\bar{u}d, \bar{d}u]$ | $\rho^0(\omega)[\bar{u}u - \bar{d}d]$, $\rho^\pm[\bar{u}d, \bar{d}u]$ |
| mass | 140 MeV | 770(780) MeV |

| Baryons composed of $u$- and $d$-quarks | | | |
|---|---|---|---|
| composition | $p[uud]$ | $n[udd]$ | $\Delta^{++}[uuu]$, $\Delta^{+}[uud]$, $\Delta^{0}[udd]$, $\Delta^{-}[ddd]$ |
| mass | 938 MeV | 939 MeV | 1232 MeV |

# ***QCD SRs: Way to Study Hadrons in Non-Perturbative QCD***

# *QCD SRs: Hadrons in npQCD*

- **Problem**: bound states in QCD?

# *QCD SRs: Hadrons in npQCD*

- **Problem**: bound states in QCD?
- QCD SR method: calculate **properties of hadrons** (masses, decay constants, magnetic moments) without considering **hadronization** or **confinement**.

# *QCD SRs: Hadrons in npQCD*

- **Problem**: bound states in QCD?
- QCD SR method: calculate **properties of hadrons** (masses, decay constants, magnetic moments) without considering **hadronization** or **confinement**.
- Invented in 1977 by **Shifman, Vainshtein & Zakharov (ITEP)** to describe $J/\psi$-meson $= c\bar{c}$-system, discovered in 1974 in $e^+e^-$-annihilation at SPEAR (SLAC) and, in parallel, in $p + Be$-collisions at BNL.

# *QCD SRs: Hadrons in npQCD*

- **Problem**: bound states in QCD?
- QCD SR method: calculate **properties of hadrons** (masses, decay constants, magnetic moments) without considering **hadronization** or **confinement**.
- Invented in 1977 by **Shifman, Vainshtein & Zakharov (ITEP)** to describe $J/\psi$-meson $= c\bar{c}$-system, discovered in 1974 in $e^+e^-$-annihilation at SPEAR (SLAC) and, in parallel, in $p + Be$-collisions at BNL.
- In 1979 used to describe light hadrons in massless QCD.

# QCD SRs: Hadrons in npQCD

- **Problem**: bound states in QCD?
- QCD SR method: calculate **properties of hadrons** (masses, decay constants, magnetic moments) without considering **hadronization** or **confinement**.
- Invented in 1977 by **Shifman, Vainshtein & Zakharov (ITEP)** to describe $J/\psi$-meson $= c\bar{c}$-system, discovered in 1974 in $e^+e^-$-annihilation at SPEAR (SLAC) and, in parallel, in $p + Be$-collisions at BNL.
- In 1979 used to describe light hadrons in massless QCD.
- **Main idea**: to calculate correlators of hadron currents $\langle 0|T\,[J_1(x)J_2(0)]\,|0\rangle$ by two ways. Sum Rule is the result of matching.

# QCD SRs: General scheme

Correlator of hadron currents via dispersion integral

$$F_{x\to q}\left[\langle 0|T\left[J_1(x)J_2(0)\right]|0\rangle\right]\left(Q^2\right) \equiv \Pi(Q^2) =$$

$$= \int\limits_0^\infty \frac{\rho_{12}\left(s\right)\, ds}{s+Q^2} + \text{“subtractions”}$$

# *QCD SRs: General scheme*

Correlator of hadron currents via dispersion integral

$$F_{x\to q}\left[\langle 0|T\left[J_1(x)J_2(0)\right]|0\rangle\right]\left(Q^2\right) \equiv \Pi(Q^2) =$$

$$= \int\limits_0^\infty \frac{\rho_{12}\left(s\right)\, ds}{s+Q^2} + \text{“subtractions”}$$

Apply Borel transform

$$B_{Q^2\to M^2}\left[\Pi(Q^2)\right] \equiv \Phi\left(M^2\right) = \int\limits_0^\infty \rho_{12}\left(s\right)\, e^{-s/M^2}\frac{ds}{M^2}$$

to suppress “higher states” and to kill “subtractions” in DR.

