High Energy QCD and Pomeron Loops

Dionysis Triantafyllopoulos

Saclay (CEA/SPhT)

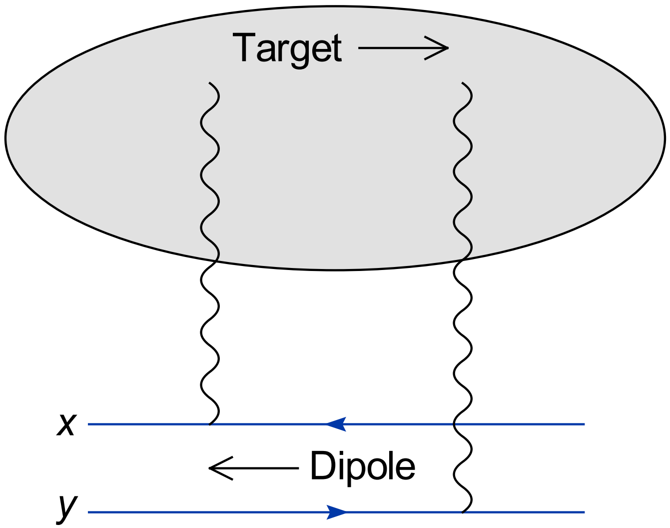
Based on : E. Iancu, D.N.T., Nucl. Phys. **A 756** (2005) 419, Phys. Lett. **B 610** (2005) 253
J.-P. Blaizot, E. Iancu, K. Itakura, D.N.T., Phys. Lett. **B 615** (2005) 221
Y. Hatta, E. Iancu, L. McLerran, A. Stasto, D.N.T., hep-ph/0504182

Outline

- Approaches to High Energy QCD ($s \to \infty,\ \Lambda^2_{\mathrm{QCD}} \ll Q^2 \ll s$)
  - Color Dipole Picture
  - Color Glass Condensate (CGC)- JIMWLK equation
  - QCD - Statistical Physics correspondence
  - Effective Action, Pomeron Vertices, Reggeized Gluons,...
- Outline:
  - The BFKL Pomeron
  - Pomeron Mergings, Saturation and the CGC
  - The Saturation Momentum
  - Pomeron Splittings and Fluctuations
  - Pomeron Loops, Evolution Equations at High Energy
  - Duality, Effective Hamiltonian
  - The Saturation Momentum Revisited

The BFKL Pomeron

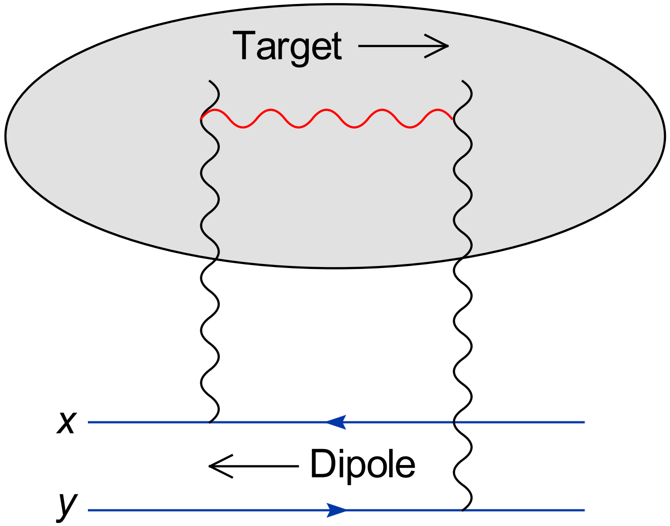
- Probe gluon distribution of generic hadron with small color dipole
  Dipole size : $\boldsymbol{r}^2 = (\boldsymbol{x} - \boldsymbol{y})^2 \ll \Lambda_{\rm QCD}^{-2}$, Gluon momentum : $Q^2 \sim 4/\boldsymbol{r}^2$



Lowest order in pQCD

The BFKL Pomeron

- Probe gluon distribution of generic hadron with small color dipole
  Dipole size : $\boldsymbol{r}^2 = (\boldsymbol{x} - \boldsymbol{y})^2 \ll \Lambda_{\rm QCD}^{-2}$, Gluon momentum : $Q^2 \sim 4/\boldsymbol{r}^2$

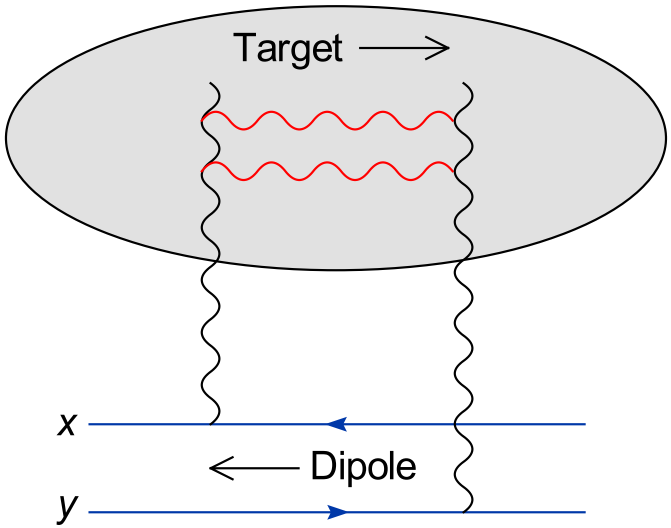


One soft gluon: $\alpha_s Y$

- $Y = \ln(1/x) = \ln(p^+/k^+)$

The BFKL Pomeron

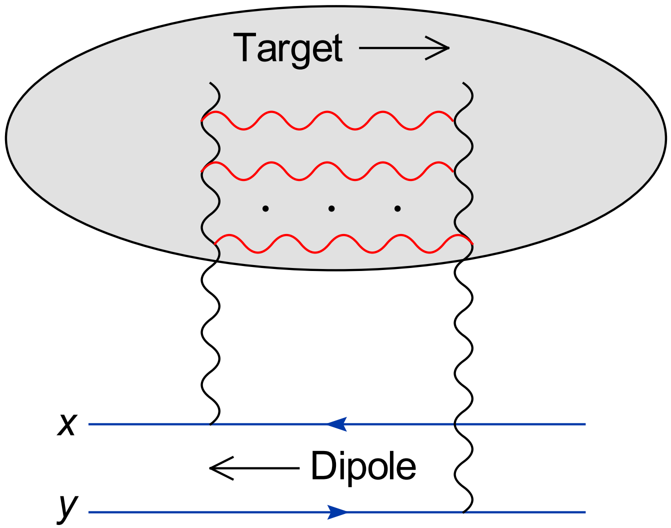
- Probe gluon distribution of generic hadron with small color dipole
  Dipole size : $\boldsymbol{r}^2 = (\boldsymbol{x}-\boldsymbol{y})^2 \ll \Lambda_{\rm QCD}^{-2}$, Gluon momentum : $Q^2 \sim 4/\boldsymbol{r}^2$



Two soft gluons: $(\alpha_s Y)^2$

The BFKL Pomeron

- Probe gluon distribution of generic hadron with small color dipole
  Dipole size : $\boldsymbol{r}^2 = (\boldsymbol{x}-\boldsymbol{y})^2 \ll \Lambda_{\rm QCD}^{-2}$, Gluon momentum : $Q^2 \sim 4/\boldsymbol{r}^2$

