Reggeized gluon states

and anomalous dimensions in QCD

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Calculation of structure function \((F_2(x, Q^2))\) in perturbative QCD

\[ x = \frac{Q^2}{2(pq)}, \quad Q^2 = -q_\mu^2, \quad M^2 = p_\mu^2 \]

in the limit of small \(x\):

\[ M^2 \ll Q^2 \ll s^2 = (p + q)^2 = Q^2(1 - x)/x \]
Calculation of structure function \( F_2(x, Q^2) \) in perturbative QCD

\[
\gamma^*(q) \rightarrow h(p) \rightarrow \gamma^*(q)
\]

where \( x = Q^2 / 2(pq) \), \( Q^2 = -q_\mu^2 \), \( M^2 = p_\mu^2 \)

in the limit of small \( x \):

\[
M^2 \ll Q^2 \ll s^2 = (p + q)^2 = Q^2(1 - x) / x
\]

- Resummation of appropriate Feynman’s Diagram ⇒ Formulation of effective field theory, in which compound states of gluons – reggeized gluons (Reggeons) – play a role of a new elementary field. •
Introduction

Calculation of structure function \( F_2(x, Q^2) \) in perturbative QCD

\[ x = Q^2/(2pq), \quad Q^2 = -q_\mu^2, \quad M^2 = p_\mu^2 \]

in the limit of small \( x \):

\[ M^2 \ll Q^2 \ll s^2 = (p + q)^2 = Q^2(1 - x)/x \]

- Resummation of appropriate Feynman’s Diagram ⇒ Formulation of effective field theory, in which compound states of gluons – reggeized gluons (Reggeons) – play a role of a new elementary field.

\[ \text{propagator} \]
\[ \frac{s^{\omega(t)}}{t} \]

\[ \text{interaction} \]

Lipatov’s vertices

Reggeized gluon states – p.3/17
Structure Function for Reggeons

Structure Function for $j \to 1$: in the limit of small $x$ and $s \to \infty$

$$\tilde{F}(j, Q^2) \equiv \int_0^1 dx x^{j-2} F(x, Q^2) = \sum_{N=2}^{\infty} \tilde{\alpha}_s^{N-2} \tilde{F}_N(j, Q^2)$$

where

$$\tilde{F}_N(j, Q^2) = \sum_q \frac{1}{j - 1 + \tilde{\alpha}_s E_N(q)} \beta_q^\gamma(Q) \beta_p^q(M)$$

$$\tilde{\alpha}_s = \alpha_s N_c / \pi$$

Reggeized gluon states – p.4/17
Structure Function for Reggeons

Structure Function for \( j \to 1 \): in the limit of small \( x \) and \( s \to \infty \)

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\tilde{F}(j, Q^2) \equiv \int_0^1 dx x^{j-2} F(x, Q^2) = \sum_{N=2}^{\infty} \tilde{F}_N(j, Q^2)
\]

where

\[
\tilde{F}_N(j, Q^2) = \sum_q \frac{1}{j - 1 + \tilde{\alpha}_s E_N(q)} \beta_{\gamma^*(Q)}^q \beta_p^q(M)
\]

\( \tilde{\alpha}_s = \alpha_s N_c / \pi \)

Compound \( N \)-Reggeon states satisfy the Schrödinger equation

\[
\mathcal{H}_N \Psi_q (\{\vec{z}_k\}) = E_N \Psi_q (\{\vec{z}_k\})
\]

\( E_N \) - “energy” of \( N \)-Reggeon state

Reggeized gluon states – p.4/17
Structure Function for Reggeons

Structure Function for $j \to 1$: in the limit of small $x$ and $s \to \infty$

$$\tilde{F}(j, Q^2) \equiv \int_0^1 dx x^{j-2} F(x, Q^2) = \sum_{N=2}^{\infty} \tilde{F}_N(j, Q^2)$$