# *QCD SRs: General scheme*

**1-st way**: Operator Product Expansion with account for **quark and gluon condensates** in QCD vacuum

$$\Phi\left(Q^2\right) = \Phi_{\text{pert}}\left(Q^2\right) + c_{GG}\,\frac{\langle(\boldsymbol{\alpha_s/\pi})\boldsymbol{GG}\rangle}{M^4} + c_{\bar{q}q}\,\frac{\boldsymbol{\alpha_s}\langle\bar{\boldsymbol{q}}\boldsymbol{q}\rangle^{\mathbf{2}}}{M^6}$$

Here $\langle\frac{\alpha_s}{\pi}G^a_{\mu\nu}G^{a\mu\nu}\rangle = 0.012\ \text{GeV}^4$, $\alpha_s\langle\bar{q}q\rangle^2 = 0.0018\ \text{GeV}^6$.

# QCD SRs: General scheme

**1-st way**: Operator Product Expansion with account for **quark and gluon condensates** in QCD vacuum

$$\Phi\left(Q^2\right) = \Phi_{\rm pert}\left(Q^2\right) + c_{GG}\,\frac{\langle(\alpha_s/\pi)GG\rangle}{M^4} + c_{\bar{q}q}\,\frac{\alpha_s\langle\bar{q}q\rangle^2}{M^6}$$

**2-nd way**: phenomenological saturation of spectral density by hadronic states

$$\rho_{\rm had}\left(s\right) = f_h^2\delta\left(s - m_h^2\right) + \rho_{\rm pert}\left(s\right)\theta\left(s - s_0\right)$$

Our model is ground state $h$ + continuum, which starts from threshold $s = s_0$.

# QCD SRs: General scheme

**1-st way**: Operator Product Expansion with account for **quark and gluon condensates** in QCD vacuum

$$\Phi\left(Q^2\right) = \Phi_{\rm pert}\left(Q^2\right) + c_{GG}\,\frac{\langle(\alpha_s/\pi)GG\rangle}{M^4} + c_{\bar{q}q}\,\frac{\alpha_s\langle\bar{q}q\rangle^2}{M^6}$$

**2-nd way**: phenomenological saturation of spectral density by hadronic states

$$\rho_{\rm had}\left(s\right) = f_h^2\,\delta\left(s - m_h^2\right) + \rho_{\rm pert}\left(s\right)\theta\left(s - s_0\right)$$

**Sum Rule**:

$$f_h^2\,e^{-m_h^2/M^2} = \int\limits_0^{s_0}\rho_{\rm pert}(s)\,e^{-s/M^2}ds + c_{GG}\frac{\langle\frac{\alpha_s}{\pi}GG\rangle}{M^2} + c_{\bar{q}q}\frac{\alpha_s\langle\bar{q}q\rangle^2}{M^4}$$

# *Borel Transform*

Borel transform is defined as

$$\Phi(M^2) = \hat{B}(Q^2 \to M^2)\Pi(Q^2) = \lim_{n\to\infty} \frac{(-Q^2)^n}{\Gamma(n)} \left[\frac{d^n}{dQ^{2n}}\Pi(Q^2)\right]_{Q^2=nM^2}$$

# *Borel Transform*

Borel transform is defined as

$$\Phi(M^2) = \hat{B}(Q^2 \to M^2)\Pi(Q^2) = \lim_{n\to\infty} \frac{(-Q^2)^n}{\Gamma(n)} \left[\frac{d^n}{dQ^{2n}}\Pi(Q^2)\right]_{Q^2=nM^2}$$

Here we list the most important examples:

| $\Pi(Q^2)$ | $\Rightarrow$ | $\Phi(M^2)$ |
|---|---|---|
| $C\,\log\left(\frac{Q^2}{\mu^2}\right)$ | $\Rightarrow$ | $-C$ |

# *Borel Transform*

Borel transform is defined as

$$\Phi(M^2) = \hat{B}(Q^2 \to M^2)\Pi(Q^2) = \lim_{n\to\infty} \frac{(-Q^2)^n}{\Gamma(n)} \left[\frac{d^n}{dQ^{2n}}\Pi(Q^2)\right]_{Q^2=nM^2}$$

Here we list the most important examples:

| $\Pi(Q^2)$ | $\Rightarrow$ | $\Phi(M^2)$ |
|---|---|---|
| $C \log\left(\frac{Q^2}{\mu^2}\right)$ | $\Rightarrow$ | $-C$ |
| $\frac{1}{Q^{2n}}$ | $\Rightarrow$ | $\frac{1}{\Gamma(n)\, M^{2n}}$ |

# *Borel Transform*

Borel transform is defined as

$$\Phi(M^2) = \hat{B}(Q^2 \to M^2)\Pi(Q^2) = \lim_{n\to\infty} \frac{(-Q^2)^n}{\Gamma(n)} \left[\frac{d^n}{dQ^{2n}}\Pi(Q^2)\right]_{Q^2=nM^2}$$