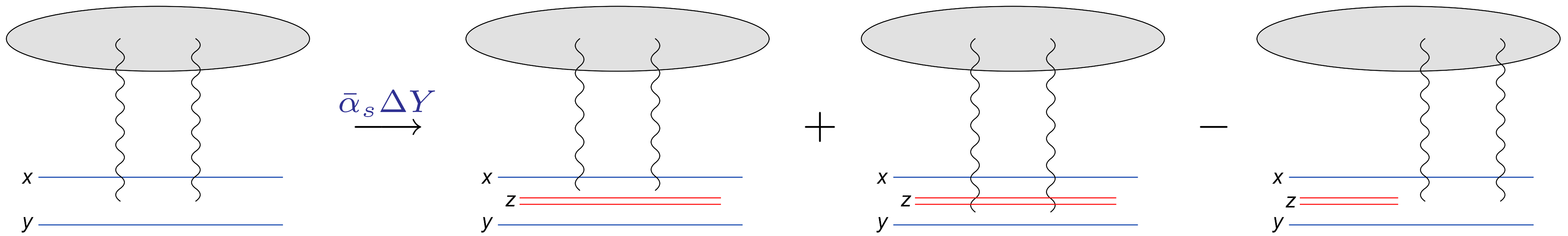


n soft gluons: $(\alpha_s Y)^n$

- Resum all $(\alpha_s Y)^n$ terms when $\alpha_s Y \gtrsim 1$
  Gluon ladder : BFKL Pomeron

The BFKL Equation (1/3)

- Equivalent to diagram resummation $\rightarrow$
  Write evolution equation for scattering amplitude T
- View soft gluon emission in projectile
  At large-N_c: Gluon $\rightarrow$ Quark-Antiquark pair
  Either daughter dipole can scatter off target



- BFKL Equation (coordinate space)

$$\frac{\mathrm{d}T_{\boldsymbol{xy}}}{\mathrm{d}Y} = \frac{\bar{\alpha}_s}{2\pi} \int \mathrm{d}^2 \boldsymbol{z} \; \underbrace{\mathcal{M}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})}_{\frac{(\boldsymbol{x}-\boldsymbol{y})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{z}-\boldsymbol{y})^2}} \; [T_{\boldsymbol{xz}} + T_{\boldsymbol{zy}} - T_{\boldsymbol{xy}}] \equiv \mathcal{K} \otimes T_{\boldsymbol{xy}}$$

  - Kernel : dipole splitting differential probability

The BFKL Equation (2/3)

Linear evolution $\rightsquigarrow$ Solve eigenvalue problem

- The easy problem: Integrate over impact parameter

$$\mathcal{K} \otimes r^{2\gamma} = [2\psi(1) - \psi(\gamma) - \psi(1-\gamma)]\; r^{2\gamma} = \chi_0(\gamma)\, r^{2\gamma}$$

- The hard problem: Fixed impact parameter
- Solution: Superposition of (evolved) eigenfunctions
  Both cases: High energy, fixed $\boldsymbol{r}^2$ same eigenvalue dominates
  Energy dependence in asymptotics

$$T \sim \alpha_s \varphi \sim \alpha_s^2\, n \sim \alpha_s^2 \exp[\omega_{\mathbb{P}} Y]$$

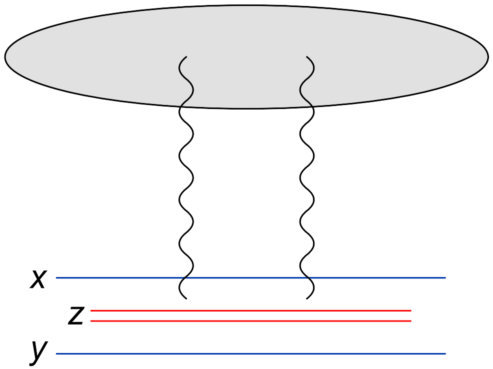
  $\omega_{\mathbb{P}} = 4\bar{\alpha}_s \ln 2 = \bar{\alpha}_s \chi_0(1/2) =$ hard pomeron intercept

- Exponential increase of gluon distribution φ, dipole density n in target

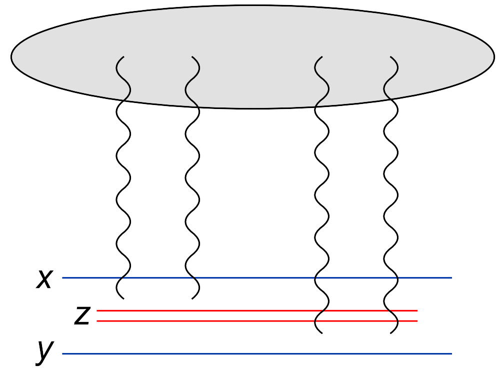
The BFKL Equation (3/3)

Pathologies of BFKL Equation

- Violation of Unitarity:
  Amplitude must satisfy $T(\boldsymbol{r},\boldsymbol{b}) \leq 1$
  Maximal allowed gluon density is
  $$\varphi \sim a^\dagger a \sim \mathcal{A}^2 \lesssim 1/g^2 \sim 1/\alpha_s$$
- Sensitivity to non-perturbative physics:
  Transverse coordinates ($\sim$ momenta) not strongly ordered
  Non-local in transverse coordinates ($\sim$ momentum) kernel $\rightsquigarrow$
  Random-walk in $\ln \boldsymbol{r}^2$ $\rightsquigarrow$ Diffusion to infrared: $\boldsymbol{r}^2 \gtrsim 1/\Lambda^2_{\rm QCD}$
  BFKL evolution is not self-consistent
- Next to leading BFKL: resum $\alpha_s(\alpha_s Y)^n$ terms
  Will not save from difficulties
  Simply adds $\mathcal{O}(\alpha_s^2)$ correction to $\omega_{\mathbb{P}}$

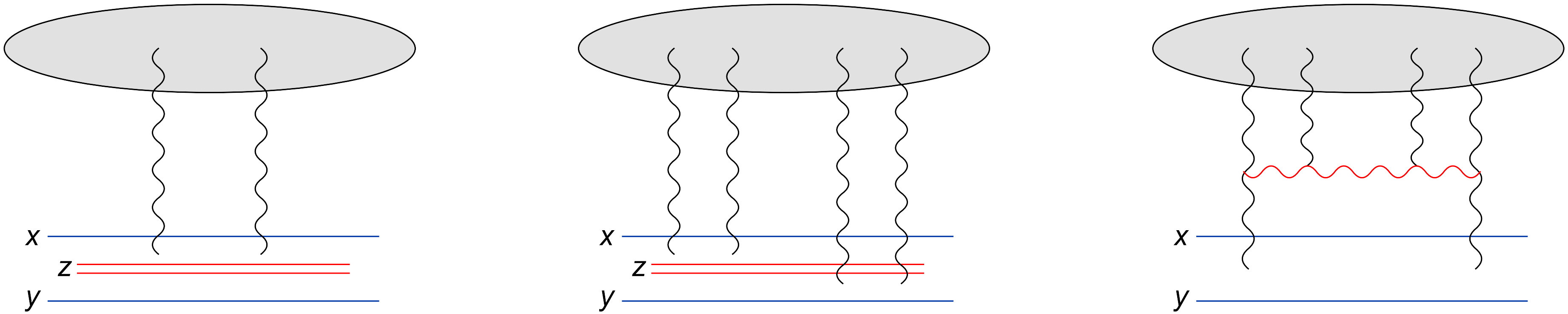


$\bar{\alpha}_s \Delta Y \mathcal{O}(\alpha_s \varphi)$



$\bar{\alpha}_s \Delta Y \mathcal{O}(\alpha_s^2 \varphi^2)$

- Second diagram small in perturbation theory $\varphi \ll 1/\alpha_s$
  Equally important at high density $\varphi \sim 1/\alpha_s \rightsquigarrow$
  $T^{(2)} \sim T$: Allow both dipoles to scatter



$\bar{\alpha}_s \Delta Y \, \mathcal{O}(\alpha_s \varphi)$ $\bar{\alpha}_s \Delta Y \, \mathcal{O}(\alpha_s^2 \varphi^2)$

- Second diagram small in perturbation theory $\varphi \ll 1/\alpha_s$
  Equally important at high density $\varphi \sim 1/\alpha_s \rightsquigarrow$
  $T^{(2)} \sim T$: Allow both dipoles to scatter
- Third diagram (equiv to second): target evolution
  Merging of two pomerons
  Gluon recombination

