Reggeized gluon states

and anomalous dimensions in QCD

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Contents

- Reggeon states in QCD
  - Reggeization
  - Structure Function
  - Hamiltonian

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  - OPE expansion in twist series
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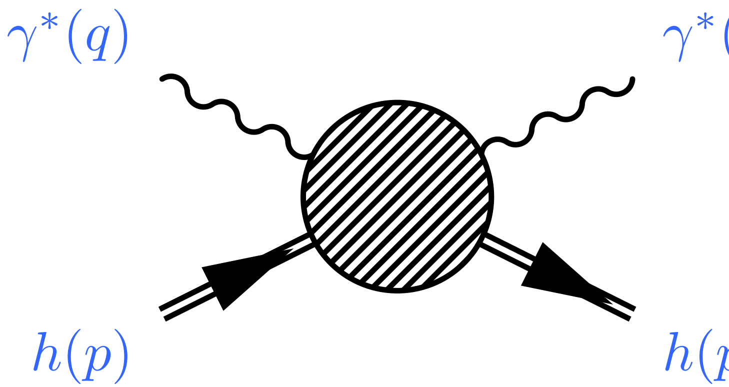
  - Analytical continuation of energy
- Numerical results

Introduction

Calculation of structure function ($F_2(x,Q^2)$) in perturbative QCD



where $x = Q^2/2(pq), \quad Q^2 = -q_\mu^2, \quad M^2 = p_\mu^2$

in the limit of small x:

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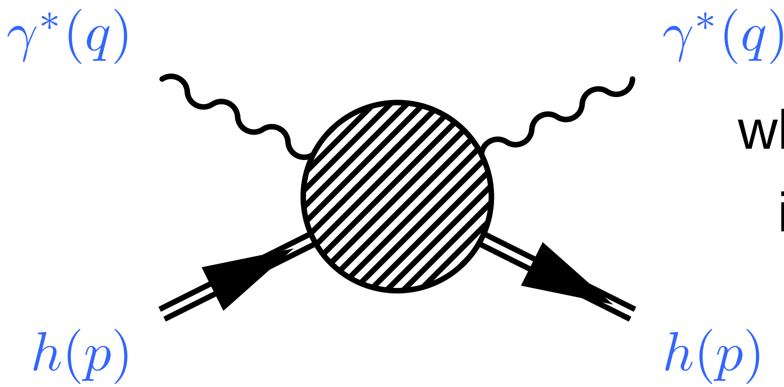

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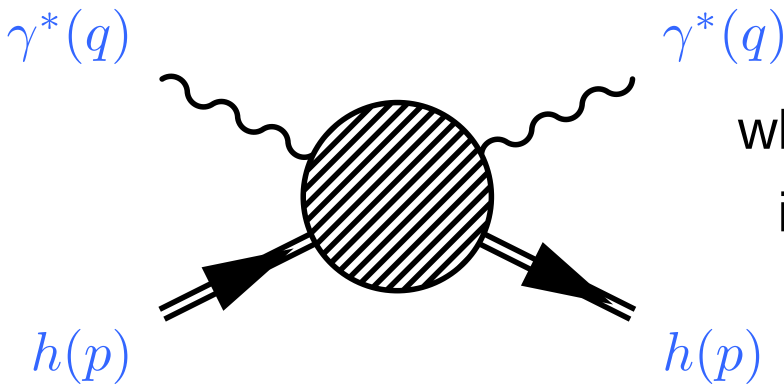

