Chaotic wave maps coupled to gravity

Chaos in General Relativity

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References

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Introduction

- Chaos in dynamical systems

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- Chaos in the context of General Relativity

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  - The chaotic geodesics

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  - The diffeomorphism invariance, e.g., – the Lyapunov exponents do not satisfy general covariance ($t \to \ln \tau$)

$$\epsilon(t) = \epsilon_0 e^{\lambda t} = \epsilon_0 \tau^{\lambda}$$

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  - The chaotic geodesics
  - Metric chaos
  - The diffeomorphism invariance, e.g., – the Lyapunov exponents do not satisfy general covariance ($t \rightarrow \ln \tau$)

$$\epsilon(t) = \epsilon_0 e^{\lambda t} = \epsilon_0 \tau^{\lambda}$$

  - The geometric method - fractals

Chaotic wave maps coupled to gravity

- The action

$$S(X) = \frac{1}{G}\int_M \frac{R}{16\pi} - \frac{f_\pi^2}{2}\int_M (\partial X)^2$$

  $X : M \rightarrow N$, $\alpha = 4\pi {f_\pi}^2 G$ (dimensionless)
  $[(\partial X)^2] = [R] = L^{-2}$

- $N = S^3$
- Einstein's equations "$\partial g \sim \alpha(\dots)$"
- The assumptions: spherical symmetry, self-similarity, equivariance

$$g_{ab} = e^{-2\tau}\hat{g}_{ab}(\rho)$$

Chaotic wave maps coupled to gravity

- The metric (spherical symmetry and self-similarity)

$$g_{ab} = \exp(-2\tau) \begin{pmatrix} \tilde{A} & \tilde{B} & 0 & 0 \\ \tilde{B} & \tilde{C} & 0 & 0 \\ 0 & 0 & \tilde{F}^2 & 0 \\ 0 & 0 & 0 & \tilde{F}^2 \sin^2\theta \end{pmatrix},$$

where

$$\begin{aligned} \tilde{A} &= \frac{\rho^2}{A}\left(1 - W^2\right), \\ \tilde{B} &= -\frac{\rho}{A}, \\ \tilde{C} &= \frac{1}{A}, \\ \tilde{F} &= \rho \end{aligned}$$

- $\frac{dx}{d\rho} = \frac{W(\rho)}{\rho}$, $D = \frac{dF(\rho)}{d\rho}$, $A = A(W, F, D)$

Chaotic wave maps coupled to gravity

- The system of ordinary differential equations

$$\begin{aligned} W' &= -1 + \alpha(1 - W^2)D^2, \\ D' &= 2\alpha W D^3 + \frac{\sin(2F)}{-1+2\alpha\sin(F)^2}\left(\alpha D^2 + \frac{1}{1-W^2}\right), \\ F' &= D \end{aligned}$$

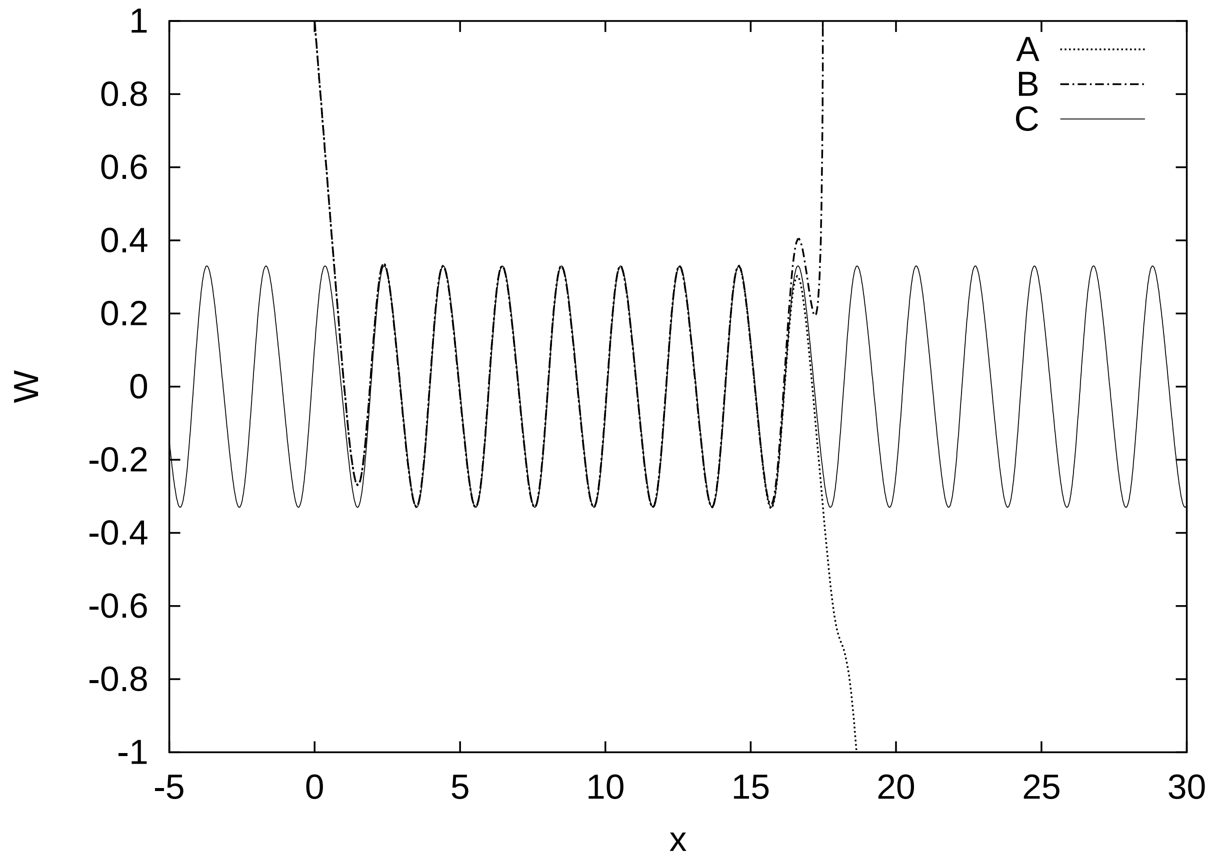
- The regularity conditions (c is discrete)

$$W(0) = 1, \quad D(0) = c, \quad F(0) = \frac{\pi}{2}$$

Chaotic wave maps coupled to gravity

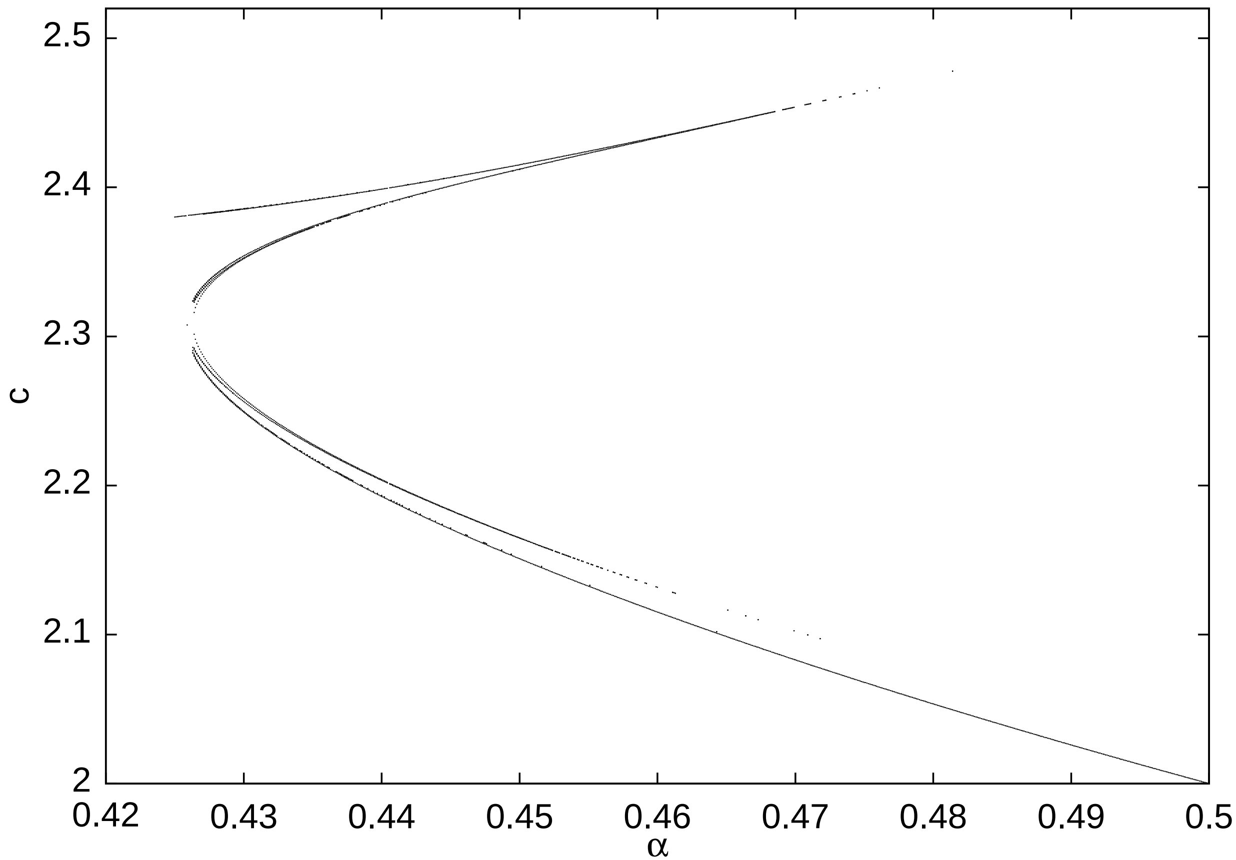
- Two families
  - type A - a naked singularity
  - type B - an apparent horizon
- At the threshold between A and B, the critical solution - type C

Type C solutions (weak coupling)