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$$\alpha_s = \alpha_s N_c / \pi$$

integrals of motion: $q = (q_2, \bar{q}_2 \ldots, q_N, \bar{q}_N)$

Compound $N$-Reggeon states satisfy the Schrödinger equation

$$H_N \Psi_q (\{\bar{z}_k\}) = E_N \Psi_q (\{\bar{z}_k\})$$

$E_N$ - “energy” of $N$–Reggeon state

Multi-colour limit, i.e. $N_c \to \infty$
QCD Hamiltonian for $N_c \to \infty$

For $N_c \to \infty$: $t_j^a t_k^a \to -\frac{N_c}{2} \delta_{j,k+1}$ where $t_j^a t_k^b$ are colour matrices

$$\mathcal{H}_N \sim \sum_{j,k=1, j>k}^N H(\vec{z}_j, \vec{z}_k) t_j^a t_k^a \mapsto \mathcal{H}_N \sim \sum_{k=0}^{N-1} H(\vec{z}_k, \vec{z}_{k+1})$$

with $\vec{z}_0 \equiv \vec{z}_N$
QCD Hamiltonian for $N_C \rightarrow \infty$

For $N_C \rightarrow \infty$: $t^a_j t^a_k \rightarrow -\frac{N_C}{2} \delta_{j,k+1}$ where $t^a_j t^b_k$ are colour matrices

$$\mathcal{H}_N \sim \sum_{j,k=1, j>k}^{N} H(\bar{z}_j, \bar{z}_k) t^a_j t^a_k \rightarrow \mathcal{H}_N \sim \sum_{k=0}^{N-1} H(\bar{z}_k, \bar{z}_{k+1})$$

with $\bar{z}_0 \equiv \bar{z}_N$

Bose Symmetry $\longrightarrow$ invariance under
cyclic and mirror permutation of particle

$\mathbb{P}\Psi(\bar{z}_1, \ldots, \bar{z}_{N-1}, \bar{z}_N) = \Psi(\bar{z}_2, \ldots, \bar{z}_N, \bar{z}_1)$

$\mathbb{M}\Psi(\bar{z}_1, \ldots, \bar{z}_{N-1}, \bar{z}_N) = \Psi(\bar{z}_N, \ldots, \bar{z}_2, \bar{z}_1)$
$SL(2, \mathbb{C})$ invariance

Holomorphic and anti-holomorphic coordinates

\[ \tilde{z}_k \equiv (x_k, y_k) \leftrightarrow \begin{cases} z_k = x_k + iy_k \\ \tilde{z}_k = x_k - iy_k \end{cases} \]

\[ H_N \sim \sum_{k=0}^{N-1} [H(z_k, z_{k+1}) + H(\tilde{z}_k, \tilde{z}_{k+1})] \]
$SL(2, \mathbb{C})$ invariance

- Holomorphic and anti-holomorphic coordinates

\[ \tilde{z}_k \equiv (x_k, y_k) \leftrightarrow \left\{ \begin{array}{l}
    z_k = x_k + i y_k \\
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\end{array} \right. \]

\[ \mathcal{H}_N \sim \sum_{k=0}^{N-1} [H(z_k, z_{k+1}) + H(\tilde{z}_k, \tilde{z}_{k+1})] \]

- $\mathcal{H}_N$ is invariant under conformal transformation $SL(2, \mathbb{C})$

\[ \tilde{z}_k \rightarrow \frac{az_k + b}{cz_k + d}, \quad \tilde{z}_k \rightarrow \frac{a\tilde{z}_k + b}{c\tilde{z}_k + d} \quad ad - bc = 1 \quad \bar{a}\bar{d} - \bar{b}\bar{c} = 1 \]
\textbf{SL}(2, \mathbb{C}) \textbf{ invariance}

- Holomorphic and anti-holomorphic coordinates

\[
\begin{align*}
\tilde{z}_k & \equiv (x_k, y_k) \leftrightarrow \begin{cases} 
z_k = x_k + iy_k \\
\tilde{z}_k = x_k - iy_k
\end{cases} \\
\mathcal{H}_N & \sim \sum_{k=0}^{N-1} [H(z_k, z_{k+1}) + H(\tilde{z}_k, \tilde{z}_{k+1})]
\end{align*}
\]

- \( \mathcal{H}_N \) is invariant under conformal transformation \( SL(2, \mathbb{C}) \)

\[
\begin{align*}
\tilde{z}_k & \rightarrow \frac{az_k+b}{cz_k+d} & \tilde{z}_k & \rightarrow \frac{a\tilde{z}_k+b}{c\tilde{z}_k+d} & ad - bc = 1 & \bar{a}d - \bar{b}c = 1
\end{align*}
\]