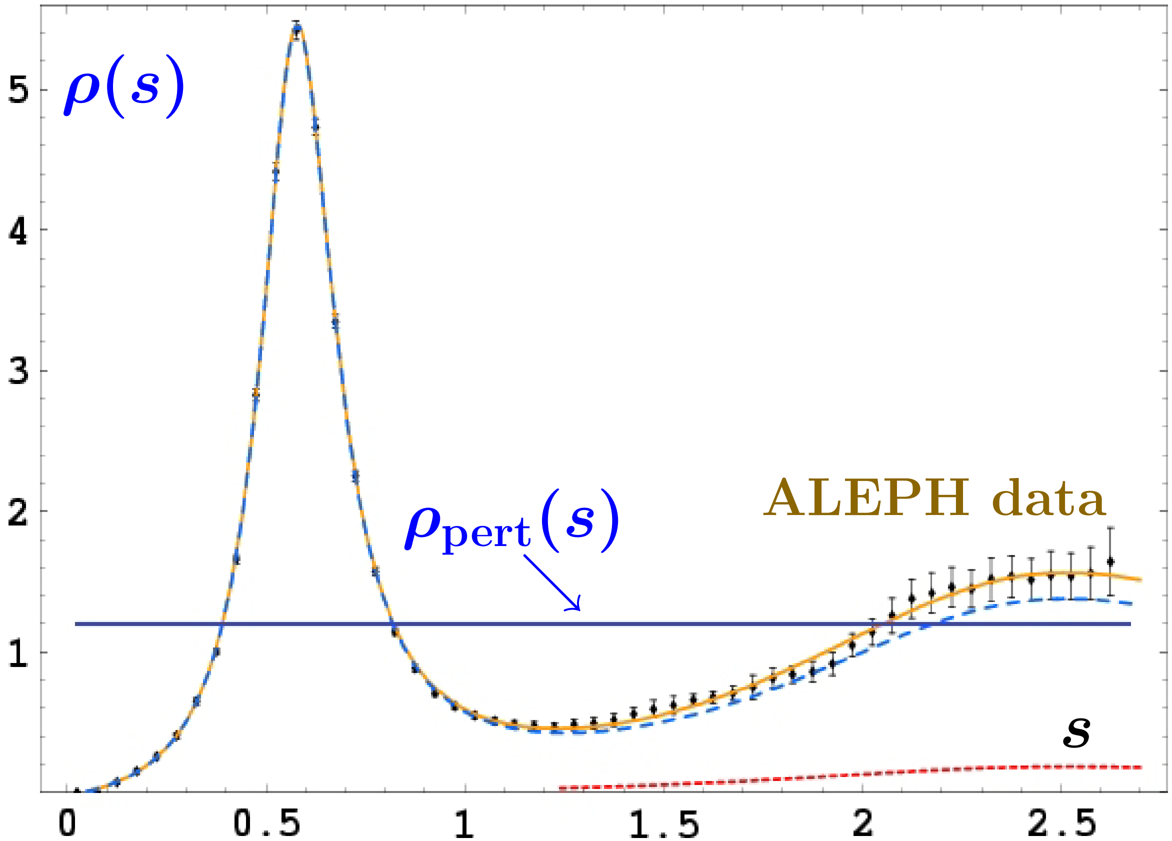
Here we list the most important examples:

| $\Pi(Q^2)$ | $\Rightarrow$ | $\Phi(M^2)$ |
|---|---|---|
| $C\,\log\left(\frac{Q^2}{\mu^2}\right)$ | $\Rightarrow$ | $-C$ |
| $\frac{1}{Q^{2n}}$ | $\Rightarrow$ | $\frac{1}{\Gamma(n)\,M^{2n}}$ |
| $\frac{1}{s+Q^2}$ | $\Rightarrow$ | $\frac{1}{M^2}\,e^{-s/M^2}$ |

# *Quark–Hadron Duality in QCD*

# *Quark-hadron Duality*

$$\int_{s_1}^{s_2} \rho_{\text{pert}}(s)ds = \int_{s_1}^{s_2} \rho_{\text{had}}(s)ds$$