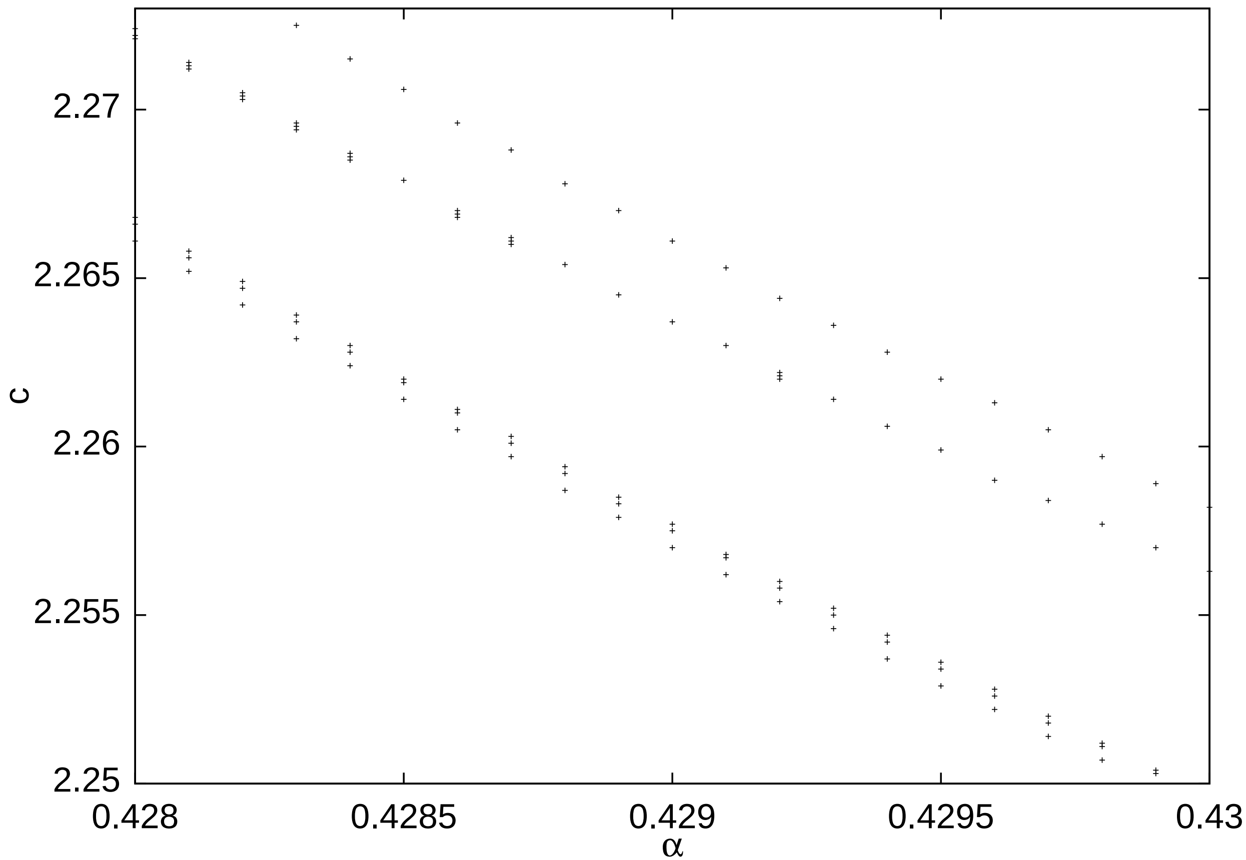
$\alpha = 0.38$; type A ($c = 2.36134$); type B ($c = 2.36135$).

The bifurcation diagram



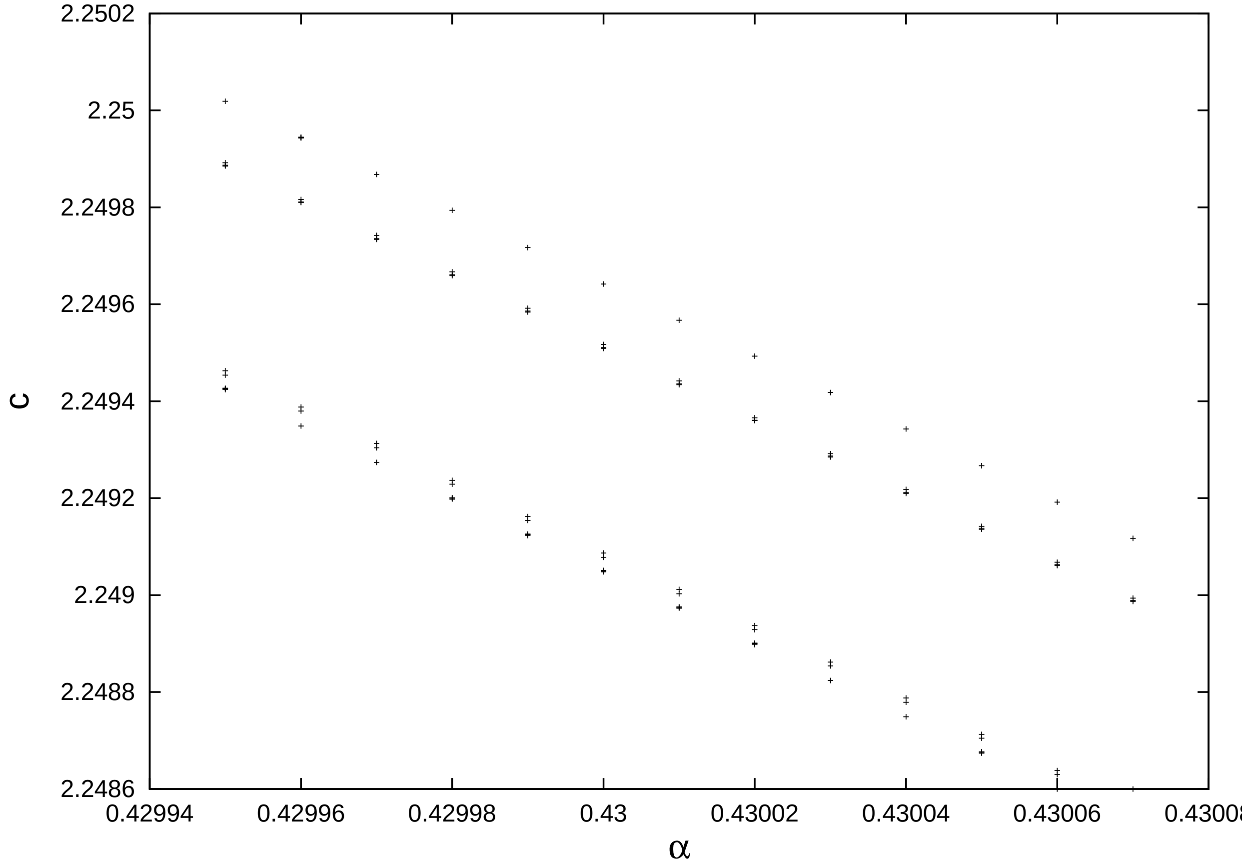
The lattice $\Delta\alpha = 10^{-4}$ i $\Delta c = 2 \cdot 10^{-4}$.

The bifurcation diagram



The bifurcation diagram enlarged 1000 times, the lattice $\Delta\alpha = 10^{-4}$ i $\Delta c = 2 \cdot 10^{-4}$.

The bifurcation diagram

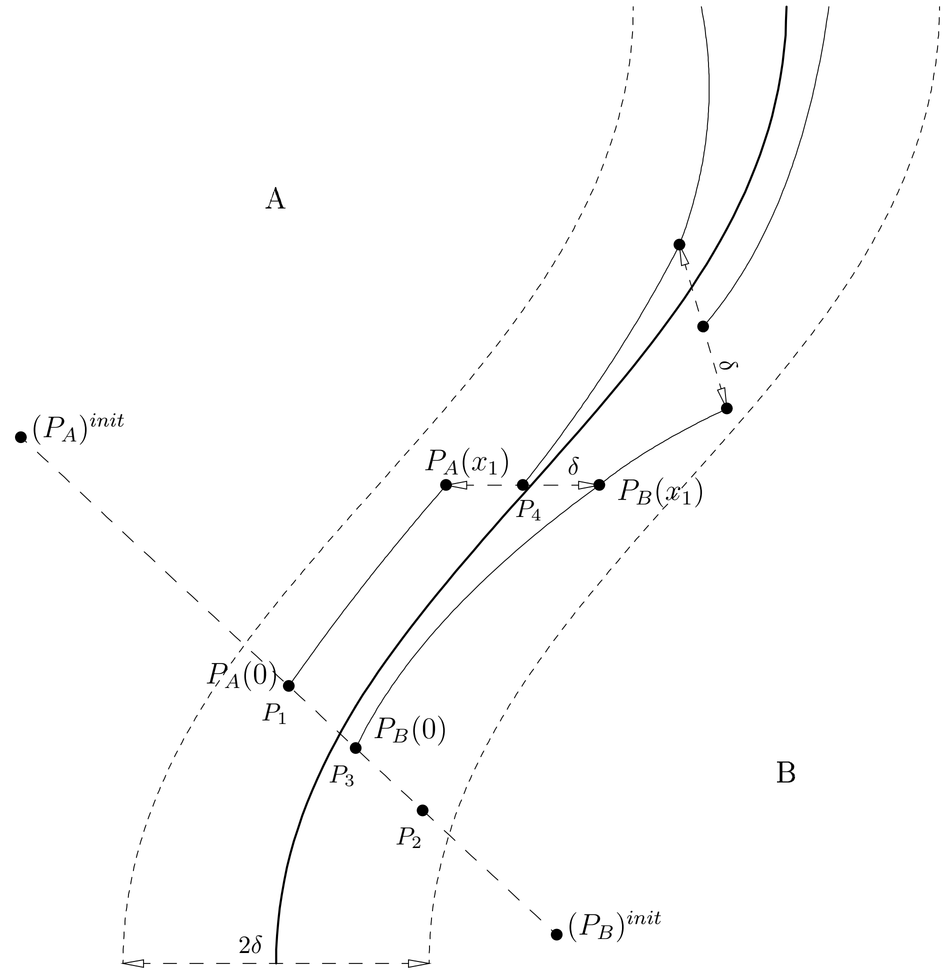


The bifurcation diagram enlarged 200000 times, the lattice $\Delta\alpha = 10^{-5}$ i $\Delta c = 10^{-6}$.