- Eigenstates \( \Psi_q(\{\tilde{z}\}; \tilde{z}_0) \rightarrow \)

\[
(cz_0 + d)^{2h}(\bar{c}\tilde{z}_0 + \bar{d})^{2\bar{h}} \prod_{k=1}^{N} (cz_k + d)^{2s}(\bar{c}\tilde{z}_k + \bar{d})^{2\bar{s}} \Psi_q(\{\tilde{z}\}; \tilde{z}_0)
\]

where \( (s = 0, \bar{s} = 1) \) are the complex spins of Reggeons and \( (h, \bar{h}) \) define a spin of \( N \)-Reggeon state

\[
h = \frac{1+n_h}{2} + i\nu_h \quad \bar{h} = \frac{1-n_h}{2} + i\nu_h \quad n_h \in \mathbb{Z} \text{ and } \nu_h \in \mathbb{R}
\]
Anomalous dimensions $\gamma_n^a(j)$

Expansion (OPE) in inverse powers of hard scale $Q$ (twist series-$n$)

$$\tilde{F}(j, Q^2) = \sum_{n=2,3,...} \frac{1}{Q^n} \sum_a C_n^a(j, \alpha_s(Q^2)) \langle p | \mathcal{O}_{n,j}^a | p \rangle$$

Reggeized gluon states – p.7/17
Anomalous dimensions $\gamma_n^a(j)$

Expansion (OPE) in inverse powers of hard scale $Q$ (twist series-$n$)

$$\tilde{F}(j, Q^2) = \sum_{n=2,3,\ldots} \frac{1}{Q^n} \sum_a C_n^a(j, \alpha_s(Q^2)) \langle p | \mathcal{O}_{n,j}^a | p \rangle$$

Wilson operators $\mathcal{O}_{n,j}^a$ satisfy

$$Q^2 \frac{d}{dQ^2} \langle p | \mathcal{O}_{n,j}^a(0) | p \rangle = \gamma_n^a(j) \langle p | \mathcal{O}_{n,j}^a(0) | p \rangle$$

where $a$ enumerates operators with the same twist.
Anomalous dimensions $\gamma_n^a(j)$

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Anomalous dimensions $\gamma_n^a(j) = \sum_{k=1}^{\infty} \gamma_{k,n}^a(j) (\alpha_s(Q^2)/\pi)^k$
Anomalous dimensions $\gamma_n^a(j)$

Expansion (OPE) in inverse powers of hard scale $Q$ (twist series-$n$)

$$\tilde{F}(j, Q^2) = \sum_{n=2,3,\ldots} \frac{1}{Q^n} \sum_a C_n^a(j, \alpha_s(Q^2)) \langle p | \mathcal{O}_{n,j}^a | p \rangle$$

Wilson operators $\mathcal{O}_{n,j}^a$ satisfy

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where $a$ enumerates operators with the same twist.

Anomalous dimensions $\gamma_n^a(j) = \sum_{k=1}^{\infty} \gamma_{k,n}^a(j) (\alpha_s(Q^2)/\pi)^k$

In the limit $j \to 1$ the moment $\tilde{F}(j, Q^2)$ takes a form

$$\tilde{F}(j, Q^2) = \frac{1}{Q^2} \sum_{n=2,3,\ldots} \sum_a \tilde{C}_n^a(j, \alpha_s(Q^2)) \left( \frac{M}{Q} \right)^{n-2-2\gamma_n^a(j)}$$
Expansion for large $Q$

Impact factors have a form

$$\beta^{q*} (Q^2) = \int d^2 z_0 \langle \Psi_{\gamma^*} | \Psi_q (z_0) \rangle$$

$$\beta^{q} (M^2) = \int d^2 z_0 \langle \Psi_q (z_0) | \Psi_p \rangle$$
Expansion for large $Q$

Impact factors have a form

$$\beta_{q^*}^q(Q^2) = \int d^2 z_0 \langle \Psi_{q^*} | \Psi_q(\vec{z}_0) \rangle \quad \beta_p^q(M^2) = \int d^2 z_0 \langle \Psi_q(\vec{z}_0) | \Psi_p \rangle$$

Scaling symmetry of Reggeon states → calculation of their dimensions

$$\beta_{q^*}^q(Q^2) = C_{q^*}^q Q^{-1-2i\nu_n} \quad \beta_p^q(M^2) = C_p^q M^{-1+2i\nu_n}$$