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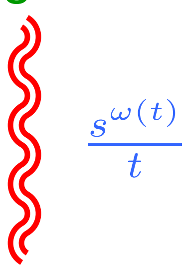
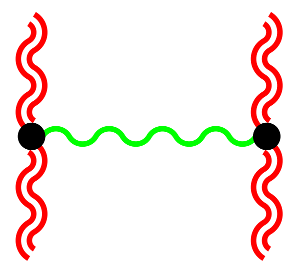
propagator



interaction



Lipatov's vertices

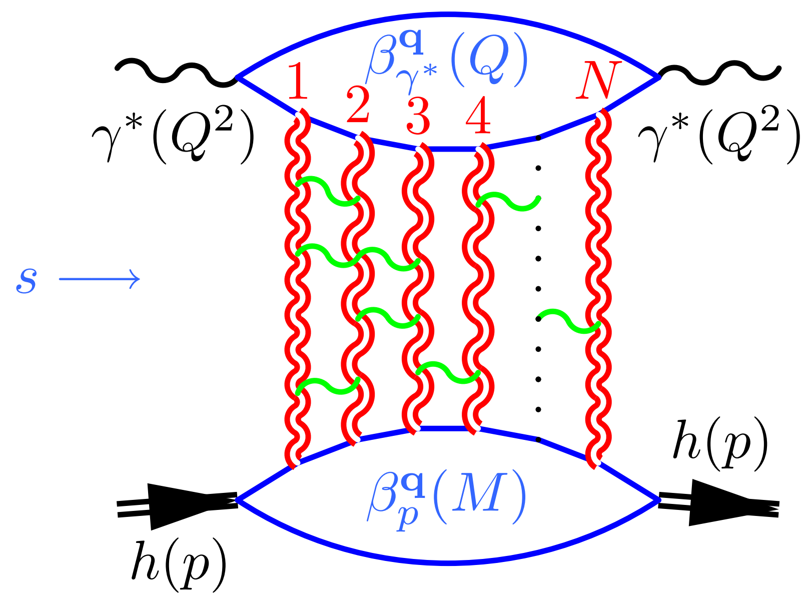
Structure Function for Reggeons

Structure Function for $j \to 1$: in the limit of small x and $s \to \infty$

$$\widetilde{F}(j, Q^2) \equiv \int_0^1 dx x^{j-2} F(x, Q^2) = \sum_{N=2}^{\infty} \bar{\alpha}_s^{N-2} \widetilde{F}_N(j, Q^2)$$

where $$\widetilde{F}_N(j, Q^2) = \sum_{\boldsymbol{q}} \frac{1}{j - 1 + \bar{\alpha}_s E_N(\boldsymbol{q})} \beta_{\gamma^*}^{\boldsymbol{q}}(Q)\, \beta_p^{\boldsymbol{q}}(M)$$

$\bar{\alpha}_s = \alpha_s N_c / \pi$



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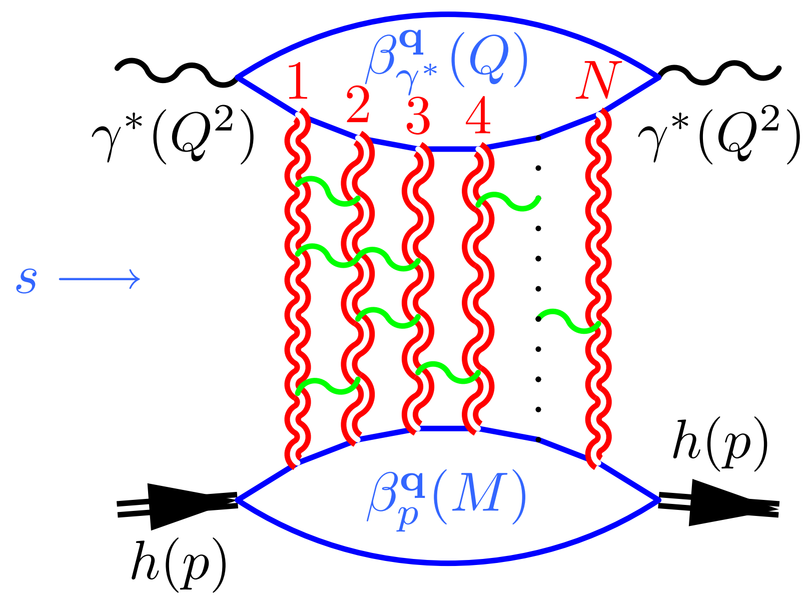
Compound N-Reggeon states

satisfy the Schrödinger equation

$$\mathcal{H}_N \Psi_{\boldsymbol{q}}(\{\vec{z}_k\}) = E_N \Psi_{\boldsymbol{q}}(\{\vec{z}_k\})$$