Saturation - Unitarity (2/2)

- First Balitsky Equation

$$\frac{\mathrm{d}T_{\boldsymbol{xy}}}{\mathrm{d}Y} = \mathcal{K}_{\mathrm{BFKL}} \otimes T_{\boldsymbol{xy}} - \frac{\bar{\alpha}_s}{2\pi} \int \mathrm{d}^2\boldsymbol{z} \mathcal{M}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) T^{(2)}_{\boldsymbol{xz};\boldsymbol{zy}}$$

- “Mean field” approximation: closed equation (Kovchegov)

$$T^{(2)}(\boldsymbol{xz}; \boldsymbol{zy}) \simeq T(\boldsymbol{x}, \boldsymbol{z}) T(\boldsymbol{z}, \boldsymbol{y})$$

- Fixed points
  - $T = 0 \rightsquigarrow$ unstable
  - $T = 1 \rightsquigarrow$ stable
- Pathologies are cured
  - Amplitude satisfies unitarity bound
  - Non-linear term cuts diffusion to the infrared
- Saturation line $Q_s^2 \approx \Lambda^2 \exp(\lambda_s Y)$ where $T(r \sim 2/Q_s) = \mathcal{O}(1)$
  Justifies weak coupling approximation: $\alpha_s(Q_s) \ll 1$

The Color Glass Condensate

- Fast moving partons with momentum $p^+ \to$ Large lifetime

$$\Delta x^+ \sim 1/p^- = 2p^+/\boldsymbol{p}^2$$

- Time scale separation:
  "Frozen" sources for slow partons with momenta $k^+ = xp^+ \ll p^+$
  Small-x gluons $\sim$ color field radiated by fast partons
- Solve Classical Yang-Mills equation $\rightsquigarrow \mathcal{A}(\rho)$ for given source ρ

$$\left(D_\nu F^{\nu\mu}\right)_a(x) = \delta^{\mu +}\rho_a(x^-, \boldsymbol{x}) \qquad \underline{\mathsf{Non-linear}}$$

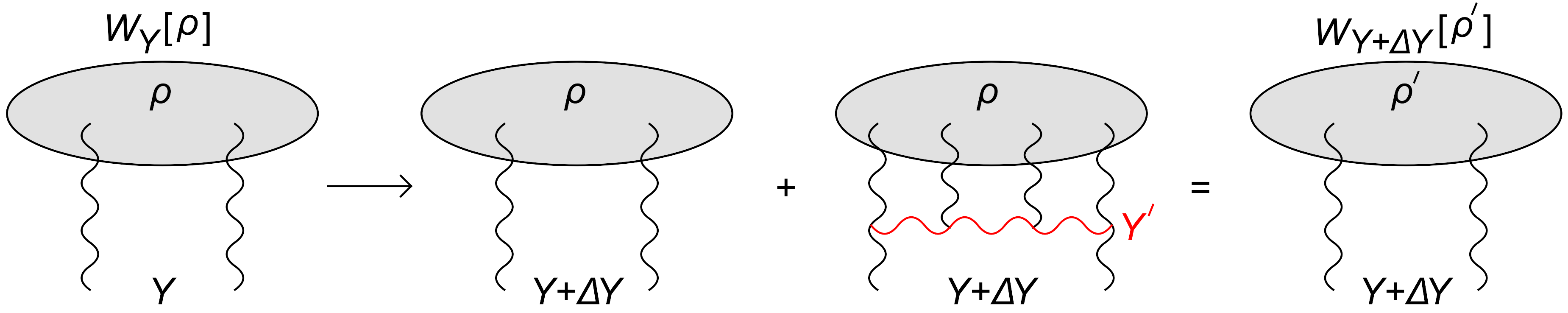
- Calculate observable $\mathcal{O}(\mathcal{A}) = \mathcal{O}(\rho)$

$$\langle O[\rho]\rangle_Y = \int \mathcal{D}\rho\, W_Y[\rho] O[\rho]$$

$W_Y[\rho]$ = probability distribution of color sources at rapidity Y

The JIMWLK Equation (RGE) (1/2)

- Increase rapidity $Y \to Y + \Delta Y$
  Previously slow modes now become fast
  Integrate to include them in source
  Obtain change of $W_Y[\rho]$
- Leading order in $\bar{\alpha}_s \ln(1/x)$
  All orders in classical fiels $\mathcal{A}[\rho] \rightsquigarrow$
  Resum $\bar{\alpha}_s Y$ terms in presence of strong color field
- Still a classical theory at $Y + \Delta Y$



The JIMWLK Equation (RGE) (2/2)

- Renormalization Group Evolution Equation (JIMWLK)

$$\frac{\partial}{\partial Y} W_Y[\rho] = -H\left[\rho, \frac{\mathrm{d}}{\mathrm{d}\rho}\right] W_Y[\rho]$$

- Hamiltonian better expressed in terms of (covariant gauge) color field $\alpha(x^-, \boldsymbol{x}) \equiv A^+(x^-, \boldsymbol{x})$

$$H = -\frac{1}{16\pi^3} \int_{\boldsymbol{uvz}} \mathcal{M}(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{z}) \left[1 + \widetilde{V}^\dagger_{\boldsymbol{u}} \widetilde{V}_{\boldsymbol{v}} - \widetilde{V}^\dagger_{\boldsymbol{u}} \widetilde{V}_{\boldsymbol{z}} - \widetilde{V}^\dagger_{\boldsymbol{z}} \widetilde{V}_{\boldsymbol{v}}\right]^{ab} \frac{\delta}{\delta \alpha^a_\infty(\boldsymbol{u})} \frac{\delta}{\delta \alpha^b_\infty(\boldsymbol{v})}$$

  $x^- \sim 1/k^+ \to \infty$: “Action” takes place in last layer of longitudinal extent

- Wilson lines arise from propagator of integrated modes

$$\widetilde{V}^\dagger_{\boldsymbol{x}}[\alpha] = \mathrm{P} \exp\left[\mathrm{i}\, g \int_{-\infty}^{\infty} \mathrm{d}x^- \alpha^a(x^-, \boldsymbol{x})\, T^a\right]$$

The “Observables”

- Hilbert space : Gauge invariant operators built from Wilson lines

$$\mathcal{O}[\alpha] = \mathrm{tr}\big(V^{\dagger}_{\boldsymbol{x}_1} V_{\boldsymbol{x}_2} V^{\dagger}_{\boldsymbol{x}_3} V_{\boldsymbol{x}_4} \dots\big)\, \mathrm{tr}\big(V^{\dagger}_{\boldsymbol{y}_1} V_{\boldsymbol{y}_2} \dots\big) \dots$$

- Indeed: Left moving quark with eikonal trajectory

$$\bar{\psi}(x')\,\gamma^{-} A^{+}(x')\,\psi(x') \to \delta^{(2)}(\boldsymbol{x}' - \boldsymbol{x})\,\delta(x'^{+})\,A^{+}(x')$$

S-matrix $\to$ Wilson line in fundamental represenation

- Scatter single dipole off target

$$\begin{aligned} S(\boldsymbol{x},\boldsymbol{y}) &= \frac{1}{N_c}\mathrm{tr}\big(V^{\dagger}_{\boldsymbol{x}} V_{\boldsymbol{y}}\big) = 1 - T(\boldsymbol{x},\boldsymbol{y}) \\ &= 1 - \frac{g^2}{4N_c}\left[\alpha^a(\boldsymbol{x}) - \alpha^a(\boldsymbol{y})\right]^2 + \mathcal{O}(g^3) \\ &\equiv 1 - T_0(\boldsymbol{x},\boldsymbol{y}) + \mathcal{O}(g^3) \end{aligned}$$

The Balitsky Equations

- Evolution of observables

$$\frac{\partial \langle \mathcal{O} \rangle}{\partial Y} = \int \mathcal{D}\alpha \, W_Y[\alpha] H \, \mathcal{O} = \langle H \, \mathcal{O} \rangle$$