$C_{q^*}$ and $C_p^q$ are dimensionless

$$\tilde{F}_N(j, Q^2) = \frac{1}{Q^2} \sum_{\ell} \sum_{n_h \geq 0} \int_{-\infty}^{\infty} d\nu_n \frac{C_{q^*}^q C_p^q}{j - 1 + \bar{\alpha}_s E_N(q) \left( \frac{M}{Q} \right)^{-1+2i\nu_n}}$$
Expansion for large $Q$

Impact factors have a form

$$\beta^q_{\gamma^*}(Q^2) = \int d^2 z_0 \langle \Psi_{\gamma^*}\Psi_q(\bar{z}_0) \rangle \quad \beta^q_p(M^2) = \int d^2 z_0 \langle \Psi_q(\bar{z}_0)\Psi_p \rangle$$

Scaling symmetry of Reggeon states $\rightarrow$ calculation of their dimensions

$$\beta^q_{\gamma^*}(Q^2) = C^q_{\gamma^*} Q^{-1-2i\nu_h}, \quad \beta^q_p(M^2) = C^q_p M^{-1+2i\nu_h}$$

$C^q_{\gamma^*}$ and $C^q_p$ are dimensionless

$$\tilde{F}_N(j, Q^2) = \frac{1}{Q^2} \sum_{\ell} \sum_{n_h \geq 0} \int_{-\infty}^{\infty} d\nu_h \frac{C^q_{\gamma^*} C^q_p}{j - 1 + \tilde{\alpha}_s E_N(q)} \left( \frac{M}{Q} \right)^{-1+2i\nu_h}$$

We integrate by summing residua

$$j - 1 + \tilde{\alpha}_s E_N(q(\nu_h(j); n_h, \ell)) = 0$$

$$\tilde{F}_N(j, Q^2) \sim \frac{1}{Q^2} \left( \frac{M}{Q} \right)^{-1+2i\nu_h(j)}$$

$$\gamma_n(j) = (n - 1)/2 - i\nu_h(j) = [n - (h(j) + \bar{h}(j))]/2$$

$$q = q(\nu_h; n_h, \ell)$$

$$\gamma_n(j) = \gamma_n^{(0)} \tilde{\alpha}_s/(j - 1) + O(\tilde{\alpha}_s^2)$$

$\ell = (\ell_1, \ldots, \ell_{2N-4})$
Expansion for large $Q$

Making use of the above equations we can combine $E_N$ from $\gamma_n(j)$:

$$E_N(q) = - \left[ \frac{c_{-1}}{\epsilon} + c_0 + c_1 \epsilon + \ldots \right]$$

where $\gamma_n(j) = -\epsilon$, i.e. $i\nu_h = i\nu_{\text{pole}}^{\text{pole}} + \epsilon$, so that

$$\gamma_n(j) = -c_{-1} \left[ \frac{\bar{\alpha}_s}{j-1} + c_0 \left( \frac{\bar{\alpha}_s}{j-1} \right)^2 + (c_1 c_{-1} + c_0^2) \left( \frac{\bar{\alpha}_s}{j-1} \right)^3 + \ldots \right]$$

and $c_k = c_k(n, n_h, \ell)$
Expansion for large $Q$

Making use of the above equations we can combine $E_N$ from $\gamma_n(j)$:

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and $c_k = c_k(n, n_h, \ell)$

Position of energy poles

$$E_N(q) \sim \frac{\gamma_n^{(0)}}{i\nu_h - (n-1)/2}$$

determines twist $n$: $i\nu_h = (n - 1)/2$ with $n \geq N + n_h$
Expansion for large $Q$

Making use of the above equations we can combine $E_N$ from $\gamma_n(j)$:

$$E_N(q) = - \left[ \frac{c_{-1}}{\epsilon} + c_0 + c_1 \epsilon + \ldots \right]$$

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and $c_k = c_k(n, n_h, \ell)$

Position of energy poles

$$E_N(q) \sim \frac{\gamma_n^{(0)}}{i\nu_h -(n-1)/2}$$

determines twist $n$: $i\nu_h = (n - 1)/2$ with $n \geq N + n_h$

For $N = 2$ [Jaroszewicz ($n = 2$), Lipatov (higher $n$)]

$$\gamma_2(j) = \frac{\bar{\alpha}_s}{j-1} + 2\zeta(3)\left( \frac{\bar{\alpha}_s}{j-1} \right)^4 + 2\zeta(5)\left( \frac{\bar{\alpha}_s}{j-1} \right)^6 + \mathcal{O}(\bar{\alpha}_s^8)$$
Analytical Continuation for $\nu_h \in \mathbb{C}$