E_N - "energy" of $N-$Reggeon state



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$\bar{\alpha}_s = \alpha_s N_c/\pi$ integrals of motion: $\boldsymbol{q} = (q_2, \bar{q}_2 \ldots, q_N, \bar{q}_N)$

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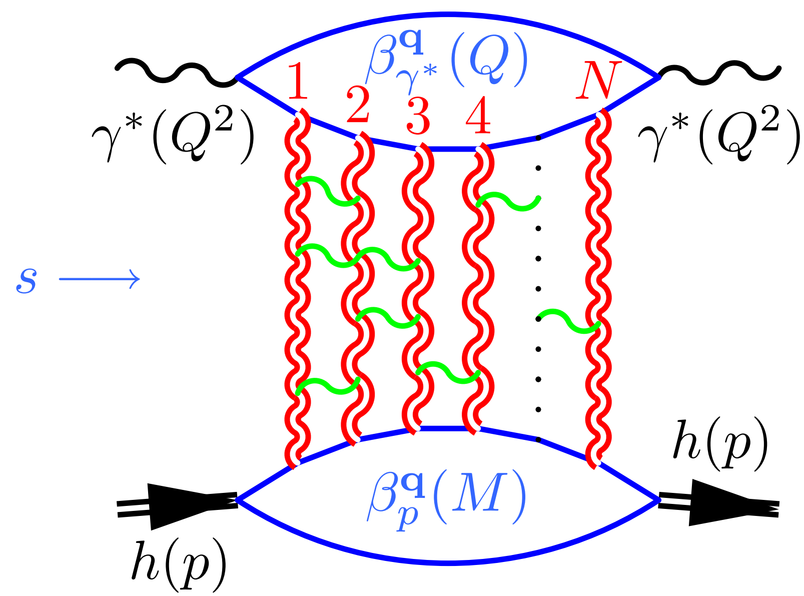
Multi-colour limit, i.e. $N_c \to \infty$



QCD Hamiltonian for $N_c \to \infty$

- For $N_c \to \infty$: $t_j^a t_k^a \to -\frac{N_c}{2}\delta_{j,k+1}$ where $t_j^a t_k^b$ are colour matrices

$$\mathcal{H}_N \sim \sum_{j,k=1,j>k}^{N} H(\vec{z}_j,\vec{z}_k) t_j^a t_k^a \longrightarrow \mathcal{H}_N \sim \sum_{k=0}^{N-1} H(\vec{z}_k,\vec{z}_{k+1})$$

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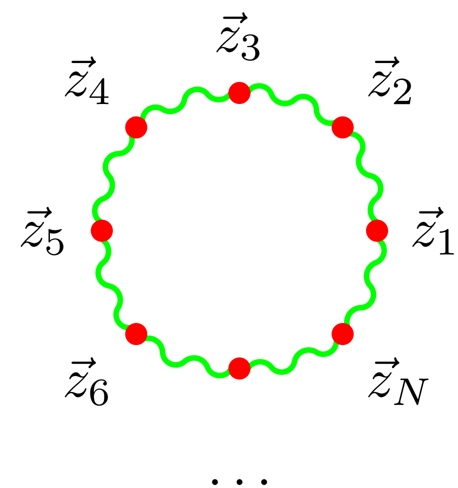
with $\vec{z}_0 \equiv \vec{z}_N$

- Bose Symmetry $\longrightarrow$ invariance under

  cyclic and mirror permutation of particle

$$\mathbb{P}\Psi(\vec{z}_1, \ldots, \vec{z}_{N-1}, \vec{z}_N) = \Psi(\vec{z}_2, \ldots, \vec{z}_N, \vec{z}_1)$$

$$\mathbb{M}\Psi(\vec{z}_1, \ldots, \vec{z}_{N-1}, \vec{z}_N) = \Psi(\vec{z}_N, \ldots, \vec{z}_2, \vec{z}_1)$$