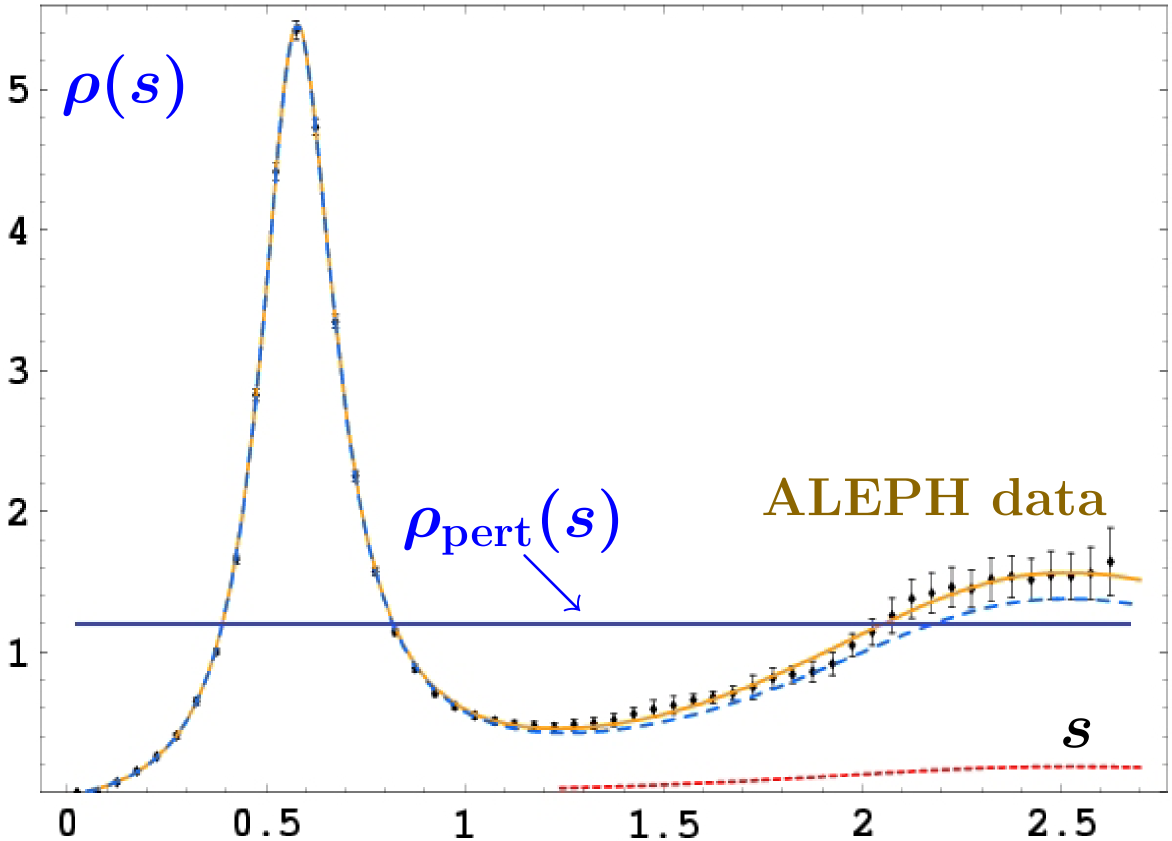



**Observations**:

**1°** Real hadron spectral density is more smooth than in HO case;

# *Quark-hadron Duality*

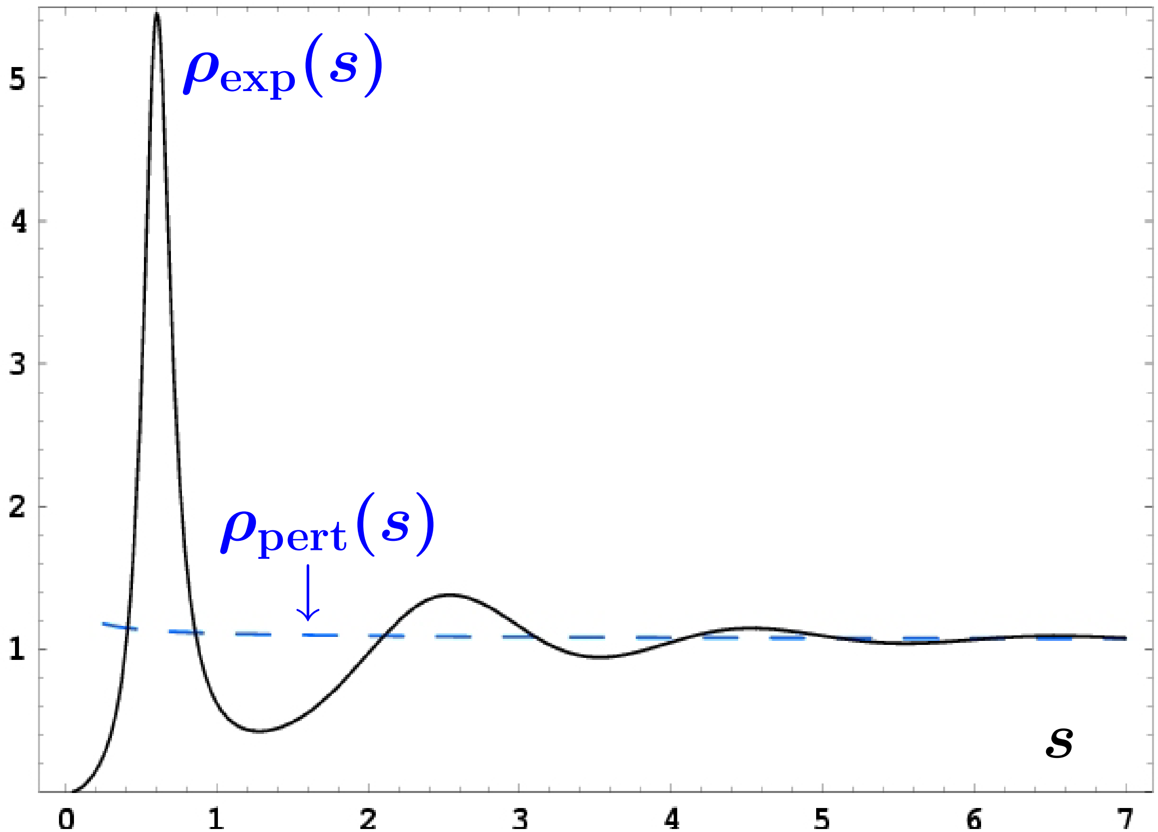
$$\int_{s_1}^{s_2} \rho_{\rm pert}(s)ds = \int_{s_1}^{s_2} \rho_{\rm had}(s)ds$$




**Observations**:
**1°** Real hadron spectral density is more smooth than in HO case;
**2°** Duality is working!

# Quark-hadron Duality

$$\int_{s_1}^{s_2} \rho_{\text{pert}}(s)ds = \int_{s_1}^{s_2} \rho_{\text{had}}(s)ds$$




**Observations**:
**1°** Real hadron spectral density is more smooth than in HO case;
**2°** Duality is working!
**3°** Asymptotics starts at $s \geq 3$ GeV$^2$

# QCD: Currents, Correlators and Spectral Densities of Real Particles

# *Currents related to $\pi$-mesons in QCD*

Currents related to $\pi^{\pm}$ meson:

$$\text{AV:} \quad J_{\mu 5}(x) = \bar{u}(x)\gamma_\mu\gamma_5 d(x)\,; \quad J^{\dagger}_{\mu 5}(x) = \bar{d}(x)\gamma_\mu\gamma_5 u(x)\,.$$

# *Currents related to $\pi$-mesons in QCD*

Currents related to $\pi^{\pm}$ meson:

$$\text{AV:} \quad J_{\mu 5}(x) = \bar{u}(x)\gamma_\mu\gamma_5 d(x)\,; \quad J^\dagger_{\mu 5}(x) = \bar{d}(x)\gamma_\mu\gamma_5 u(x)\,.$$

$$\text{PS:} \quad J_5(x) = i\,\bar{u}(x)\gamma_5 d(x)\,; \quad J^\dagger_5(x) = i\,\bar{d}(x)\gamma_5 u(x)\,.$$

# *Currents related to $\pi$-mesons in QCD*

Currents related to $\pi^{\pm}$ meson:

$$\text{AV:} \quad J_{\mu 5}(x) = \bar{u}(x)\gamma_\mu\gamma_5 d(x)\,; \qquad J^\dagger_{\mu 5}(x) = \bar{d}(x)\gamma_\mu\gamma_5 u(x)\,.$$

$$\text{PS:} \quad J_5(x) = i\,\bar{u}(x)\gamma_5 d(x)\,; \qquad J^\dagger_5(x) = i\,\bar{d}(x)\gamma_5 u(x)\,.$$