Type C solutions

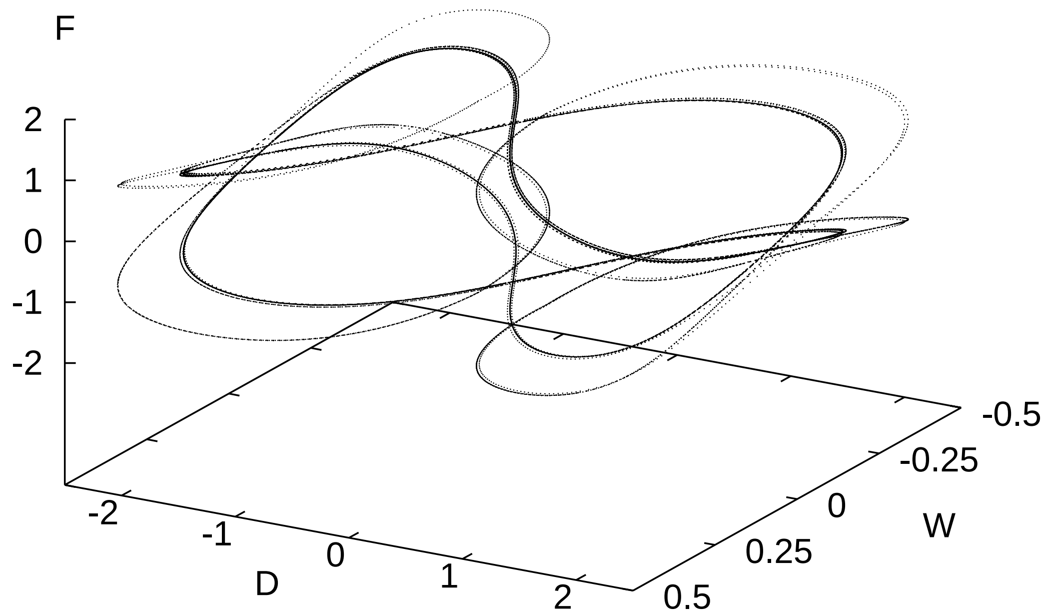
- The methods of construction
  - Straddle trajectories
  - Poincar'e–Lindstedt series (only weak coupling)

Straddle trajectories



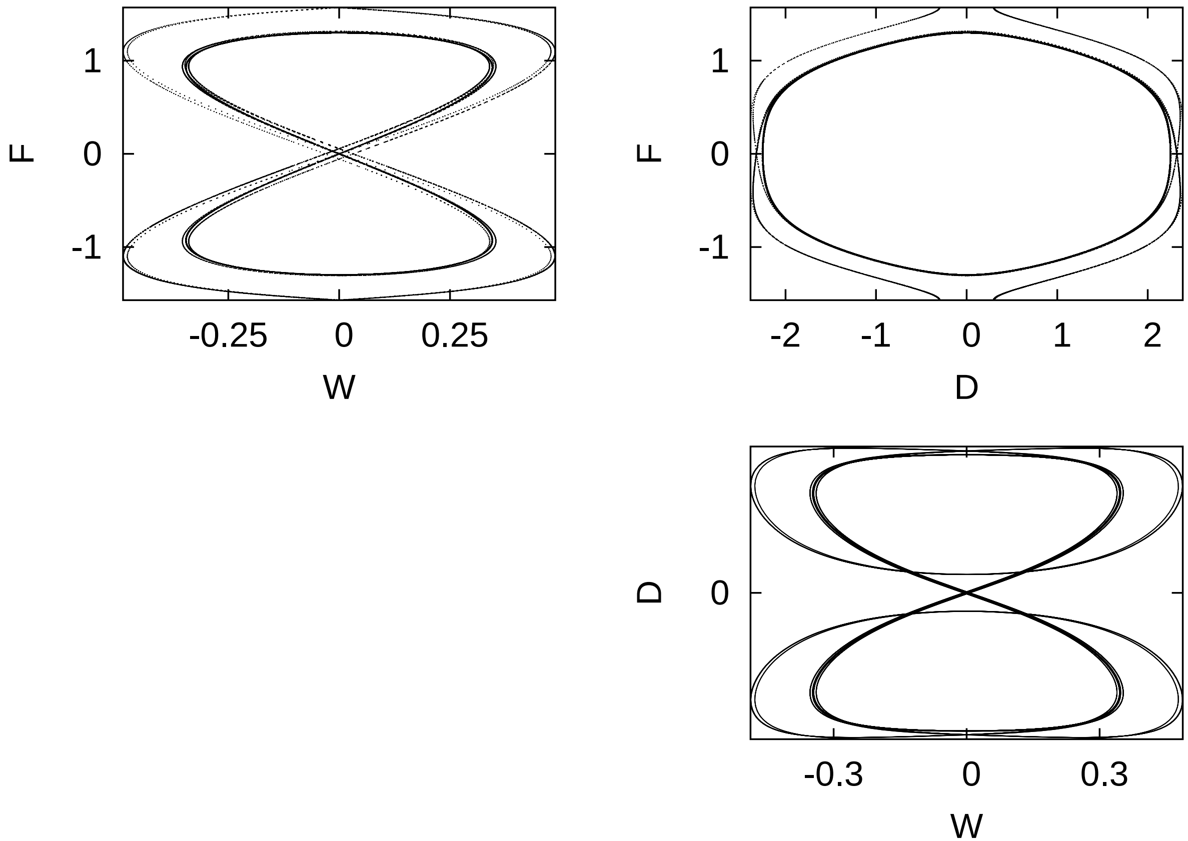
The straddle orbit method

Type C solutions (strong coupling)



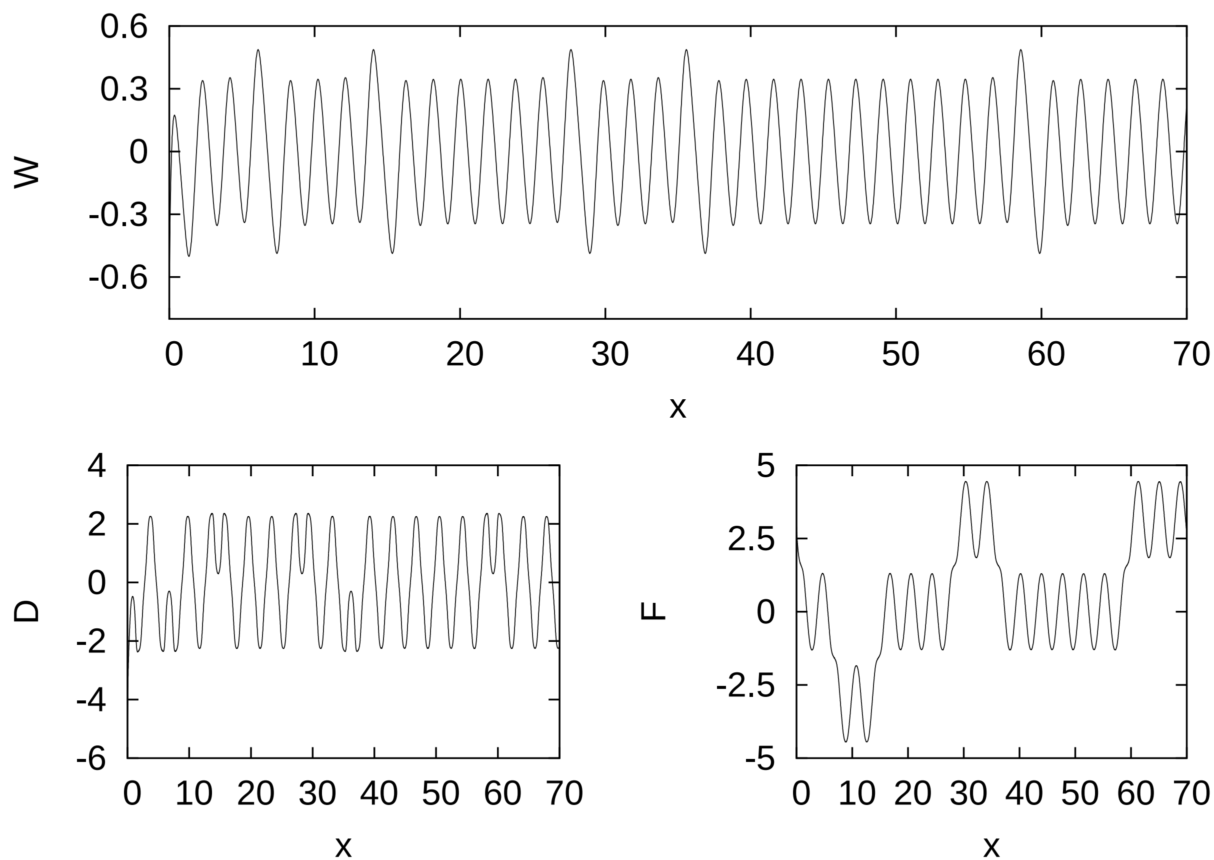
The C attractor $\alpha = 0.43$.

Type C solutions (strong coupling)



The projection of the C attractor $\alpha = 0.43$.