$SL(2,\mathbb{C})$ invariance

- Holomorophic and anti-holomorphic coordinates

$$\vec{z}_k \equiv (x_k, y_k) \leftrightarrow \begin{cases} z_k = x_k + iy_k \\ \bar{z}_k = x_k - iy_k \end{cases}$$

$$\mathcal{H}_N \sim \sum_{k=0}^{N-1} \left[H(z_k, z_{k+1}) + H(\bar{z}_k, \bar{z}_{k+1}) \right]$$

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- $\mathcal{H}_N$ is invariant under conformal transformation $SL(2,\mathbb{C})$

$$\vec{z}_k \longrightarrow \frac{a z_k + b}{c z_k + d} \qquad \bar{\vec{z}}_k \longrightarrow \frac{\bar{a}\bar{z}_k + \bar{b}}{\bar{c}\bar{z}_k + \bar{d}} \qquad ad - bc = 1 \quad \bar{a}\bar{d} - \bar{b}\bar{c} = 1$$

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- Eigenstates $\Psi_q(\{\vec{z}\}; \vec{z}_0) \rightarrow$

$$(c z_0 + d)^{2h} (\bar{c}\bar{z}_0 + \bar{d})^{2\bar{h}} \prod_{k=1}^{N} (c z_k + d)^{2s} (\bar{c}\bar{z}_k + \bar{d})^{2\bar{s}} \Psi_{\boldsymbol{q}}(\{\vec{z}\}; \vec{z}_0)$$

where $(s = 0, \bar{s} = 1)$ are the complex spins of Reggeons and $(h, \bar{h})$ define a spin of N-Reggeon state

$$h = \frac{1+n_h}{2} + i\nu_h \qquad \bar{h} = \frac{1-n_h}{2} + i\nu_h \qquad n_h \in \mathbb{Z} \text{ and } \nu_h \in \mathbb{R}$$

Anomalous dimensions $\gamma_n^a(j)$

Expansion (OPE) in inverse powers of hard scale Q (twist series-n)

$$\widetilde{F}(j, Q^2) = \sum_{n=2,3,\ldots} \frac{1}{Q^n} \sum_a C_n^a(j, \alpha_s(Q^2)) \, \langle p \, | \mathcal{O}_{n,j}^a | p \rangle$$

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Wilson operators $\mathcal{O}_{n,j}^a$ satisfy

$$Q^2 \frac{d}{dQ^2} \langle p \,| \mathcal{O}_{n,j}^a(0) | p \rangle = \gamma_n^a(j) \, \langle p \,| \mathcal{O}_{n,j}^a(0) | p \rangle$$

where a enumerates operators with the same twist.

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In the limit $j \to 1$ the moment $\widetilde{F}(j,Q^2)$ takes a form

$$\widetilde{F}(j,Q^2) = \frac{1}{Q^2} \sum_{n=2,3,\ldots} \sum_a \tilde{C}_n^a(j,\alpha_s(Q^2)) \, \left(\frac{M}{Q}\right)^{n-2-2\gamma_n^a(j)}$$

Expansion for large Q

Impact factors have a form

$\beta^{\boldsymbol{q}}_{\gamma^*}(Q^2) = \int d^2 z_0 \, \langle \Psi_{\gamma^*} | \Psi_{\boldsymbol{q}}(\vec{z}_0) \rangle \qquad \beta^{\boldsymbol{q}}_{p}(M^2) = \int d^2 z_0 \, \langle \Psi_{\boldsymbol{q}}(\vec{z}_0) | \Psi_p \rangle$

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Scaling symmetry of Reggeon states $\to$ calculation of their dimensions

$$\beta^{\boldsymbol{q}}_{\gamma^*}(Q^2) = C^{\boldsymbol{q}}_{\gamma^*} Q^{-1-2i\nu_h} \, , \qquad \beta^{\boldsymbol{q}}_{p}(M^2) = C^{\boldsymbol{q}}_{p} \, M^{-1+2i\nu_h}$$