Note that Dirac equation $i\,\hat{D}\,q(x) = m_q\,q(x)$ gives relation:

$$\partial^\mu J_{\mu 5}(x) = (m_u + m_d)\ J_5(x)\,. \qquad (*)$$

# *Currents related to $\pi$-mesons in QCD*

Currents related to $\pi^{\pm}$ meson:

$$\text{AV:} \quad J_{\mu 5}(x) = \bar{u}(x)\gamma_\mu\gamma_5 d(x)\,; \quad J^{\dagger}_{\mu 5}(x) = \bar{d}(x)\gamma_\mu\gamma_5 u(x)\,.$$

$$\text{PS:} \quad J_5(x) = i\,\bar{u}(x)\gamma_5 d(x)\,; \quad J^{\dagger}_5(x) = i\,\bar{d}(x)\gamma_5 u(x)\,.$$

Note that Dirac equation $i\,\hat{D}\,q(x) = m_q\,q(x)$ gives relation:

$$\partial^\mu J_{\mu 5}(x) = (m_u + m_d)\; J_5(x)\,. \qquad (*)$$

Decay constant $f_\pi$ of physical pion $\pi(P)$ is defined via

$$\langle 0 \big| J_{\mu 5}(0) \big| \pi(P)\rangle = i\, f_\pi\, P_\mu\,.$$

It was measured in decay $\pi \to \mu\nu_\mu$ to be $f_\pi = 132$ MeV.

# *Currents related to $\pi$-mesons in QCD*

Currents related to $\pi^{\pm}$ meson:

$$\text{AV:} \quad J_{\mu 5}(x) = \bar{u}(x)\gamma_\mu\gamma_5 d(x)\,; \qquad J^\dagger_{\mu 5}(x) = \bar{d}(x)\gamma_\mu\gamma_5 u(x)\,.$$

$$\text{PS:} \quad J_5(x) = i\,\bar{u}(x)\gamma_5 d(x)\,; \qquad J^\dagger_5(x) = i\,\bar{d}(x)\gamma_5 u(x)\,.$$

Note that Dirac equation $i\,\hat{D}\,q(x) \;=\; m_q\,q(x)$ gives relation:

$$\partial^\mu J_{\mu 5}(x) = (m_u + m_d)\; J_5(x)\,. \qquad (*)$$

Decay constant $f_\pi$ of physical pion $\pi(P)$ is defined via

$$\langle 0\big|J_{\mu 5}(0)\big|\pi(P)\rangle = i\,f_\pi\,P_\mu\,.$$

Eq. $(*)$ then gives $\langle 0\big|J_5(0)\big|\pi(P)\rangle = \dfrac{f_\pi\,m_\pi^2}{m_u + m_d}$.

# *Currents related to vector mesons in QCD*

Currents related to $\rho^{\pm}$ meson:

$$J_\mu(x) = \bar{u}(x)\gamma_\mu d(x)\,; \quad J^{\dagger}{}_\mu(x) = \bar{d}(x)\gamma_\mu u(x)\,.$$

Decay constant $f_\rho$ of physical $\rho^{\pm}(P,\varepsilon)$-meson with polarization $\varepsilon$ and momentum $P$, satisfying $(P\,\varepsilon) = 0$ and $(\varepsilon,\varepsilon) = -1$,

$$\langle 0 \big| J_\mu(0) \big| \rho(P,\varepsilon)\rangle = f_\rho\, m_\rho\, \varepsilon_\mu\,.$$

Decay $\rho^0 \to e^+e^-$ allows to measure $f_{\rho^0} = 150$ MeV, that gives $f_{\rho^{\pm}} = 210$ MeV.

# *Vector current correlator* $\Pi_{\mu\nu}$

Lorentz invariance and vector current conservation dictate

$$\Pi_{\mu\nu}(q) = i\int d^4x\, e^{iqx} \langle 0 | T\, [J^\mu(x) J_\nu(0)]\, | 0 \rangle = \left[ q_\mu\, q_\nu - g_{\mu\nu}\, q^2 \right]\, \Pi(q)\,.$$

# *Vector current correlator* $\Pi_{\mu\nu}$

Lorentz invariance and vector current conservation dictate

Inserting $\hat{1}$ in between currents we obtain

$$\Pi(q) = \frac{-i}{3q^2} \sum_{X(p)} \int_0^{\infty} dt\, e^{iq_0 t} \int d^3\vec{x}\, e^{-i\vec{q}\vec{x}} \langle 0 \big| J^\mu(x) \big| X(p) \rangle \langle X(p) \big| J^\dagger_\mu(0) \big| 0 \rangle$$

$$+ \frac{-i}{3q^2} \sum_{X(p)} \int_{-\infty}^{0} dt\, e^{iq_0 t} \int d^3\vec{x}\, e^{-i\vec{q}\vec{x}} \langle 0 \big| J^\dagger_\mu(0) \big| X(p) \rangle \langle X(p) \big| J^\mu(x) \big| 0 \rangle$$