After analytical continuation

Reggeon wave-function $\Psi_q(\{\bar{z}\}; \bar{z}_0)$ is not normalizable

$$
\langle \Psi_q(\bar{z}_0) | \Psi_{q'}(\bar{z}_{0}') \rangle \equiv \int \prod_{k=1}^{N} \, d^2 z_k \, \Psi_q(\{\bar{z}\}; \bar{z}_0) (\Psi_{q'}(\{\bar{z}\}; \bar{z}_0'))^* = \delta^{(2)}(z_0 - z_0') \, \delta_{qq'}
$$

where $\delta_{qq'} \equiv \delta(\nu_h - \nu'_h) \delta_{n_h n'_h} \delta_{\ell \ell'}$
After analytical continuation

Reggeon wave-function $\Psi_q(\{\vec{z}\}; \vec{z}_0)$ is not normalizable

$$\langle \Psi_q(\vec{z}_0)|\Psi_q'(\vec{z}'_0)\rangle \equiv \int \prod_{k=1}^{N} d^2 z_k \Psi_q(\{\vec{z}\}; \vec{z}_0)(\Psi_q'(\{\vec{z}\}; \vec{z}'_0))^* = \delta^{(2)}(z_0 - z'_0) \delta_{qq'}$$

where $\delta_{qq'} \equiv \delta(\nu_h - \nu'_h) \delta_{n_h n'_h} \delta_{\ell \ell'}$

quantum conditions (for $q$) are relaxed: $\bar{q}_k \neq q_k^*, \bar{h} \neq 1 - h^*$
After analytical continuation

- Reggeon wave-function \( \Psi_q(\{\vec{z}\}; \vec{z}_0) \) is not normalizable

\[
\langle \Psi_q(\vec{z}_0)|\Psi_{q'}(\vec{z}'_0) \rangle \equiv \int \prod_{k=1}^{N} d^2 z_k \Psi_q(\{\vec{z}\}; \vec{z}_0)(\Psi_{q'}(\{\vec{z}\}; \vec{z}'_0))^* = \delta^{(2)}(z_0 - z'_0) \delta_{qq'}
\]

where \( \delta_{qq'} \equiv \delta(\nu_h - \nu_{h'})\delta_{n_h n_{h'}}\delta_{\ell \ell'} \)

- Quantum conditions (for \( q \)) are relaxed: \( \bar{q}_k \neq q^*_k, \bar{h} \neq 1 - h^* \)

For \( N = 2 \):

\[
E_2(\nu_h, n_h) = \psi\left(\frac{1+|n_h|}{2} + i\nu_h\right) + \psi\left(\frac{1+|n_h|}{2} - i\nu_h\right) - 2\psi(1)
\]

Poles in \( i\nu_h = \pm(n - 1)/2 \) without cuts

Leading twist \( n = 2 \) for \( \nu_h = -i/2 \)

\[\Psi(x) = \frac{d}{dx} \Gamma(x)\]
Analytical Continuation for $\nu_h \in \mathbb{C}$

After analytical continuation

- Reggeon wave-function $\Psi_q(\{\bar{z}\}; \bar{z}_0)$ is not normalizable

$$\langle \Psi_q(\bar{z}_0)|\Psi_q'(\bar{z}_0')\rangle \equiv \int \prod_{k=1}^{N} d^2z_k \Psi_q(\{\bar{z}\}; \bar{z}_0) (\Psi_q'(\{\bar{z}\}; \bar{z}_0'))^* = \delta^{(2)}(z_0 - z_0') \delta_{qq'}$$

where $\delta_{qq'} \equiv \delta(\nu_h - \nu_h')\delta_{n_h n_h'} \delta_{\ell \ell'}$

- Quantum conditions (for $q$) are relaxed: $\bar{q}_k \neq q_k^*, \bar{h} \neq 1 - h^*$