- First Balitsky equation

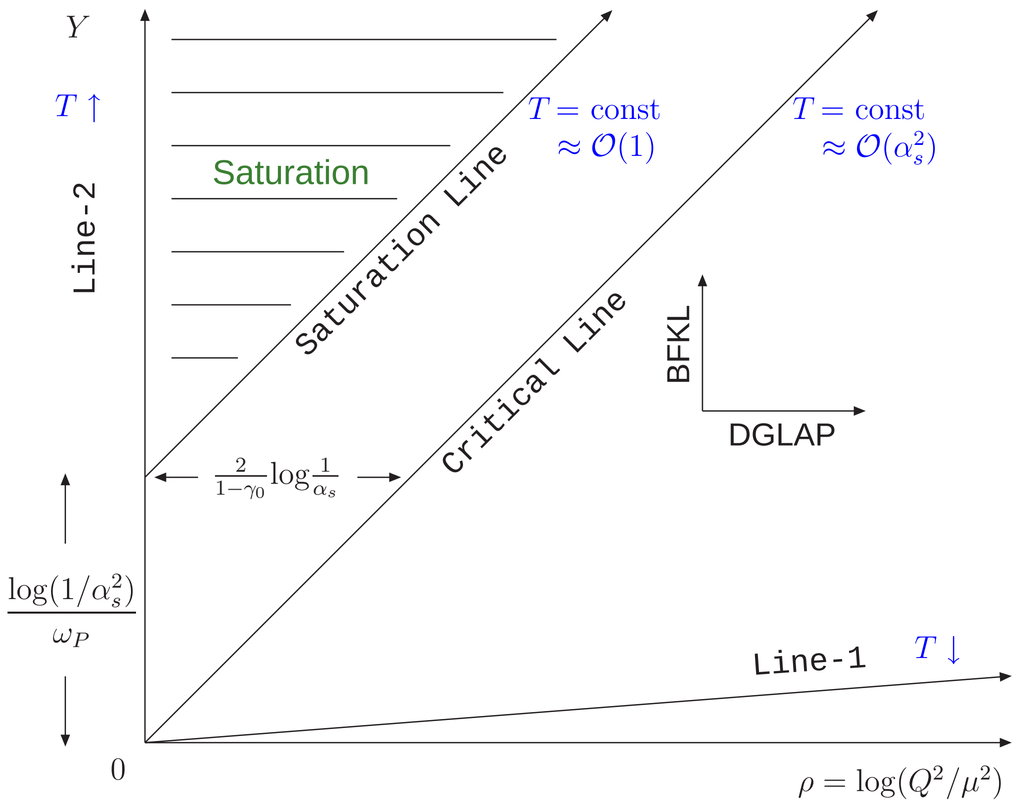
$$\frac{\partial \langle S_{\boldsymbol{xy}} \rangle}{\partial Y} = \frac{\bar{\alpha}_s}{2\pi} \int_{\boldsymbol{z}} \mathcal{M}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) \left[\langle S_{\boldsymbol{xz}} S_{\boldsymbol{xz}} \rangle - \langle S_{\boldsymbol{xy}} \rangle \right]$$

- Second Balitsky Equation

$$\frac{\partial \langle S_{\boldsymbol{xz}} S_{\boldsymbol{zy}} \rangle}{\partial Y} = \left\langle \frac{\partial S_{\boldsymbol{xz}}}{\partial Y} S_{\boldsymbol{zy}} \right\rangle + \left\langle S_{\boldsymbol{xz}} \frac{\partial S_{\boldsymbol{zy}}}{\partial Y} \right\rangle + \mathcal{O}\left(\frac{\mathrm{tr}(6V)}{N_c^3} \right)$$

- Projectile evolution
  One dipole $\rightarrow$ two dipoles $\rightarrow$ three dipoles + non-dipolar state $\rightarrow \cdots$
- Infinite hierarchy, factorization not justified, but consistent at large-N_c

The Saturation Momentum (1/3)



- Increase momentum, increase rapidity so that $T = \text{const}$
- Line-1: DGLAP $\gamma \to 0$
- Line-2: Hard Pomeron Intercept $\gamma = 1/2$
- Saturation-Line: $0 < \gamma_s < 1/2$

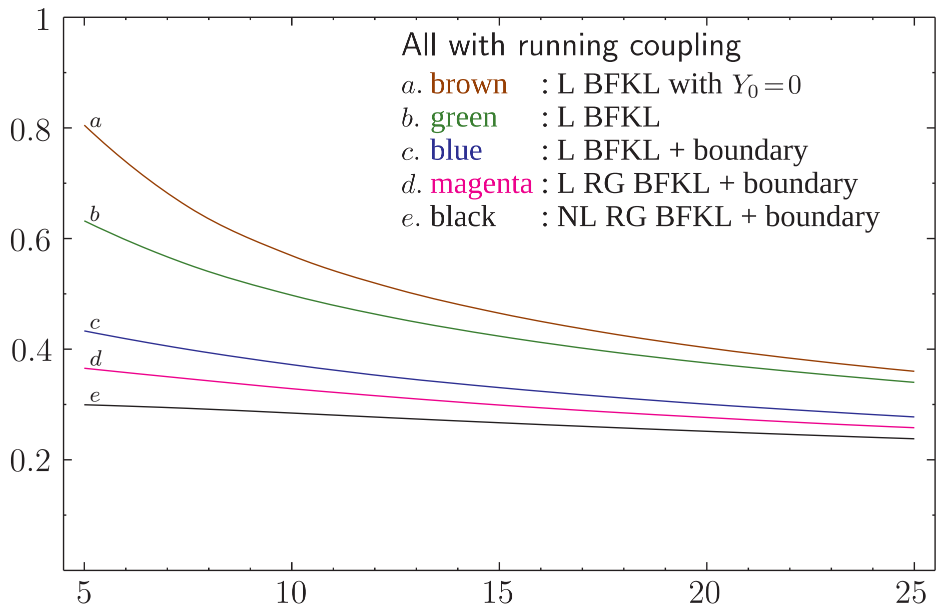
The Saturation Momentum (2/3)

- Can we use BFKL dynamics?
- Yes, but put absorptive boundary $\rightsquigarrow$
  Cuts diffusive paths to saturation, mimics non-linear term
- Require constant T and saddle point
  - $\chi_0(\gamma_s) + (1-\gamma_s)\chi_0'(\gamma_s) = 0 \Rightarrow \gamma_s = 0.372$
  - $T = \left(\frac{Q_s^2}{Q^2}\right)^{1-\gamma_s} \left(\ln \frac{Q^2}{Q_s^2} + \mathrm{c}\right)$ "Scaling form"

    Exact eigenfunction, valid up to diffusion radius $\sim \sqrt{Y}$ towards UV
  - $\lambda_s \equiv \frac{\mathrm{d} \ln Q_s^2}{\mathrm{d}Y} = \bar{\alpha}_s \frac{\chi_0(\gamma_s)}{1-\gamma_s} - \frac{3}{2(1-\gamma_s)} \frac{1}{Y} = 4.88 \bar{\alpha}_s - \frac{2.39}{Y}$
- Confirmed by rigorous analysis of non-linear equations
  (Calculated $1/Y^{3/2}$ term)
- Full JIMWLK on lattice: Almost same $\rightsquigarrow$ Factorization (at large-N_c)

The Saturation Momentum (3/3)