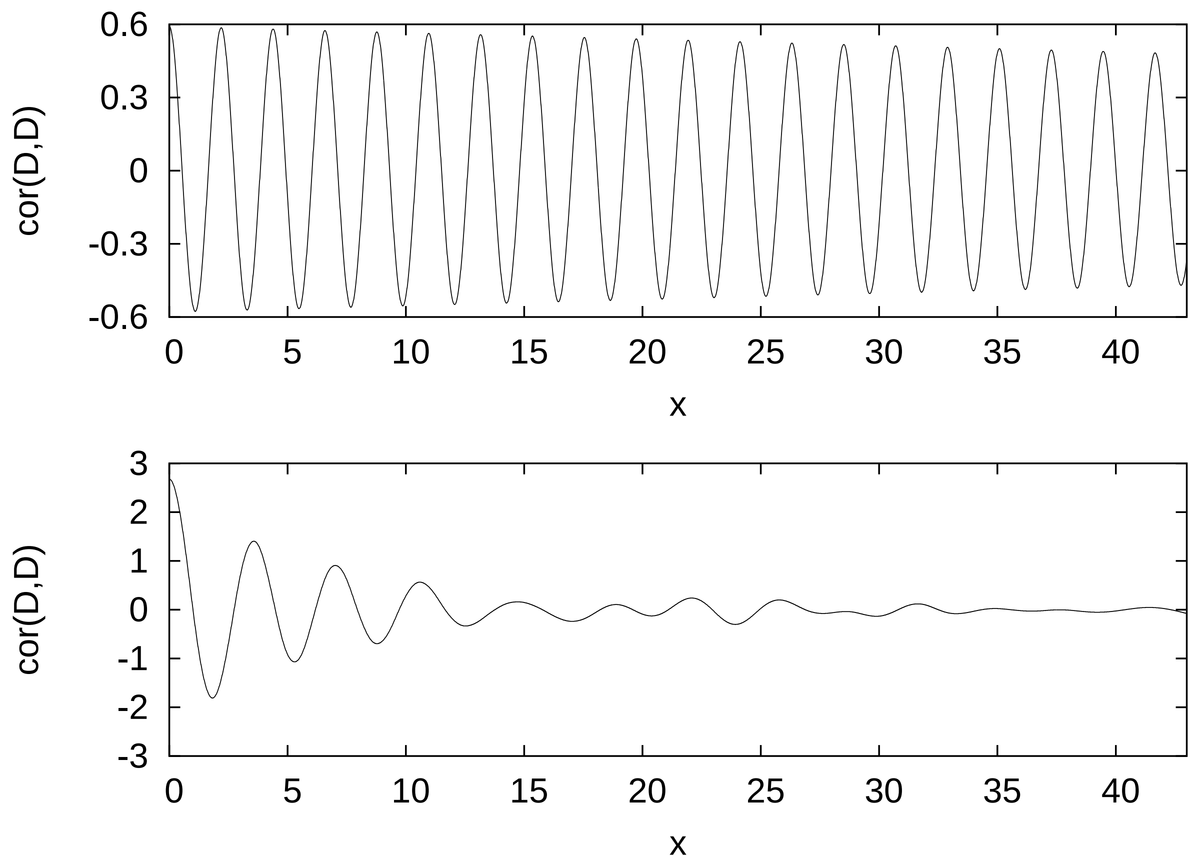
Type C solutions (strong coupling)



The C attractor $\alpha = 0.43$.

The autocorrelation function

- $cor(D,D)(x) = \int_{-\infty}^{\infty} D(x+\xi)D(\xi)d\xi$



The autocorrelation function $\alpha = 0.42$.

Fractal dimension

- The capacity dimension d

$$d = -\lim_{\epsilon \to 0^+} \frac{\ln N(\epsilon)}{\ln \epsilon}$$

  - Relation to uncertainty of the phase space f

$$d = D - a,$$

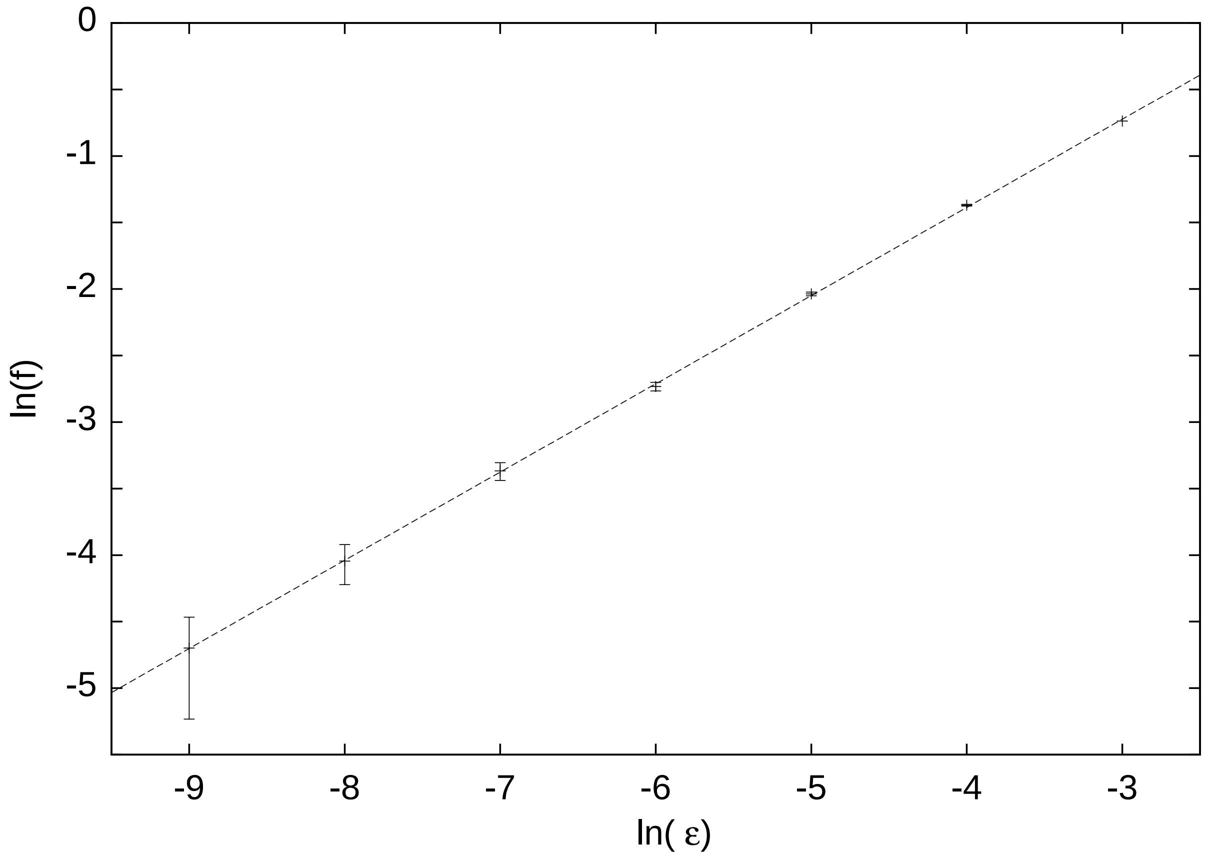
$$f(\epsilon) \sim \epsilon^a$$

  in the limit

$$\lim_{\epsilon \to 0} \frac{\ln f(\epsilon)}{\ln \epsilon} = a,$$

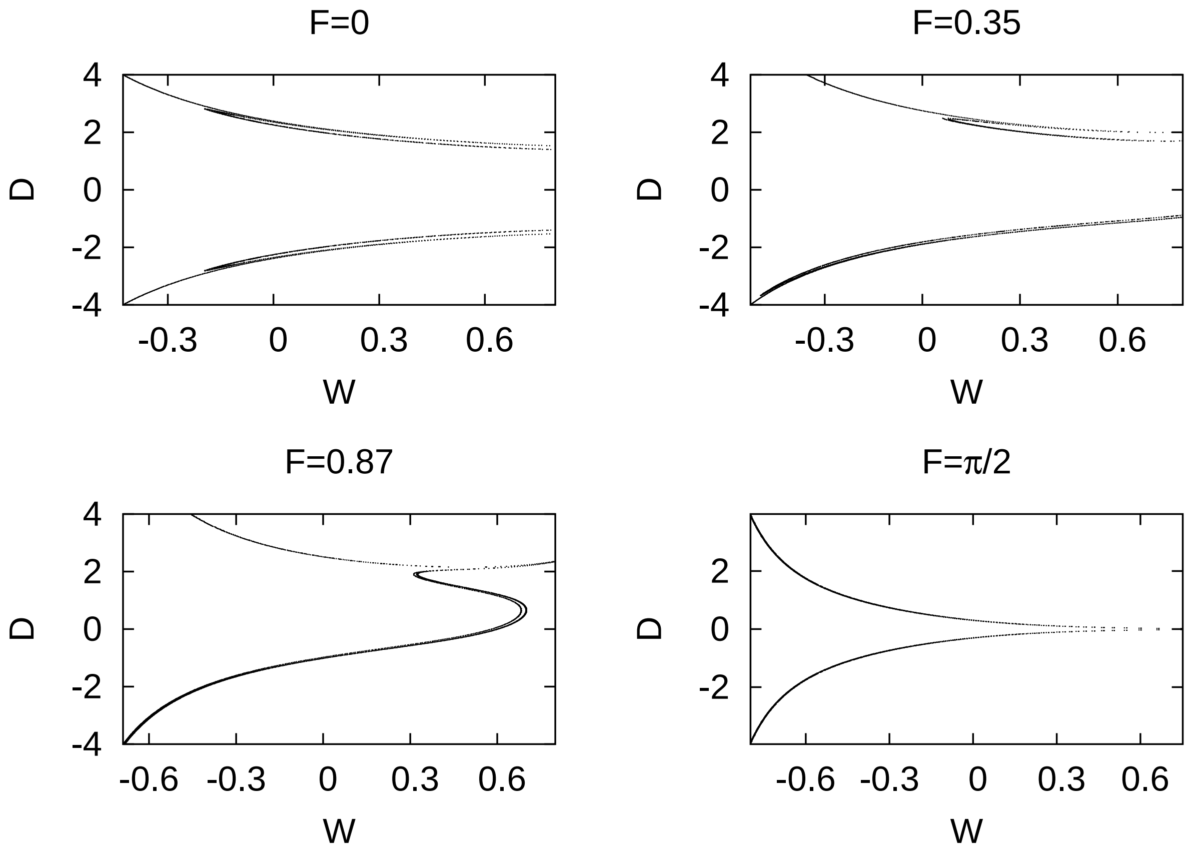
  where D is a dimension of the phase space
- Independence on discretization

Fractal dimension



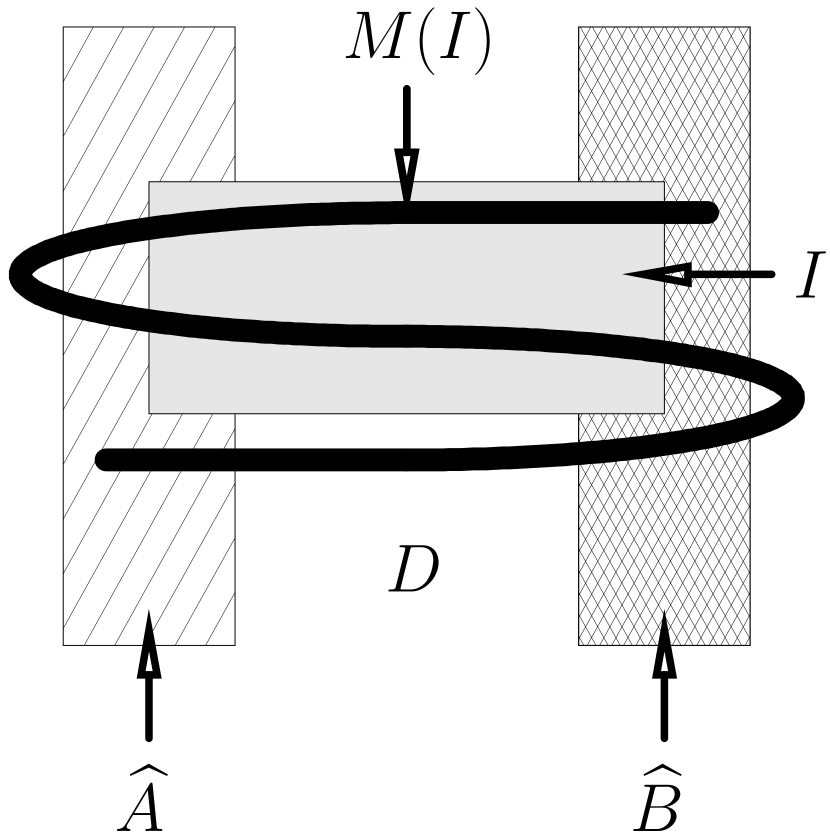
The slope $a = 0.663 \pm 0.003$ (for $\alpha = 0.4264$) implies $d = 0.337 \pm 0.003$.