For $N = 2$: $E_2(\nu_h, n_h) = \psi\left(\frac{1+|n_h|}{2} + i\nu_h\right) + \psi\left(\frac{1+|n_h|}{2} - i\nu_h\right) - 2\psi(1)$

Poles in $i\nu_h = \pm(n - 1)/2$ without cuts

Leading twist $n = 2$ for $\nu_h = -i/2$

For $N \geq 3$: no unique formula – multi-valued function

Poles at $i\nu_h = (n - 1)/2$ with $n \geq N + n_h$

Cuts like $E^\pm_N \sim a_k \pm b_k \sqrt{\nu_{br,k} - \nu_h}$
Energy $E_N(\nu_h; n_h, \ell)$ for higher $N$

\[
E_N = \varepsilon(h, q) + \varepsilon(h, -q) + (\varepsilon(1 - \bar{h}^*, \bar{q}^*))^* + (\varepsilon(1 - \bar{h}^*, -\bar{q}^*))^*
\]

where $\varepsilon(h, q) = i \frac{d}{d\epsilon} \ln [\epsilon^N Q(i + \epsilon; h, q)] \bigg|_{\epsilon=0}$ and $-q \equiv \{(−1)^k q_k\}$
Energy \( E_N(\nu_h; n_h, \ell) \) for higher \( N \)

\[ E_N = \varepsilon(h, q) + \varepsilon(h, -q) + (\varepsilon(1 - \bar{h}^*, \bar{q}^*))^* + (\varepsilon(1 - \bar{h}^*, -\bar{q}^*))^* \]

where \( \varepsilon(h, q) = i \frac{d}{de} \ln [\varepsilon^{N} Q(i + \varepsilon; h, q)] \bigg|_{\varepsilon=0} \) and \( -q \equiv \{(-1)^k q_k \} \)

Baxter chiral blocks \( Q(u; h, q) \) can be obtained by

\[ Q(u; h, q) = \frac{1}{\Gamma(-h)} \int_0^1 dz z^{iu-1} Q_1(z) \]

where

\[ \left[ (z \partial_z)^{N} z + (z \partial_z)^{N} z^{-1} - 2(z \partial_z)^{N} + \sum_{k=2}^{N} i^k q_k (z \partial_z)^{N-k} \right] Q_1(z) = 0 \]

with asymptotic \( Q_1(z) \sim (1 - z)^{-h-1} \) for \( z = 1 \)

Reggeized gluon states – p.11/17
Energy $E_N(n_h; \nu_h, \ell)$ for higher $N$

\[ E_N = \varepsilon(h, q) + \varepsilon(h, -q) + (\varepsilon(1 - \bar{h}^*, \bar{q}^*))^* + (\varepsilon(1 - \bar{h}^*, -\bar{q}^*))^* \]

where $\varepsilon(h, q) = i \frac{d}{de} \ln \left[ e^N Q(i + \varepsilon; h, q) \right] \bigg|_{\varepsilon=0}$ and $-q \equiv \{(1)^{k} q_{k}\}$

Baxter chiral blocks $Q(u; h, q)$ can be obtained by

\[ Q(u; h, q) = \frac{1}{\Gamma(-h)} \int_{0}^{1} dz z^{iu - 1} Q_1(z) \]

where

\[ \left[ (z \partial_z)^N z + (z \partial_z)^N z^{-1} - 2(z \partial_z)^N - \sum_{k=2}^{N} i^{k} q_{k} (z \partial_z)^{N-k} \right] Q_1(z) = 0 \]

with asymptotic $Q_1(z) \sim (1 - z)^{-h-1}$ for $z = 1$

Quantization conditions (for $q$) $\Psi_q(\{ z \}; \bar{z}_0)$ single-valuedness

\[ Q(i + \varepsilon; h, q) Q(i - \varepsilon; \bar{h}, -\bar{q}) - Q(i + \varepsilon; 1 - h, q) Q(i - \varepsilon; 1 - \bar{h}, -\bar{q}) = O(\varepsilon^0) \]

keeping in mind $Q(i + \varepsilon; h, q) = \frac{1}{\varepsilon^N} - i \frac{\varepsilon(h, q)}{\varepsilon_{N-1}} + O \left( \frac{1}{\varepsilon^{N-2}} \right)$ we obtain

$2N - 1$ equations for $2(N - 2)$ charges $q_k$ and $\bar{q}_k$ ($k = 3, \ldots, N$)

and remaining $q_2 = -h(h - 1)$ and $\bar{q}_2 = -\bar{h}(\bar{h} - 1)$

Reggeized gluon states – p.11/17
Energy spectrum $N = 3$ for Reggeon states $E_3(\nu_h; n_h, \ell)$ for $n_h = 0$ and $\ell = (0, \ell_2)$, with $\ell_2 = 2, 4, \ldots, 14$ from down to up (on the left).