# Vector current correlator $\Pi_{\mu\nu}$

Lorentz invariance and vector current conservation dictate

Inserting $\hat{1}$ in between currents we obtain

$$\Pi(q) = \frac{-i}{3q^2} \sum_{X(p)} \int_0^\infty dt\, e^{iq_0 t} \int d^3\vec{x}\, e^{-i\vec{q}\vec{x}} \langle 0|J^\mu(x)|X(p)\rangle\langle X(p)|J^\dagger_\mu(0)|0\rangle$$

$$+ \frac{-i}{3q^2} \sum_{X(p)} \int_{-\infty}^0 dt\, e^{iq_0 t} \int d^3\vec{x}\, e^{-i\vec{q}\vec{x}} \langle 0|J^\dagger_\mu(0)|X(p)\rangle\langle X(p)|J^\mu(x)|0\rangle$$

$$= \frac{-i\,(2\pi)^3}{3q^2} \sum_{X(p)} \delta(\vec{p}-\vec{q})\,\theta(p_0)\left|\langle 0|J_\mu(0)|X(p)\rangle\right|^2$$

$$\times \int_0^\infty dt\,\left[e^{i(q_0-p_0)t} + e^{-i(q_0+p_0)t}\right]$$

# *Vector current correlator* $\Pi_{\mu\nu}$

Then $\Pi(q^2) = \frac{-i\,(2\pi)^3}{3q^2} \sum_{X(p)} \delta(\vec{p}-\vec{q}) \left|\langle 0 \middle| J_\mu(0) \middle| X(p)\rangle\right|^2 \times$

$$\times \int_0^\infty dt \left[ e^{i(q_0-p_0)t} + e^{-i(q_0+p_0)t} \right].$$

We have the following identities

$$\int_0^\infty dt\, e^{\pm i\alpha t} = \pi\,\delta(\alpha) \pm i\,\mathcal{P}\frac{1}{\alpha}.$$

# *Vector current correlator* $\Pi_{\mu\nu}$

Then $\Pi(q^2) = \dfrac{-i\,(2\pi)^3}{3q^2} \sum\limits_{X(p)} \delta(\vec{p} - \vec{q}) \left|\langle 0 | J_\mu(0) | X(p)\rangle\right|^2 \times$

$$\times \int_0^\infty dt\, \left[e^{i(q_0 - p_0)t} + e^{-i(q_0 + p_0)t}\right].$$

We have the following identities

$$\int_0^\infty dt\, e^{\pm i\alpha t} = \pi\,\delta(\alpha) \pm i\,\mathcal{P}\frac{1}{\alpha}\,.$$

After all substitutions:

$$\mathbf{Im}\,\Pi(q^2) = -\pi\,\frac{(2\pi)^3}{3q^2} \sum_{X(p)} \delta(\vec{p} - \vec{q})\,\delta(p_0 - |q_0|)\left|\langle 0 | J_\mu(0) | X(p)\rangle\right|^2$$

# *Vector current correlator* $\Pi_{\mu\nu}$

So, we have $\boxed{\frac{1}{\pi}\,\mathbf{Im}\,\Pi(q^2) = \rho(q^2)\theta(|q_0|) = \rho(q^2)}$, with

$$\rho(q^2)\,\theta(q_0) = \frac{-(2\pi)^3}{3q^2}\sum_{X(p)} \delta^{(4)}(q-p)\,\theta(p_0)\left|\langle 0|\,J_\mu(0)\,|X(p)\rangle\right|^2 .$$

# *Vector current correlator* $\Pi_{\mu\nu}$

So, we have $\boxed{\frac{1}{\pi}\,\mathbf{Im}\,\Pi(q^2) = \rho(q^2)\theta(|q_0|) = \rho(q^2)}$, with

$$\rho(q^2)\,\theta(q_0) = \frac{-(2\pi)^3}{3q^2}\sum_{X(p)}\delta^{(4)}(q-p)\,\theta(p_0)\Big|\langle 0|J_\mu(0)|X(p)\rangle\Big|^2 .$$