Horseshoe dynamics



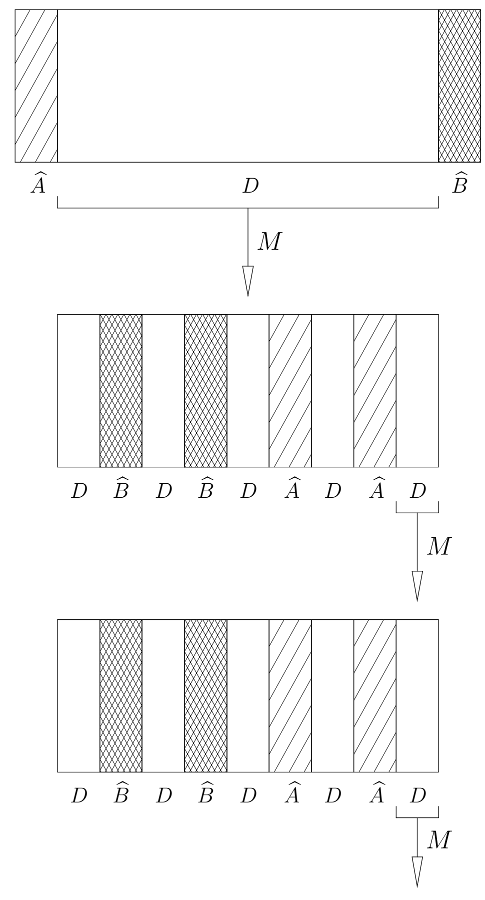
The basin boundary for $\alpha = 0.43$.

Horseshoe dynamics



The fractal basin boundary and horseshoe

Horseshoe dynamics



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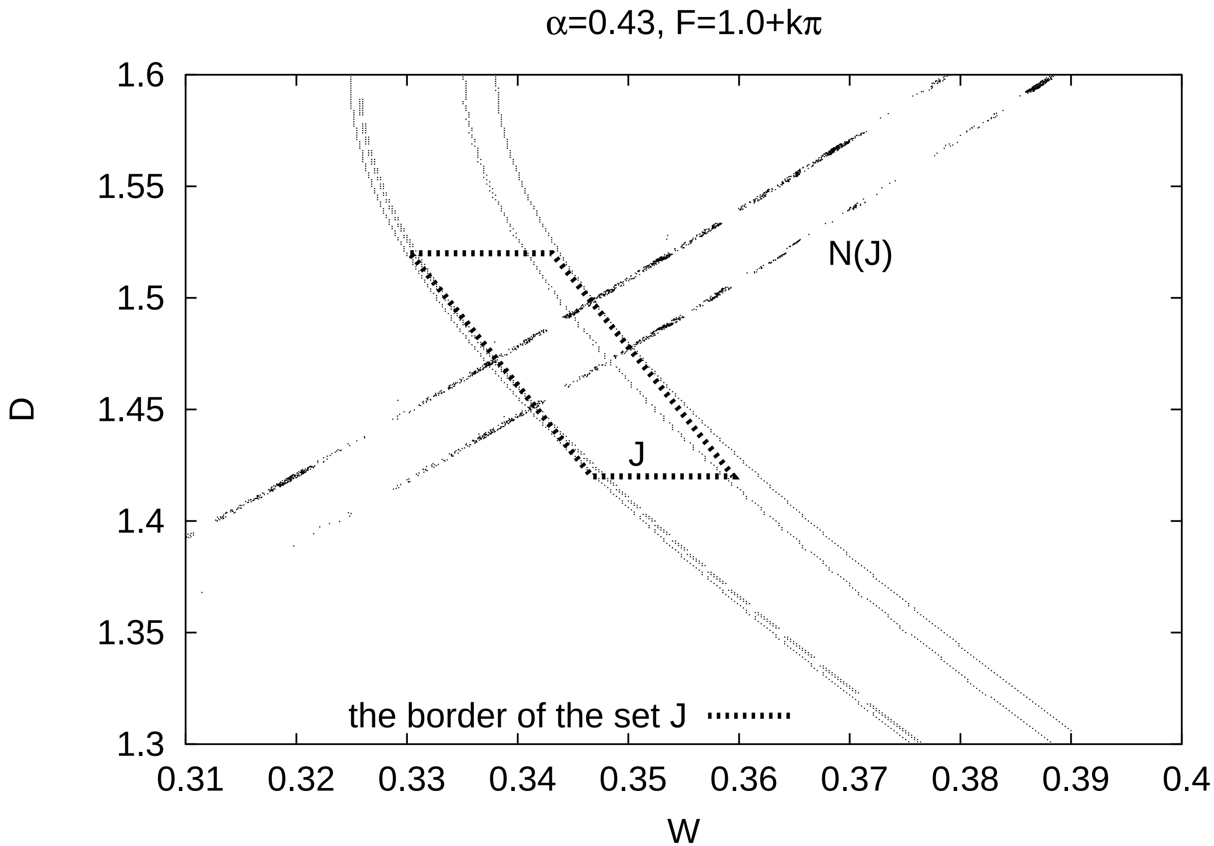
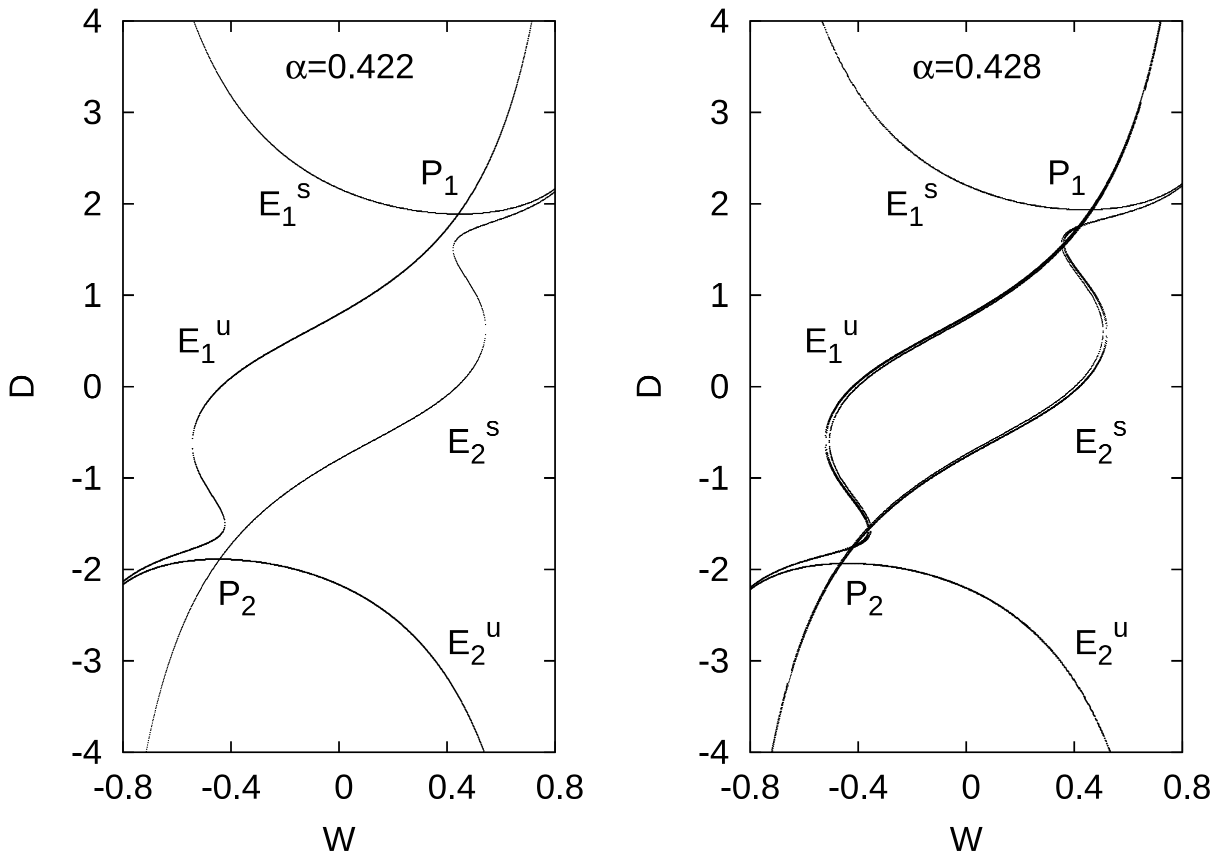


Horseshoe dynamics of wave maps coupled to gravity

Heteroclinic intersection



The transversal intersection of the stable and unstable manifold

Causal structure

- The Carter–Penrose's diagrams
  - The numerical solution - a problem
  - The C attractor - Poincaré–Lindstedt series (weak coupling)
  - Kinematics of spherically symmetric self-similar space-times
    - The decomposition $g_{ab} = e^{-2\tau} \hat{g}_{ab}$
    - The kinematical and dynamical parts $K(\tau, \rho) = e^{4\tau} \hat{K}(\rho)$