For $\bar{q}_3 + q_3 = 0$
Spectral surfaces for $N = 3$ and $n_h = 0$

Energy spectrum $N = 3$ for Reggeon states $E_3(\nu_h; n_h, \ell)$ for $n_h = 0$ and $\ell = (0, \ell_2)$, with $\ell_2 = 2, 4, \ldots, 14$ from down to up (on the left).

Analytical continuation of energy along imaginary axis $\nu_h$ (on the right).

Branching points are denoted by circles and $i\nu_{bieg} = (n - 1)/2$ where twist $n = 4, 6, 8, \ldots$

For all these surfaces $\bar{q}_3 + q_3 = 0$
\[ N = 3 \text{ case} \]

For \( N = 3 \) and \( n_h = 0 \), with \( \bar{q}_3 + q_3 = 0 \)

- poles in \( i\nu_h = (n - 1)/2 \), where twist \( n = 4, 6, 8, \ldots \)
- branching points, not only for \( \text{Re}(\nu_h) = 0 \), join surfaces with the same quantum numbers
- contributions from cuts simplify each other
For $N = 3$ and $n_h = 0$, with $\bar{q}_3 + q_3 = 0$

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For $N = 3$ and $n_h = 1$ we have spectral surfaces for $q_3(\nu_h) = 0$

- states (with $q_3 = 0$) are descendant of $N = 2$ states
- energy $E_3 = E_2(\nu_h; n_h = 1)$ is single-valuedness function
- in $\nu_h = 0$ we have physical ground state for $N = 3$ ( $E_3 = 0$)
- Poles are situated at $i\nu_h = 1, 2, \ldots$ for $n = 3, 5, 7, \ldots$
Energy pole structure

For $N = 3$, $n_h = 1$ and $q_3(v_h) = 0$ with $n = 3$:

$$E_{3,d}(2 + \epsilon) = \epsilon^{-1} + 1 - \epsilon - (2\zeta(3) - 1) \epsilon^2 + \ldots$$

$$\gamma_3^{(N=3)}(j) = \frac{\bar{\alpha}_s}{j - 1} - \left(\frac{\bar{\alpha}_s}{j - 1}\right)^2 + (2\zeta(3) + 1)\left(\frac{\bar{\alpha}_s}{j - 1}\right)^4 + \ldots$$
Energy pole structure

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For $N = 3$, $n_h = 0$ and $q_3 + \bar{q}_3 = 0$ with $n = 4$:

$$E_{3}(2 + \epsilon) = \epsilon^{-1} + \frac{1}{2} - \frac{1}{2} \epsilon + 1.7021\epsilon^2 + \ldots$$

$$\gamma_{4}^{(N=3)}(j) = \frac{\tilde{\alpha}_s}{j - 1} - \frac{1}{2}\left(\frac{\tilde{\alpha}_s}{j - 1}\right)^2 - \frac{1}{4}\left(\frac{\tilde{\alpha}_s}{j - 1}\right)^3 - 1.0771\left(\frac{\tilde{\alpha}_s}{j - 1}\right)^4 + \ldots$$
Coefficients of Laurent expansion $\gamma(h) = -\epsilon \rightarrow 0$

\[
E_3(2 + \epsilon) = \epsilon^{-1} + \frac{1}{2} - \frac{1}{2} \epsilon + 1.7021 \epsilon^2 + \ldots
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E_3(3 + \epsilon) = 2 \epsilon^{-1} + \frac{15}{8} - 1.6172 \epsilon + 0.719 \epsilon^2 + \ldots
\]

\[
E_3^{(a)}(4 + \epsilon) = \epsilon^{-1} + \frac{11}{12} - 0.6806 \epsilon - 1.966 \epsilon^2 + \ldots
\]

\[
E_3^{(b)}(4 + \epsilon) = 2 \epsilon^{-1} + \frac{15}{4} - 3.2187 \epsilon + 3.430 \epsilon^2 + \ldots
\]