- Next to leading BFKL (collinearly improved) + boundary



- Coupling decreases along saturation line $\rightsquigarrow$ Running is dominant effect
  Analytic expression (Line-c)

$$\lambda_s = \frac{1.80}{\sqrt{(Y+Y_0)}} - \frac{0.893}{(Y+Y_0)^{5/6}}$$

- Full NLO result: Close to phenomenology $\lambda_s \simeq 0.3$

Deficiencies of Balitsky-JIMWLK

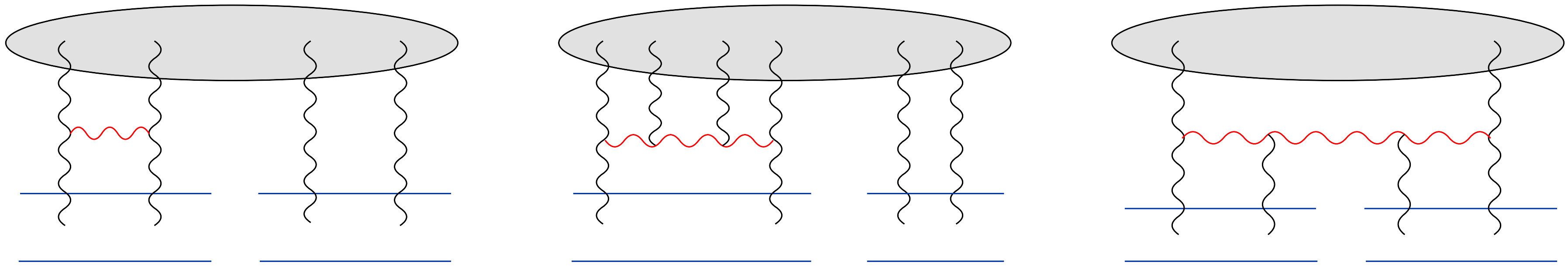
- Extreme sensitivity to the UV:
  Reconstructing solution in two (or more) steps by completeness
  Contributions from momenta up to $\ln(Q^2/Q_s^2) \lesssim \sqrt{\bar{\alpha}_s \chi_0''(\gamma_s) Y}$
  Embarrassing: Some orders of magnitude in Q^2
- Violation of Unitarity (!)
  $\mathcal{O}(1) \sim T \sim \frac{1}{\alpha_s^2} T_1 T_2 \qquad \text{and for} \quad T_1 < \alpha_s^2 \quad \text{then} \quad T_2 > 1$
- Absence of Pomeron splittings:

$$\frac{\mathrm{d}\alpha^n}{\mathrm{d}Y} = H_{\mathrm{JIMWLK}}\, \alpha^n \sim \underbrace{\alpha\, \alpha ... \alpha}_{\geq 2} \frac{\delta}{\delta \alpha} \frac{\delta}{\delta \alpha} \sim \alpha^m \qquad \text{with} \quad m \geq n$$

  Two ladders merge, but how could we have them in the first place?
  - Nucleus target $\rightsquigarrow$ Many sources $\rightsquigarrow$ Many BFKL pomerons
    No more dynamics needed - Initial condition to be lost at high energy
  - Pomeron Splittings

The Missing Diagram(s)

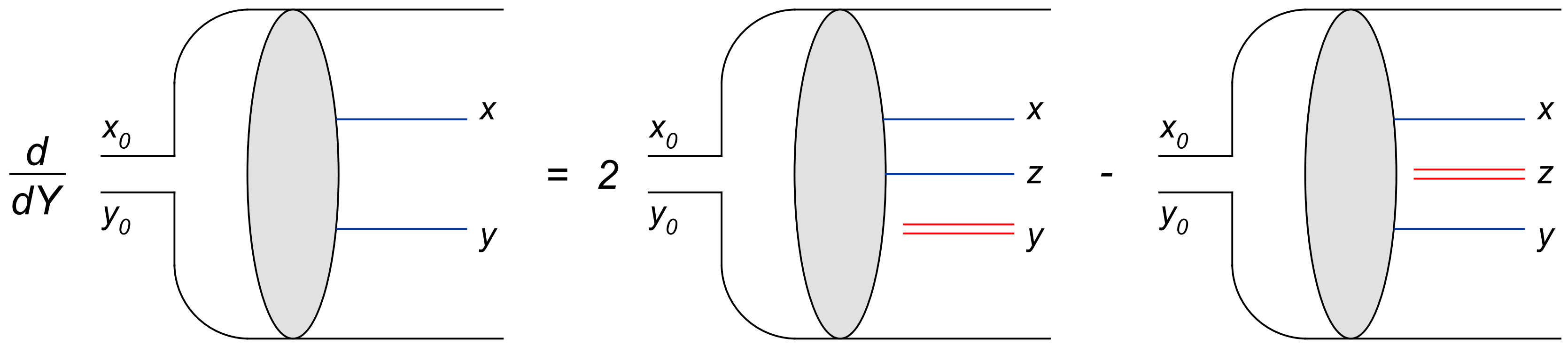
- Diagrammatic illustration of splitting



$\bar{\alpha}_s \Delta Y \mathcal{O}(\alpha_s^2 \varphi^2)$ $\bar{\alpha}_s \Delta Y \mathcal{O}(\alpha_s^3 \varphi^3)$ $\bar{\alpha}_s \Delta Y \mathcal{O}(\alpha_s^3 \varphi)$

- Third diagram not included in JIMWLK
- Important when $\varphi \sim \alpha_s \Rightarrow T \sim \alpha_s^2$, where JIMWLK has problems
- Low density region $\rightsquigarrow$ fluctuations
- “Measure” fluctuations: probe with two dipoles
- First Balitsky equation remains unchanged

- Evolution of dipole density



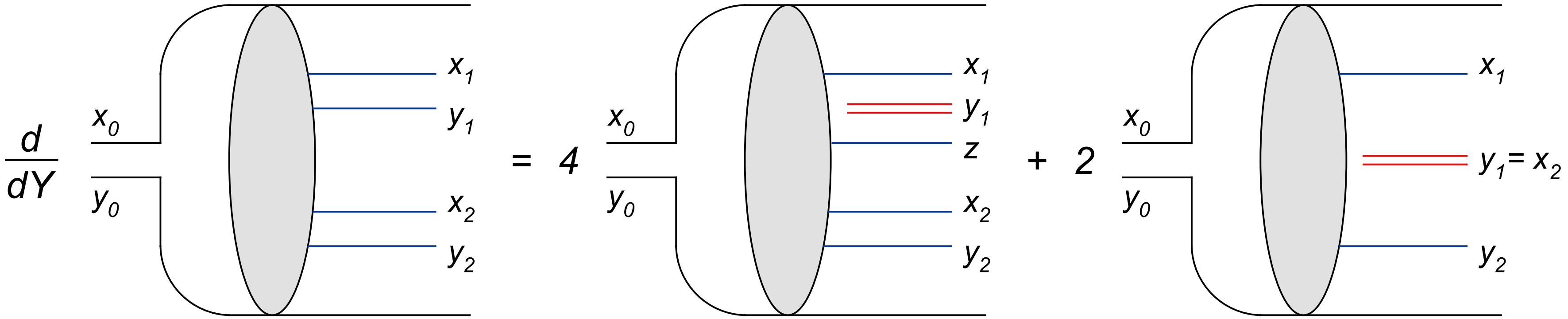
$$\frac{\partial\, n(\boldsymbol{x},\boldsymbol{y})}{\partial Y} = \frac{\bar{\alpha}_s}{2\pi}\int_{\boldsymbol{z}} \big[-\mathcal{M}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})\, n(\boldsymbol{x},\boldsymbol{y}) + \mathcal{M}(\boldsymbol{x},\boldsymbol{z},\boldsymbol{y})\, n(\boldsymbol{x},\boldsymbol{z}) + \mathcal{M}(\boldsymbol{z},\boldsymbol{y},\boldsymbol{x})\, n(\boldsymbol{z},\boldsymbol{y}) \big]$$

$$\equiv \frac{\bar{\alpha}_s}{2\pi}\int_{\boldsymbol{z}} \mathcal{K}_{\boldsymbol{xyz}} \otimes n(\boldsymbol{x},\boldsymbol{y})$$

- BFKL Equation for density

The Color Dipole Picture (2/2)