Penrose's diagrams

- The lowest order of the series

$$W(x) = \alpha^{\frac{3}{2}} \sin \frac{2x}{\sqrt{\alpha}} + 0(\alpha^2),$$

$$A(x) = \frac{1}{2}(1-\alpha) + O(\alpha^2)$$

- The metric

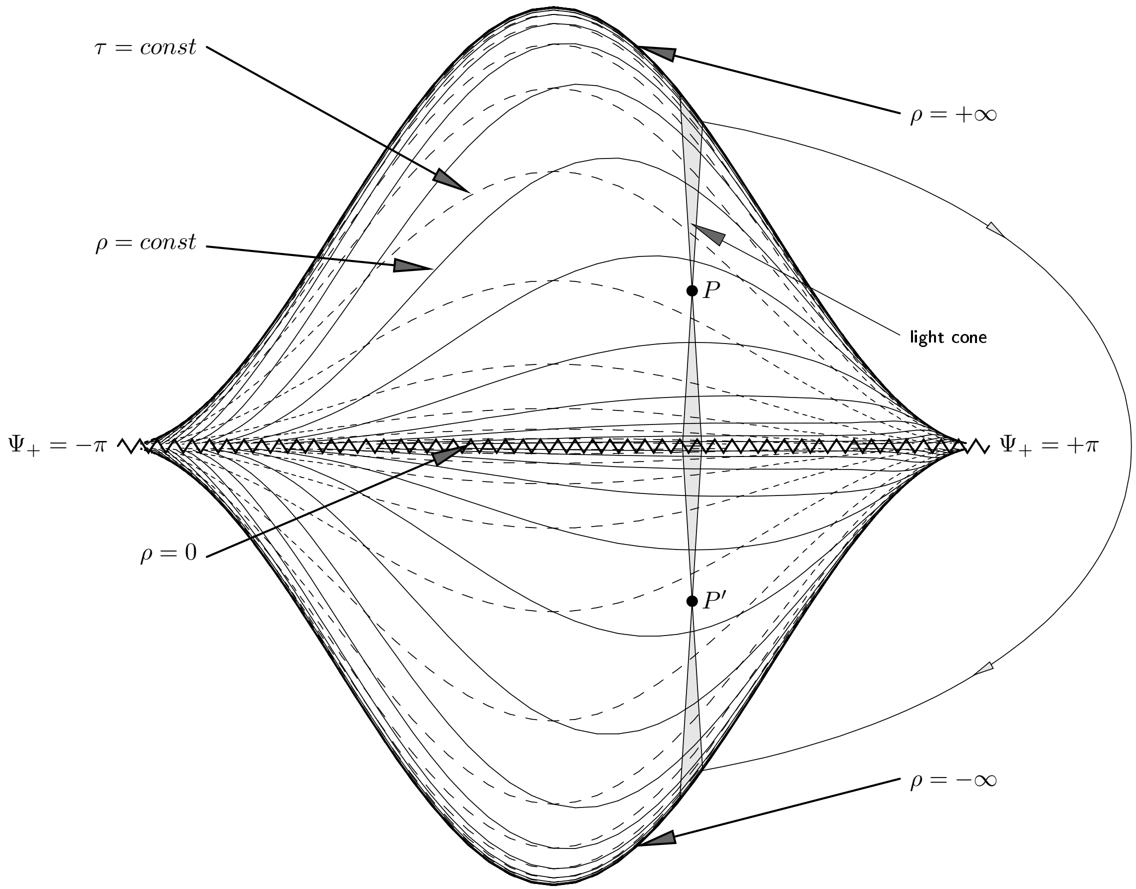
$$gabdx^a dx^b = \frac{2e^{-2\tau}}{1-\alpha}\left(\left(1 - 4\frac{\alpha^3|\rho|^{4\alpha+2}}{(1+|\rho|^{4\alpha})^2}\right)d\tau^2 - 2\rho d\tau d\rho + d\rho^2 + \frac{1-\alpha}{2}\rho^2 d\Omega^2\right)$$

Penrose's diagrams

- Double null coordinates

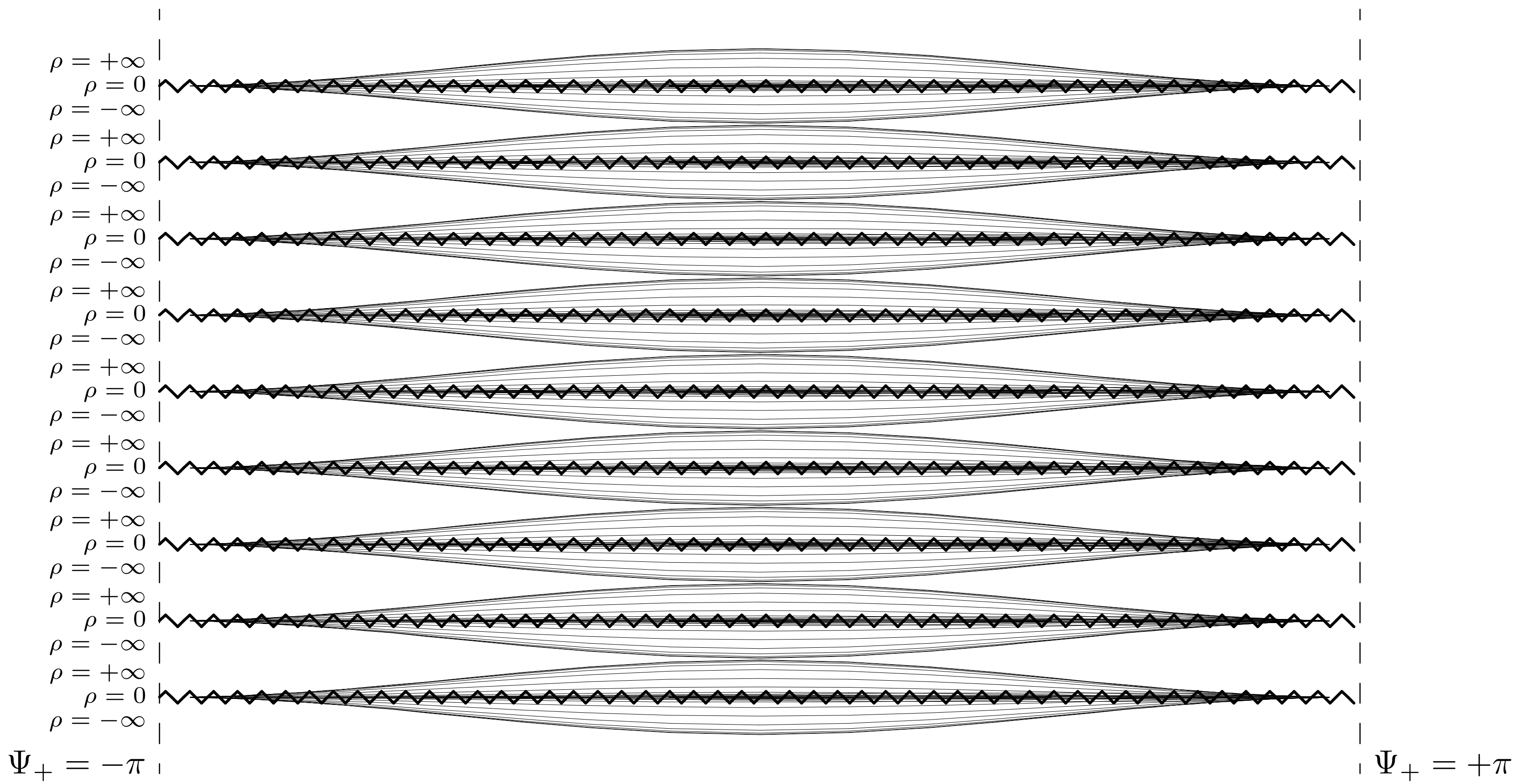
$$\begin{aligned}\Psi_\pm(\tau,\rho) &= \arctan\left(\frac{2}{\sqrt{1-\alpha^3}}\alpha^{\frac{3}{2}}\arctan\left(\frac{-sgn(\rho)|\rho|^{2\alpha}-\alpha^{\frac{3}{2}}}{\sqrt{1-\alpha^3}}\right)+2\alpha(\ln|\rho|-\tau)\right)\\ &\pm \arctan\left(\frac{2}{\sqrt{1-\alpha^3}}\alpha^{\frac{3}{2}}\arctan\left(\frac{+sgn(\rho)|\rho|^{2\alpha}-\alpha^{\frac{3}{2}}}{\sqrt{1-\alpha^3}}\right)+2\alpha(\ln|\rho|-\tau)\right)\end{aligned}$$