$C^{\boldsymbol{q}}_{\gamma^*}$ and $C^{\boldsymbol{q}}_{p}$ are dimensionless

$$\widetilde{F}_N(j, Q^2) = \frac{1}{Q^2} \sum_{\boldsymbol{\ell}} \sum_{n_h \geq 0} \int_{-\infty}^{\infty} d\nu_h \frac{C^{\boldsymbol{q}}_{\gamma^*} \, C^{\boldsymbol{q}}_{p}}{j - 1 + \bar{\alpha}_s E_N(\boldsymbol{q})} \left(\frac{M}{Q} \right)^{-1+2i\nu_h}$$

Expansion for large Q

$$j > j_N = 1 - \overline{\alpha}_s \min{}_{q} E_N(\boldsymbol{q})$$

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We integrate by summing residua

$$j - 1 + \bar{\alpha}_s E_N(\boldsymbol{q}(\nu_h(j); n_h, \boldsymbol{\ell})) = 0$$

$$\widetilde{F}_N(j, Q^2) \sim \frac{1}{Q^2} \left(\frac{M}{Q} \right)^{-1+2i\nu_h(j)}$$

$$\gamma_n(j) = (n-1)/2 - i\nu_h(j) = [n - (h(j) + \bar{h}(j))]/2 \qquad \boldsymbol{q} = \boldsymbol{q}(\nu_h; n_h, \boldsymbol{\ell})$$

$$\gamma_n(j) = \gamma_n^{(0)} \bar{\alpha}_s / (j-1) + \mathcal{O}(\bar{\alpha}_s^2) \qquad \boldsymbol{\ell} = (\ell_1, \ldots, \ell_{2N-4})$$

Expansion for large Q

Making use of the above equations

we can combine E_N from $\gamma_n(j)$:

$$E_N(\boldsymbol{q}) = -\left[\frac{c_{-1}}{\epsilon} + c_0 + c_1\,\epsilon + \ldots\right]$$

where $\gamma_n(j) = -\epsilon$, i.e. $i\nu_h = i\nu_h^{\mathrm{pole}} + \epsilon$, so that

$$\gamma_n(j) = -c_{-1}\left[\frac{\bar{\alpha}_s}{j-1} + c_0\left(\frac{\bar{\alpha}_s}{j-1}\right)^2 + \left(c_1 c_{-1} + c_0^2\right)\left(\frac{\bar{\alpha}_s}{j-1}\right)^3 + \ldots\right]$$

and $c_k = c_k(n, n_h, \boldsymbol{\ell})$

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Position of energy poles

$$E_N(\boldsymbol{q}) \sim \frac{\gamma_n^{(0)}}{i\nu_h - (n-1)/2}$$

determines twist n: $i\nu_h = (n-1)/2$ with $n \geq N + n_h$

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For $N = 2$ [Jaroszewicz ($n = 2$), Lipatov (higher n)]

$$\gamma_2(j) = \frac{\bar{\alpha}_s}{j-1} + 2\zeta(3)\left(\frac{\bar{\alpha}_s}{j-1}\right)^4 + 2\zeta(5)\left(\frac{\bar{\alpha}_s}{j-1}\right)^6 + \mathcal{O}(\bar{\alpha}_s^8)$$

Analytical Continuation for $\nu_h \in \mathbb{C}$

After analytical continuation

- Reggeon wave-function $\Psi_{\boldsymbol{q}}(\{\vec{z}\};\vec{z}_0)$ is not normarizable

$$\langle \Psi_{\boldsymbol{q}}(\vec{z}_0)|\Psi_{\boldsymbol{q}'}(\vec{z}_0')\rangle \equiv \int \prod_{k=1}^{N} d^2 z_k \Psi_{\boldsymbol{q}}(\{\vec{z}\};\vec{z}_0)\big(\Psi_{\boldsymbol{q}'}(\{\vec{z}\};\vec{z}_0')\big)^* = \delta^{(2)}(z_0 - z_0')\,\delta_{\boldsymbol{q}\boldsymbol{q}'}$$

where $\delta_{\boldsymbol{q}\boldsymbol{q}'} \equiv \delta(\nu_h - \nu_h')\delta_{n_h n_h'}\delta_{\ell\ell'}$