- Evolution of dipole-pair density



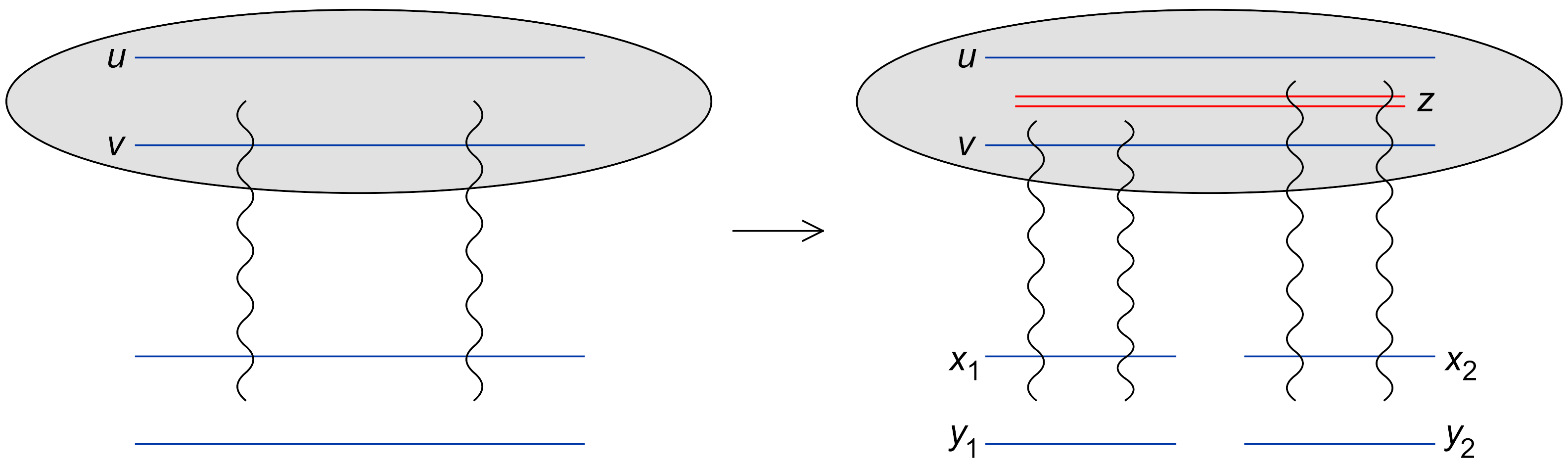
$$\frac{\partial\, n^{(2)}(\boldsymbol{x}_1,\boldsymbol{y}_1;\boldsymbol{x}_2,\boldsymbol{y}_2)}{\partial Y} = \frac{\bar{\alpha}_s}{2\pi}\left[\int_{\boldsymbol{z}} \mathcal{K}_{\boldsymbol{x}_1\boldsymbol{y}_1\boldsymbol{z}} \otimes n^{(2)}(\boldsymbol{x}_1,\boldsymbol{y}_1;\boldsymbol{x}_2,\boldsymbol{y}_2) \right.$$

$$\left. + \mathcal{M}(\boldsymbol{x}_1,\boldsymbol{y}_2,\boldsymbol{x}_2)\, n(\boldsymbol{x}_1,\boldsymbol{y}_2)\, \delta^{(2)}(\boldsymbol{x}_2-\boldsymbol{y}_1)\right] + 1 \leftrightarrow 2$$

- Multi-dipole density equations not consistent with factorization
  At low density $n^{(2)} \sim n$, rather than $n^{(2)} \sim n^2$

Pomeron Splittings

- Measure BOTH child dipoles $\rightsquigarrow$
  Evolution equation for dipole-pair scattering at large-N_c



$$\left.\frac{\partial T^{(2)}_{\boldsymbol{x}_1\boldsymbol{y}_1;\boldsymbol{x}_2\boldsymbol{y}_2}}{\partial Y}\right|_{\rm split} = \left(\frac{\alpha_s}{2\pi}\right)^2 \frac{\bar{\alpha}_s}{2\pi}\int_{\boldsymbol{uvz}} \mathcal{M}(\boldsymbol{u},\boldsymbol{v},\boldsymbol{z})\, \underbrace{\mathcal{A}_0(\boldsymbol{x}_1,\boldsymbol{y}_1|\boldsymbol{u},\boldsymbol{z})\mathcal{A}_0(\boldsymbol{x}_2,\boldsymbol{y}_2|\boldsymbol{z},\boldsymbol{v})}_{\rm Dip-Dip\ Scatt}\, \underbrace{\nabla^2_{\boldsymbol{u}}\nabla^2_{\boldsymbol{v}} T_{\boldsymbol{uv}}}_{\sim \rm Dip-density}$$

- Low density fluctuations are the seed for higher-point correlations
- Equivalent to Bartels' $1 \rightarrow 2$ Vertex

The (Large-N_c) Equations

- At large-N_c, only $1 \to 2$ process $\rightsquigarrow T^{(n)} \to T^{(n+1)}$
- Structure of the Equations (adding large-N_c Balitsky)

$$\frac{\mathrm{d}T}{\mathrm{d}Y} = T - T^{(2)}$$

$$\frac{\mathrm{d}T^{(2)}}{\mathrm{d}Y} = T^{(2)} - T^{(3)} + T$$

$$\cdot \quad \cdot \quad \cdot$$

$$\frac{\mathrm{d}T^{(n)}}{\mathrm{d}Y} = \underbrace{T^{(n)}}_{\text{BFKL}} - \underbrace{T^{(n+1)}}_{\text{merging}} + \underbrace{T^{(n-1)}}_{\text{splitting}}$$

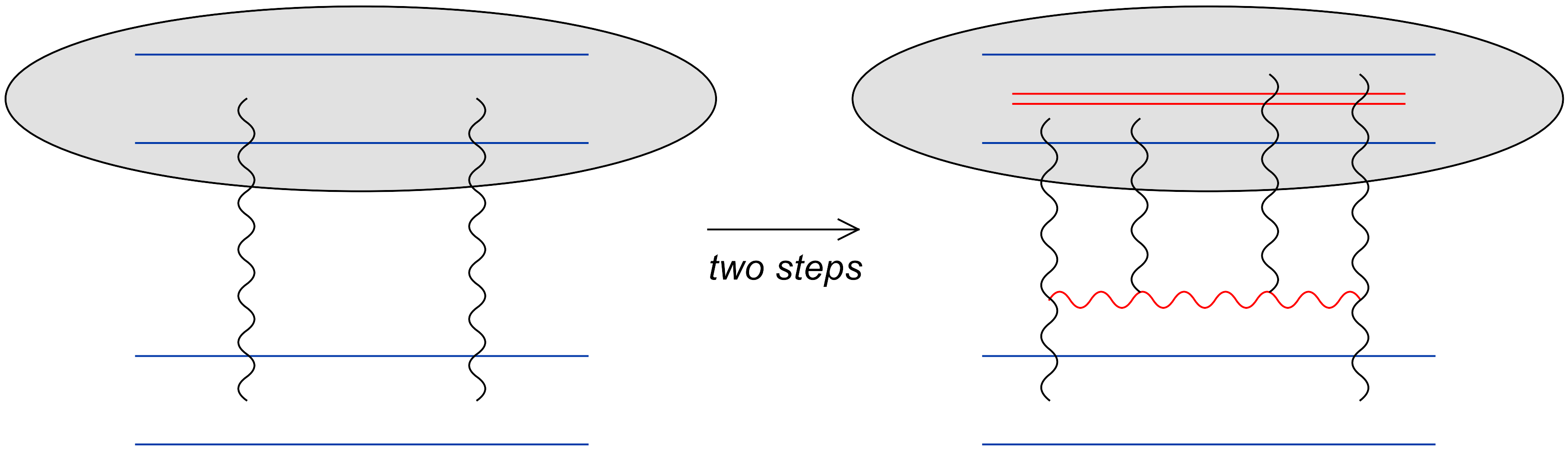
- Can be summarized in a Langevin Equation