Penrose's diagrams



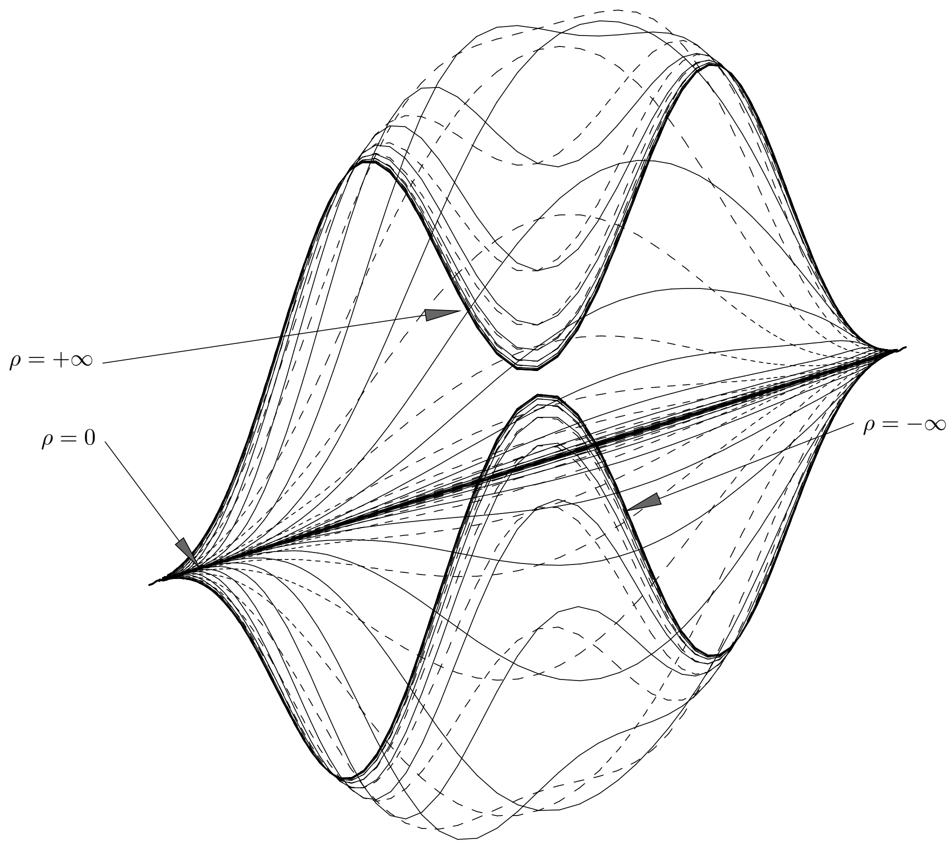
The part of the C attractor

Penrose's diagrams



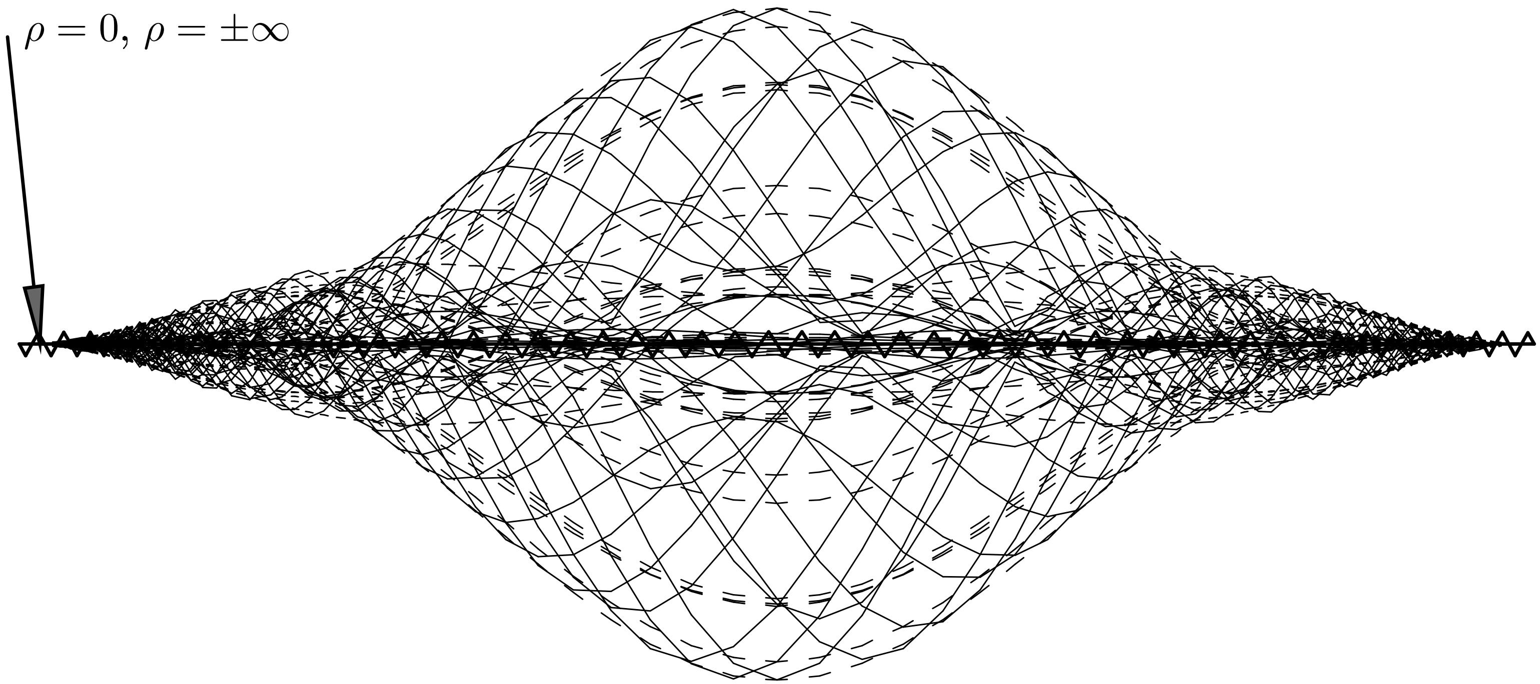
The whole C attractor

Penrose's diagrams



The part of the C attractor embeded in additional dimension

Penrose's diagrams



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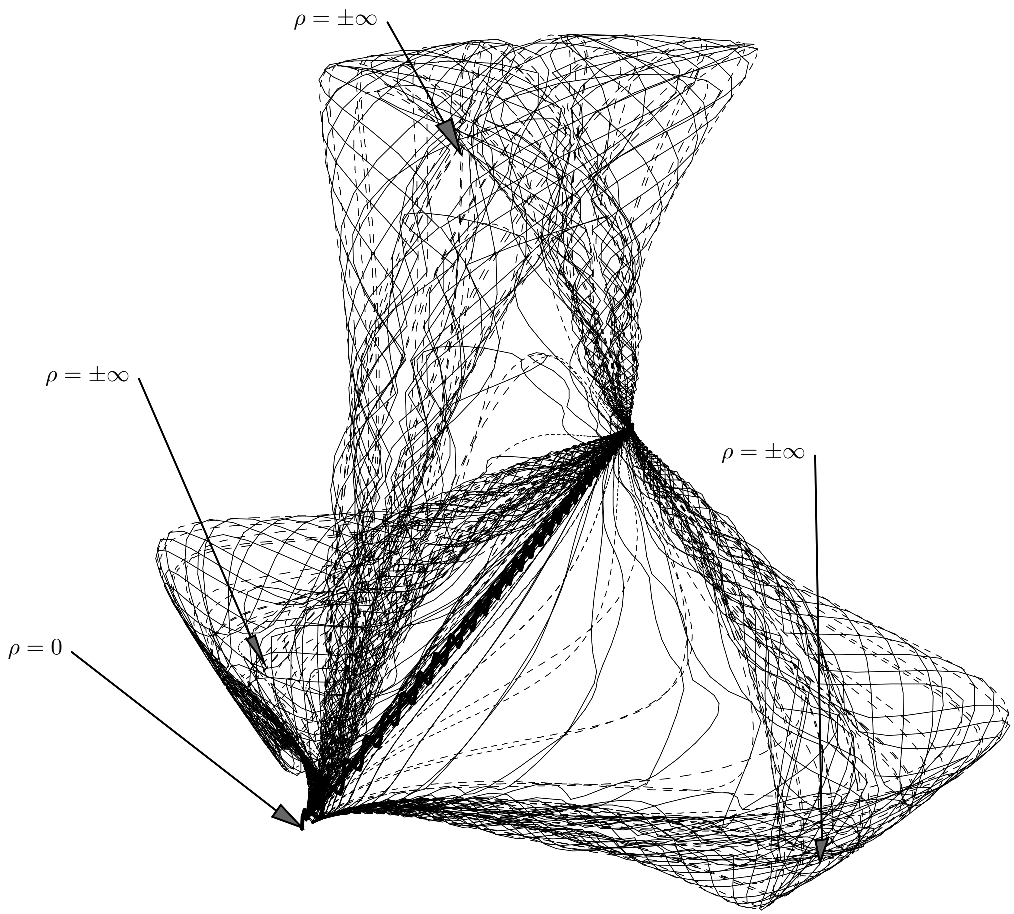


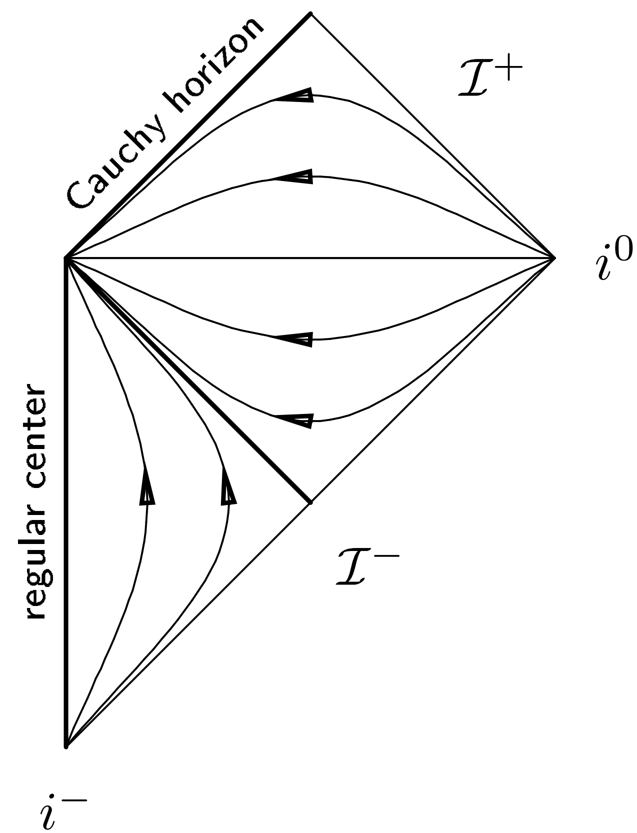
The C attractor embeded in additional dimension