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For $N = 2$: $E_2(\nu_h, n_h) = \psi\left(\frac{1+|n_h|}{2} + i\nu_h\right) + \psi\left(\frac{1+|n_h|}{2} - i\nu_h\right) - 2\psi(1)$

Poles in $i\nu_h = \pm(n-1)/2$ without cuts

$\Psi(x) = \frac{d}{dx}\Gamma(x)$

leading twist $n = 2$ for $\nu_h = -i/2$

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For $N \geq 3$: no unique formula – multi-valued function

Poles at $i\nu_h = (n-1)/2$ with $n \geq N + n_h$

Cuts like $E_N^{\pm} \sim a_k \pm b_k \sqrt{\nu_{\mathrm{br},k} - \nu_h}$

Energy $E_N(\nu_h; n_h, \boldsymbol{\ell})$ for higher N

- $E_N = \varepsilon(h,q) + \varepsilon(h,-q) + \left(\varepsilon(1-\bar{h}^*, \bar{q}^*)\right)^* + \left(\varepsilon(1-\bar{h}^*, -\bar{q}^*)\right)^*$

  where $\varepsilon(h,q) = i\frac{d}{d\epsilon}\ln\left[\epsilon^N Q(i+\epsilon; h, q)\right]\Big|_{\epsilon=0}$ and $-q \equiv \{(-1)^k q_k\}$

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- Baxter chiral blocks $Q(u;h,q)$ can be obtained by

  $Q(u;h,q) = \frac{1}{\Gamma(-h)} \int_0^1 dz\, z^{iu-1} Q_1(z)$ where

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- Quantization conditions (for $\boldsymbol{q}$) $\longleftarrow$ $\Psi_{\boldsymbol{q}}(\{\vec{z}\}; \vec{z}_0)$ single-valuedness

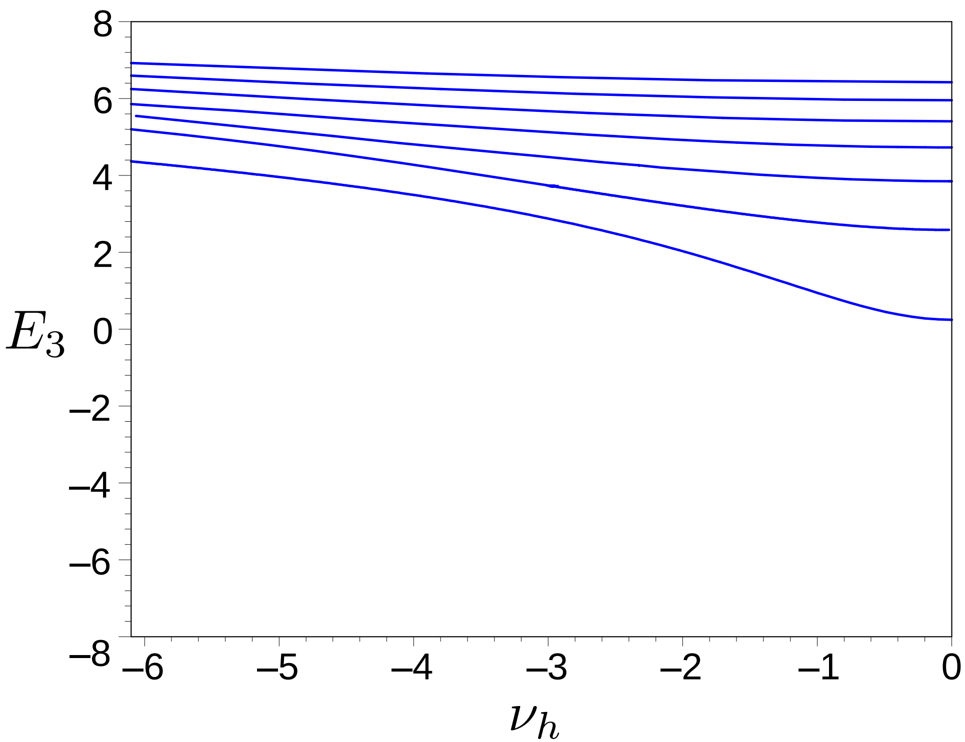
  $$Q(i+\epsilon; h, q) Q(i-\epsilon; \bar{h}, -\bar{q}) - Q(i+\epsilon; 1-h, q) Q(i-\epsilon; 1-\bar{h}, -\bar{q}) = \mathcal{O}(\epsilon^0)$$