  Certain approximations $\rightsquigarrow$ stochastic-FKPP equation

$$\frac{\mathrm{d}T}{\mathrm{d}Y} = T - T^2 + \sqrt{T}\,\nu \quad \text{with} \quad \langle \nu(Y)\nu(Y')\rangle = \delta(Y - Y')$$

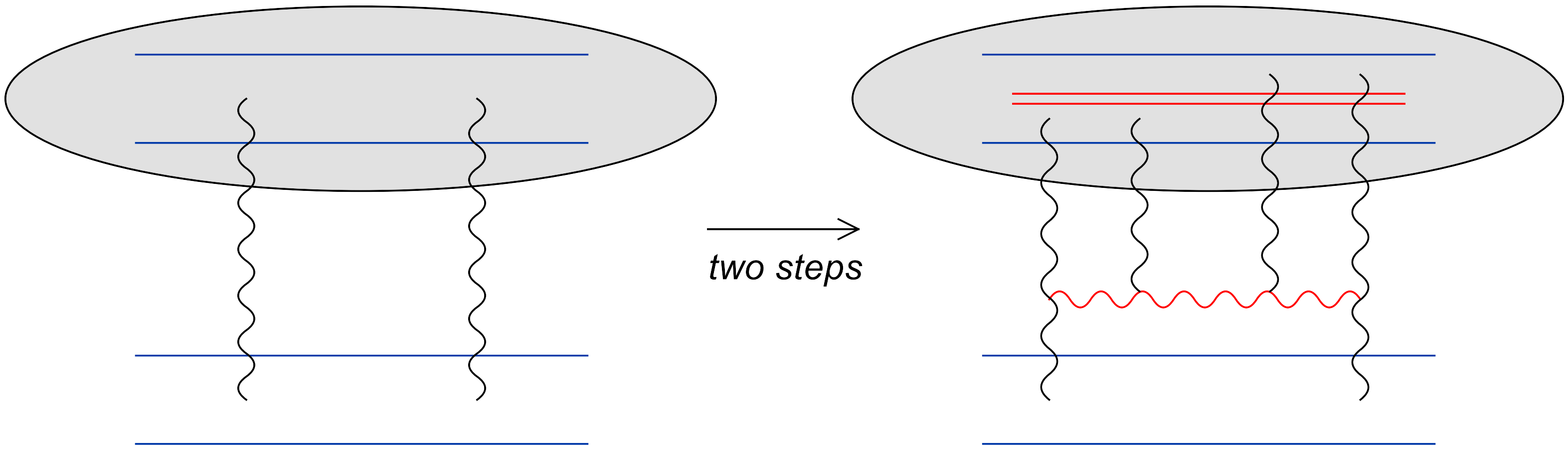
Pomeron Loops

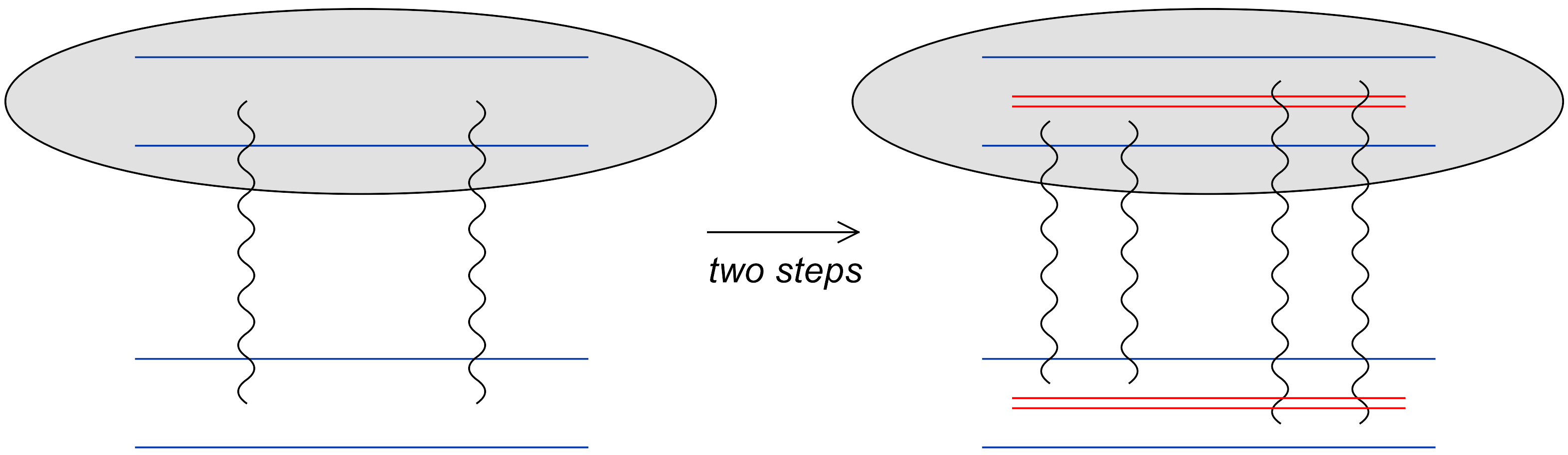
- Splittings + Mergings $\longrightarrow$ Loops



- This is a simple loop
  Pomeron Loops will be built through evolution
- On can construct “effective pomeron vertices"
  But system is non-linear $\rightsquigarrow$ Need to solve hierarchy

Duality (1/3)





- Merging in target $\leftrightarrow$ splitting in projectile
  Pomeron Loop manifestly symmetric
- Effective theory with both splittings and mergings is self-dual
  High density $\leftrightarrow$ Low density or Saturation $\leftrightarrow$ Fluctuation Duality
- Splitting Hamiltonian

$$\bar{H} \sim \alpha\, \alpha\, \underbrace{\frac{\delta}{\delta\alpha}\, \frac{\delta}{\delta\alpha} \cdots \frac{\delta}{\delta\alpha}}_{\geq 2}$$

Hamiltonian Approach

- A proposed "splitting" Hamiltonian at large-N_c

$$H^{\dagger}_{1\to 2} = -\frac{g^2}{16N_c^3}\,\frac{\bar{\alpha}_s}{2\pi}\int_{\boldsymbol{uvz}} \mathcal{M}(\boldsymbol{u},\boldsymbol{v},\boldsymbol{z})\,\rho^a_{\boldsymbol{u}}\,\rho^a_{\boldsymbol{v}}\left[\frac{\delta}{\delta\rho^b_{\boldsymbol{u}}}-\frac{\delta}{\delta\rho^b_{\boldsymbol{z}}}\right]^2\left[\frac{\delta}{\delta\rho^c_{\boldsymbol{z}}}-\frac{\delta}{\delta\rho^c_{\boldsymbol{v}}}\right]^2$$

  Two ρ's, four $\delta/\delta\rho'$s $\rightsquigarrow$ $1 \to 2$ process

- Charge density is related to dipole density

$$\rho^a(\boldsymbol{x})\,\rho^a(\boldsymbol{y}) = -g^2 N_c\,\bar{n}(\boldsymbol{x},\boldsymbol{y})$$

  Acting on $\bar{n}(\boldsymbol{x}_1,\boldsymbol{y}_1)\,\bar{n}(\boldsymbol{x}_2,\boldsymbol{y}_2)$ $\rightsquigarrow$ Splitting term in evolution

- Assume two-gluon exchange in scattering
  Acting on $T_0^{(2)}(\boldsymbol{x}_1,\boldsymbol{y}_1;\boldsymbol{x}_2,\boldsymbol{y}_2) \equiv T_0(\boldsymbol{x}_1,\boldsymbol{y}_1)\,T_0(\boldsymbol{x}_2,\boldsymbol{y}_2)$
  $\rightsquigarrow$ Splitting term in evolution

Duality (2/3)

- Scattering of two evolved dipoles in two gluon exchange

$$\langle S \rangle_Y = \int \mathcal{D}\alpha_{\rm R}\, \mathcal{D}\alpha_{\rm L}\, W_{Y-y}[\alpha_{\rm R}]\, W_y[\alpha_{\rm L}]\, \underbrace{\exp\left[{\rm i} \int_{\boldsymbol{z}} \rho^a_{\rm L}(\boldsymbol{z})\, \alpha^a_{\rm R}(\boldsymbol{z}) \right]}_{S}$$

  - S symmetric under ${\rm R} \leftrightarrow {\rm L}$; use $\nabla^2 \alpha_{\rm R/L} = -\rho_{\rm R/L}$, integrate by parts

- Lorentz (boost) invariance requires

$$\frac{{\rm d}\, \langle S \rangle_Y}{{\rm d}y} = 0 \quad \Rightarrow \quad H\left[\alpha, \frac{\delta}{{\rm i}\, \delta\alpha}\right] = H^{\dagger}\left[\frac{\delta}{{\rm i}\, \delta\rho}, \rho\right]$$

- Conceptual problem (with no answer):
  Splitting in R-wavefunction $\leftrightarrow$ Merging in L-wavefunction
  This is large-N_c, do we understand dipole “recombination”?