Penrose's diagrams

- The table of objects

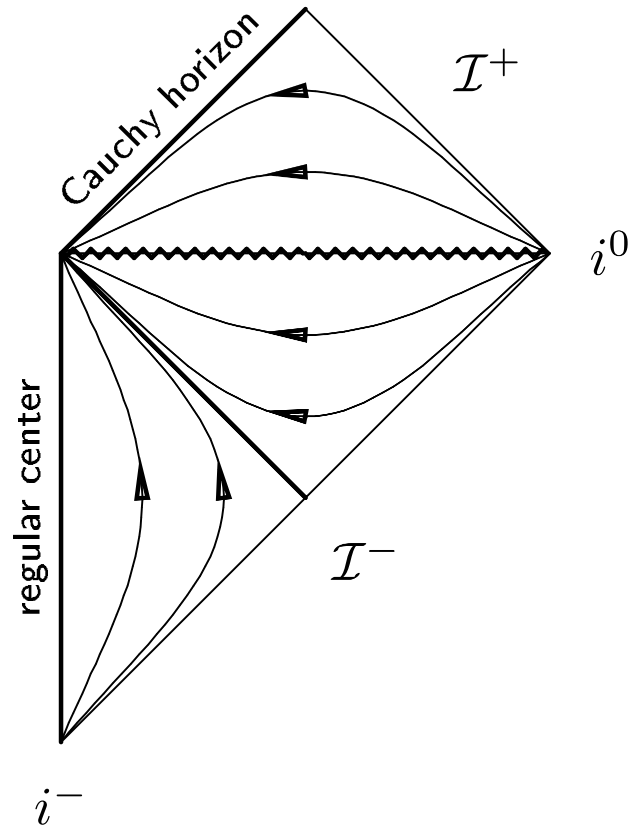
x	W	A(W,F,D)	object
$x(\rho = 0)$	$+\infty$	$+1$	regular center
$x(\rho = 0)$	0	$0 < A < 1$	spatial singularity
$x(\rho = +1)$	$+1$	$0 < A < 1$	past self-similarity horizon
$x(\rho_A < 0)$	-1	$0 < A < 1$	future self-similarity horizon
$x(\rho_B > 0)$	$+1$	0	apparent horizon
$x(\rho = \pm\infty)$	0	$0 < A < 1$	gluing point

Penrose's diagrams – type A



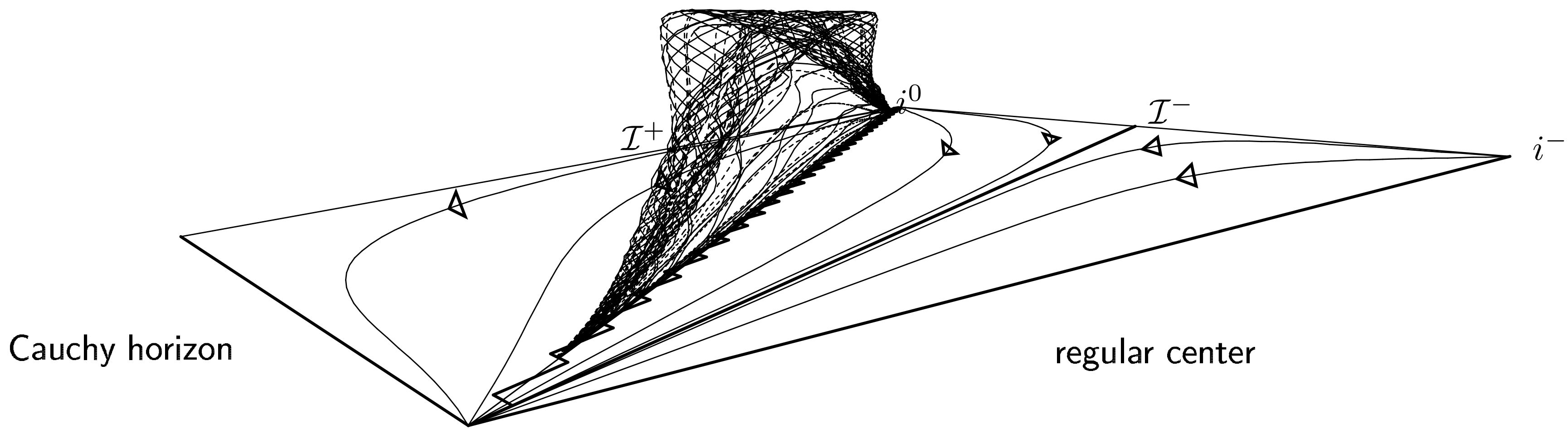
The type A solutions without a spatial singularity.

Penrose's diagrams – type A



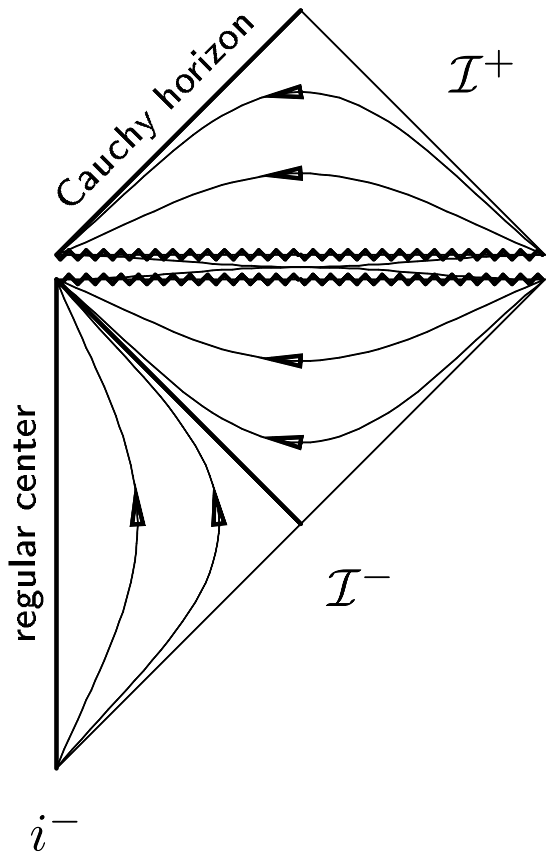
The type A solutions with a single spatial singularity.

Penrose's diagrams – type A



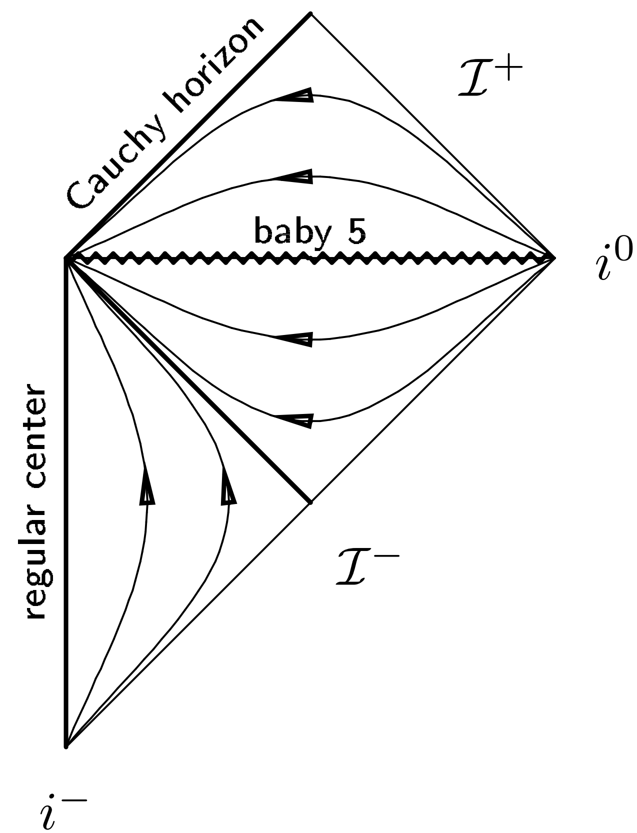
The type A solutions with two spatial singularities.

Penrose's diagrams – type A



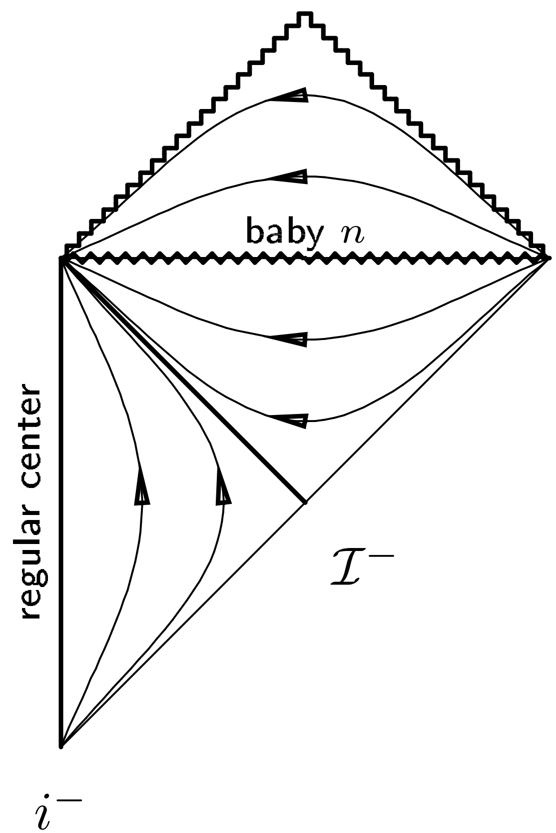
The type A solutions with two spatial singularities.

Penrose's diagrams – type A



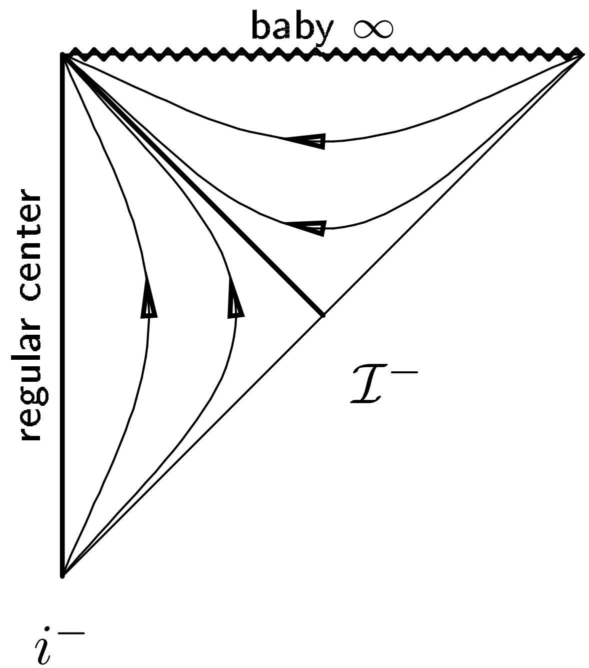
The type A solutions with five spatial singularities.

Penrose's diagrams – type B



The type B solutions with n spatial singularities.

Penrose's diagrams – type C



The type C solution.

The summary

- The strong arguments for existence of chaos
- The causal structure
  - Oscillatory model
  - Chaos is not directly detectable by a physical observer