  keeping in mind $Q(i+\epsilon; h, q) = \frac{1}{\epsilon^N} - i\frac{\varepsilon(h,q)}{\epsilon^{N-1}} + \mathcal{O}\left(\frac{1}{\epsilon^{N-2}}\right)$ we obtain

  $2N-1$ equations for $2(N-2)$ charges q_k and $\bar{q}_k$ **(**$k = 3, \ldots, N$**)**

  and remaining $q_2 = -h(h-1)$ and $\bar{q}_2 = -\bar{h}(\bar{h}-1)$

Spectral surfaces for $N = 3$ and $n_h = 0$



Energy spectrum $N = 3$ for Reggeon states $E_3(\nu_h; n_h, \boldsymbol{\ell})$ for $n_h = 0$ and $\boldsymbol{\ell} = (0, \ell_2)$, with $\ell_2 = 2,\ 4,\ \ldots, 14$ from down to up (on the left).

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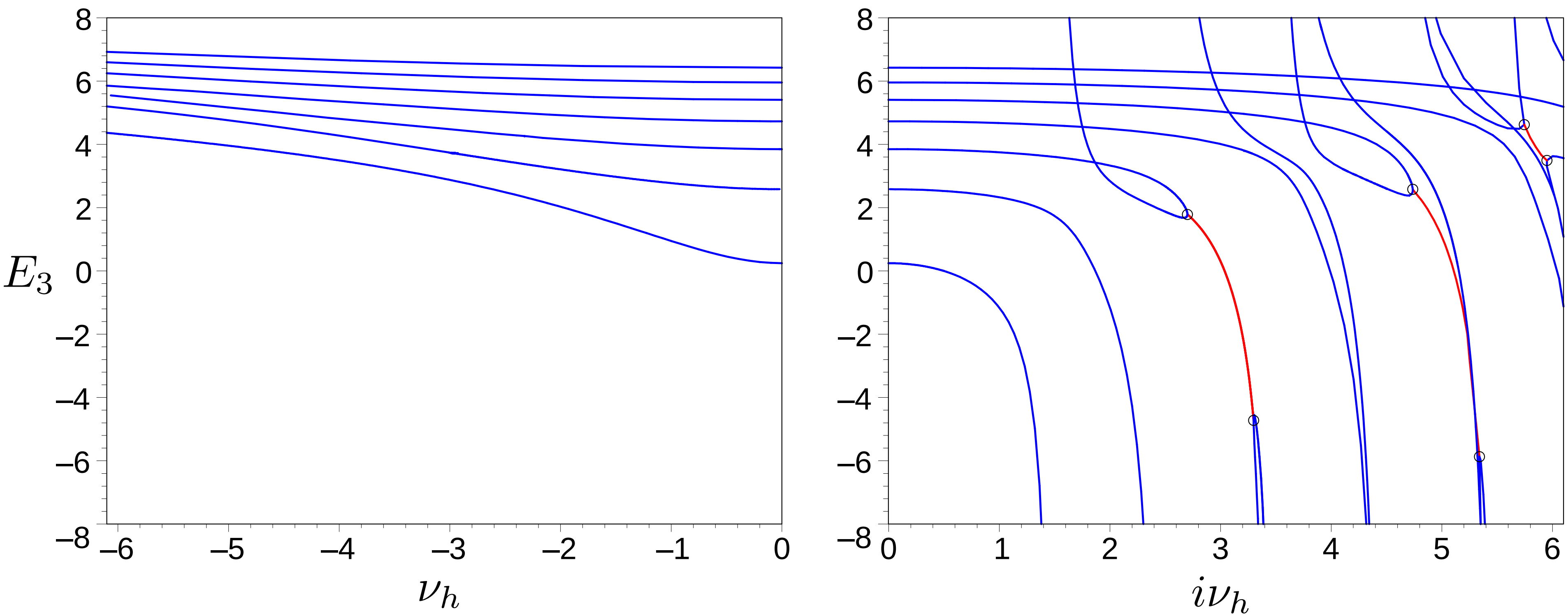

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Analytical continuation of energy along imaginary axis ν_h (on the right).