\[
E_3^{(a)}(5 + \epsilon) = 2 \epsilon^{-1} + \frac{125}{48} - 2.0687 \epsilon + 1.047 \epsilon^2 + \ldots
\]

\[
E_3^{(b)}(5 + \epsilon) = 2 \epsilon^{-1} + \frac{53}{12} - 2.4225 \epsilon + 0.247 \epsilon^2 + \ldots
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Poles have a form ($R = 2$ (or 1 for even $h$)):

$$E_3(h + \epsilon) = R\epsilon^{-1} + 2\gamma(h) + O(\epsilon)$$

$\gamma(h)$ is energy of Heisenberg model with $SL(2, \mathbb{R})$ spin
Leading twist for higher $N$

For even $N$:

$n_{min} = N$ and corresponds to $i \nu_h = (N - 1)/2$ for $n_h = 0$

Pole Residuum equals $(N - 2)$

$\gamma_N^{(N)}(j) = (N - 2) \frac{\bar{\alpha}_s}{j=1} + O(\bar{\alpha}_s^2)$
Leading twist for higher $N$

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Pole Residuum equals $(N - 2)$

$$\gamma^{(N)}_N(j) = (N - 2)\frac{\bar{\alpha}_s}{j-1} + \mathcal{O}(\bar{\alpha}_s^2)$$

For odd $N$:

$n_h = 0$: $n_{min} = (N + 1)$ and corresponds to $i\nu_h = N/2$,

i.e.: $E_5(3 + \epsilon) = \frac{3}{\epsilon} + \frac{7}{6} + \ldots$

$n_h = 1$: $n_{min} = N$ and corresponds to pole at $i\nu_h = (N - 1)/2$,

e.g. $E_{5,a}(3 + \epsilon) = \frac{3 + \sqrt{5}}{2\epsilon} + 1.36180 + \ldots$
Leading twist for higher $N$

For even $N$:

$n_{min} = N$ and corresponds to $i\nu_h = (N - 1)/2$ for $n_h = 0$

Pole Residuum equals $(N - 2)$

$$\gamma_N^{(N)}(j) = (N - 2)\frac{\bar{\alpha_s}}{j-1} + O(\bar{\alpha_s}^2)$$

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Residuum has more complex form
Leading twist for higher $N$

For even $N$:

$n_{\text{min}} = N$ and corresponds to $i\nu_h = (N - 1)/2$ for $n_h = 0$

Pole Residuum equals $(N - 2)$

$$\gamma_N^{(N)}(j) = (N - 2)\frac{\bar{\alpha}_s}{j-1} + \mathcal{O}(\bar{\alpha}_s^2)$$

For odd $N$:

$n_h = 0$: $n_{\text{min}} = (N + 1)$ and corresponds to $i\nu_h = N/2$,

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$n_h = 1$: $n_{\text{min}} = N$ and corresponds to pole at $i\nu_h = (N - 1)/2$,

e.g. $E_{5,d}(3 + \epsilon) = \frac{3 + \sqrt{5}}{2\epsilon} + 1.36180 + \ldots$

Residuum has more complex form

Pole is situated on the same surface as the state with minimal energy $E_N$
Summary

- We performed (OPE) expansion of $N$-Reggeon state contribution to $\tilde{F}(j, Q^2)$ for $j \to 1$
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Summary

- We performed (OPE) expansion of $N$-Reggeon state contribution to $\tilde{F}(j, Q^2)$ for $j \to 1$

- After analytical continuation we calculated relations between anomalous dimensions $\gamma_n^a(j)$ and Laurent expansion for $E_N$

- $E_2$ - single-valued function,
  $E_{N>2}$ - multi-valued function with cuts
We performed (OPE) expansion of $N$-Reggeon state contribution to $\tilde{F}(j, Q^2)$ for $j \to 1$.

After analytical continuation we calculated relations between anomalous dimensions $\gamma_n^a(j)$ and Laurent expansion for $E_N$.

$E_2$ - single-valued function,
$E_{N>2}$ - multi-valued function with cuts.

In the Regge limit leading contribution to $\tilde{F}(j, Q^2)$ possesses twist $n = N$:

- for even $N$ comes from $n_h = 0$
- for odd $N$ comes from $n_h = 1$