Lorentz invariance dictates

$$\langle 0|J^\mu(x)|X(p)\rangle = [A\,p_\mu + B\,\varepsilon_\mu]\;e^{-ipx}$$

with $p\cdot\varepsilon = 0$, and therefore $\varepsilon\cdot\varepsilon = -1$. From current conservation it follows $A = 0$, i. e. ($B = f_X\,m_X$)

$$\langle 0|J^\mu(x)|X(p)\rangle\langle X(p)|J^\dagger_\mu(x)|0\rangle = \big|f_X\big|^2\,m_X^2\varepsilon^2 = -\big|f_X\big|^2\,m_X^2 \le 0\,.$$

# *Vector current correlator* $\Pi_{\mu\nu}$

So, we have $\boxed{\frac{1}{\pi}\,\mathbf{Im}\,\Pi(q^2) = \rho(q^2)\theta(|q_0|) = \rho(q^2)}$, with

$$\rho(q^2)\,\theta(q_0) = \frac{-(2\pi)^3}{3q^2}\sum_{X(p)} \delta^{(4)}(q-p)\,\theta(p_0)\Big|\langle 0|J_\mu(0)|X(p)\rangle\Big|^2 .$$

Lorentz invariance and current conservation dictate

$$\langle 0|J^\mu(x)|X(p)\rangle\langle X(p)|J^\mu(x)|0\rangle = -|f_X|^2\, m_X^2 \le 0\,,$$

that gives us

$$\boxed{\rho(q^2) = \frac{-(2\pi)^3}{3q^2}\sum_{X(p)} \delta^{(4)}(q-p)\,\theta(p_0)\Big|\langle 0|J_\mu(0)|X(p)\rangle\Big|^2 \ge 0}$$

# *Vector current correlator* $\Pi_{\mu\nu}$

So, we have $\boxed{\frac{1}{\pi}\,\mathbf{Im}\,\Pi(q^2) = \rho(q^2)\theta(|q_0|) = \rho(q^2)}$, with

$$\rho(q^2)\,\theta(q_0) = \frac{-(2\pi)^3}{3q^2}\sum_{X(p)}\delta^{(4)}(q-p)\,\theta(p_0)\left|\langle 0|J_\mu(0)|X(p)\rangle\right|^2 .$$

Lorentz invariance and current conservation dictate

$$\langle 0|J^\mu(x)|X(p)\rangle\langle X(p)|J^\mu(x)|0\rangle = -\left|f_X\right|^2 m_X^2 \leq 0\,,$$

that gives us

$$\boxed{\rho(s) = \sum_X \left|f_X\right|^2 \delta(s - m_X^2) \geq 0}$$

# *Spectral density of correlators* $\Pi_{\mu\nu}$ *and* $\Pi^+_{\mu\nu}$

So, we have

$$\frac{1}{\pi}\,\mathbf{Im}\,\Pi(q^2) = \rho(q^2)\theta(|q_0|) = \rho(q^2)$$

If we consider correlator

$$\Pi^+_{\mu\nu}(q) = i\int d^4x\, e^{iqx}\langle 0|J^\mu(x)J_\nu(0)|0\rangle = \left[q_\mu\, q_\nu - g_{\mu\nu}\, q^2\right]\,\Pi^+(q)\,.$$

then

$$\frac{1}{\pi}\,\mathbf{Im}\,\Pi^+(q^2) = \rho(q^2)\theta(q_0)$$

# *Spectral density of correlators* $\Pi_{\mu\nu}$ *and* $\Pi^{+}_{\mu\nu}$

So, we have $$\frac{1}{\pi}\,\mathbf{Im}\,\Pi(q^2) = \rho(q^2)\theta(|q_0|) = \rho(q^2)$$

If we consider correlator

$$\Pi^{+}_{\mu\nu}(q) = i \int d^4x\, e^{iqx} \langle 0 \big| J^{\mu}(x) J_{\nu}(0) \big| 0 \rangle = \left[q_{\mu}\, q_{\nu} - g_{\mu\nu}\, q^2\right]\, \Pi^{+}(q)\,.$$

then $$\frac{1}{\pi}\,\mathbf{Im}\,\Pi^{+}(q^2) = \rho(q^2)\theta(q_0)$$

Now we can say why we put $T$-product in correlators
– then spectral densities, defined only by **real particles**,
are **Lorentz invariant** and **depend only on $q^2$**!