Branching points are denoted by circles and $i\nu_{bieg} = (n-1)/2$ where twist $n = 4, 6, 8, \ldots$

For all these surfaces $\bar{q}_3 + q_3 = 0$

$N = 3$ case

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- poles in $i\nu_h = (n-1)/2$, where twist $n = 4, 6, 8, \ldots$
- branching points, not only for $Re(\nu_h) = 0$, join surfaces with the same quantum numbers
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For $N = 3$ and $n_h = 1$ we have spectral surfaces for $q_3(\nu_h) = 0$

- states (with $q_3 = 0$) are descendant of $N = 2$ states
- energy $E_3 = E_2(\nu_h; n_h = 1)$ is single-valuedness function
- in $\nu_h = 0$ we have physical ground state for $N = 3$ ($E_3 = 0$)
- Poles are situated at $i\nu_h = 1, 2, \ldots$ for $n = 3, 5, 7, \ldots$

Energy pole structure

- For $N = 3$, $n_h = 1$ and $q_3(v_h) = 0$ with $n = 3$:

$$E_{3,\mathrm{d}}(2+\epsilon) = \epsilon^{-1} + 1 - \epsilon - (2\zeta(3) - 1)\,\epsilon^2 + \ldots$$

$$\gamma_3^{(N=3)}(j) = \frac{\bar{\alpha}_s}{j-1} - \left(\frac{\bar{\alpha}_s}{j-1}\right)^2 + (2\zeta(3)+1)\left(\frac{\bar{\alpha}_s}{j-1}\right)^4 + \ldots$$

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- For $N = 3$, $n_h = 0$ and $q_3 + \bar{q}_3 = 0$ with $n = 4$:

$$E_3(2+\epsilon) = \epsilon^{-1} + \frac{1}{2} - \frac{1}{2}\,\epsilon + 1.7021\,\epsilon^2 + \ldots$$

$$\gamma_4^{(N=3)}(j) = \frac{\bar{\alpha}_s}{j-1} - \frac{1}{2}\left(\frac{\bar{\alpha}_s}{j-1}\right)^2 - \frac{1}{4}\left(\frac{\bar{\alpha}_s}{j-1}\right)^3 - 1.0771\left(\frac{\bar{\alpha}_s}{j-1}\right)^4 + \ldots$$

Coefficients of Laurent expansion $\gamma(h) = -\epsilon \to 0$

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Poles have a form ($R = 2$ (or 1 for *even* h)):

$$E_3(h+\epsilon) = R\epsilon^{-1} + 2\gamma(h) + \mathcal{O}(\epsilon)$$

$\gamma(h)$ is energy of Heisenberg model with $SL(2,\mathbb{R})$ spin

Leading twist for higher N

- For *even* N:

  $n_{min} = N$ and corresponds to $i\nu_h = (N-1)/2$ for $n_h = 0$

  Pole Residuum equals $(N-2)$

  $\gamma_N^{(N)}(j) = (N-2)\frac{\bar{\alpha}_s}{j-1} + \mathcal{O}(\bar{\alpha}_s^2)$

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Residuum has more complex form

Pole is situated on the same surface as the state with minimal energy E_N

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- E_2 - single-valued function,
  $E_{N>2}$ - multi-valued function with cuts
- In the Regge limit leading contribution to $\widetilde{F}(j,Q^2)$ possesses twist $n = N$:
  - for *even* N comes from $n_h = 0$
  - for *odd* N comes from $n_h = 1$