Duality (3/3)

- From duality condition

$$H^{\dagger}_{2\to 1} = \frac{g^2}{16N_c^3}\,\frac{\bar{\alpha}_s}{2\pi}\int_{\boldsymbol{uvz}} \mathcal{M}(\boldsymbol{u},\boldsymbol{v},\boldsymbol{z})\,[\alpha^a_{\boldsymbol{u}} - \alpha^a_{\boldsymbol{z}}]^2[\alpha^b_{\boldsymbol{u}} - \alpha^b_{\boldsymbol{z}}]^2\,\frac{\delta}{\delta\alpha^c_{\boldsymbol{u}}}\,\frac{\delta}{\delta\alpha^c_{\boldsymbol{v}}}$$

- Gives correct equations of motion with Hilbert space T_0's

$$H^{\dagger}_{2\to 1}\,T_0(\boldsymbol{x},\boldsymbol{y}) = \frac{\bar{\alpha}_s}{2\pi}\int_{\boldsymbol{z}} \mathcal{M}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})\,T_0^{(2)}(\boldsymbol{x},\boldsymbol{z};\boldsymbol{z},\boldsymbol{y})$$

- Can show that BFKL part $H^{\dagger}_0$ is self-dual
- Self-dual Hamiltonian generating correct evolution at large N_c

$$H^{\dagger} = H^{\dagger}_0 + H^{\dagger}_{1\to 2} + H^{\dagger}_{2\to 1}$$

Dual of JIMWLK

- JIMWLK : Expressed in terms of Wilson lines
  Involves $n \to 1$ processes with n arbitrary, finite N_c
  High density limit of full Hamiltonian
- Low density limit of full H is dual of JIMWLK

$$\frac{\delta}{\mathrm{i}\,\delta\alpha} \to \rho, \qquad \alpha \to \frac{\delta}{\mathrm{i}\,\delta\rho}, \qquad x^- \to x^+$$

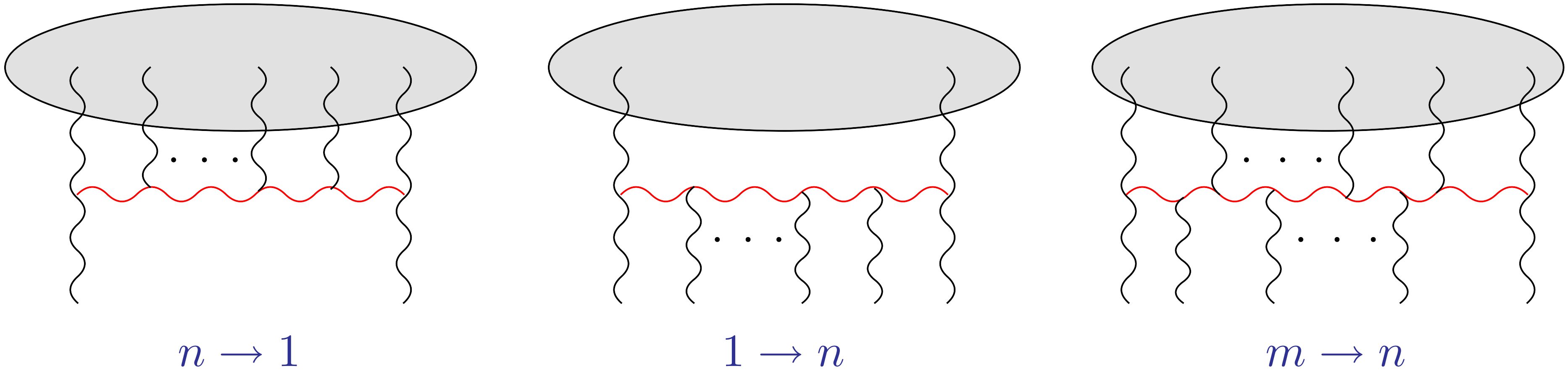
- Wilson lines, $1 \to n$ processes, finite N_c

$$\bar{H} = \frac{1}{16\pi^3} \int_{\boldsymbol{uvz}} \mathcal{M}(\boldsymbol{u},\boldsymbol{v},\boldsymbol{z})\, \rho^a_\infty(\boldsymbol{u})\rho^b_\infty(\boldsymbol{v}) \left[1 + W_{\boldsymbol{u}} W^\dagger_{\boldsymbol{v}} - W_{\boldsymbol{u}} W^\dagger_{\boldsymbol{z}} - W_{\boldsymbol{z}} W^\dagger_{\boldsymbol{v}}\right]^{ab}$$

$$W_{\boldsymbol{x}} = \mathrm{P}\exp\left[g \int_{-\infty}^{\infty} \mathrm{d}x^+ \frac{\delta}{\delta\rho^a(x^+,\boldsymbol{x})} T^a\right]$$

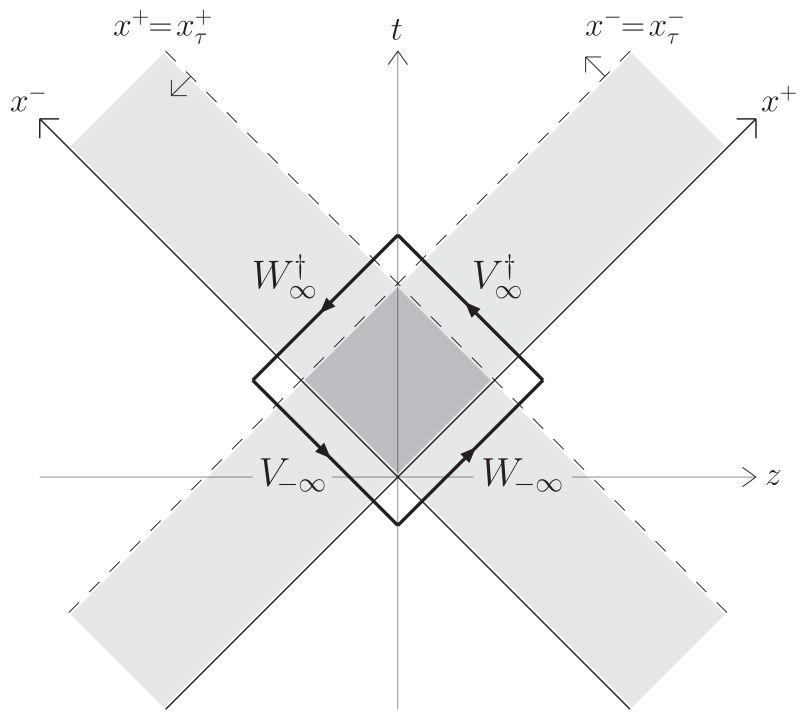
- Full Hamiltonian: NOT any simple interpolation of high and low density

Full Hamiltonian (1/2)



- Resume $n \to m$ in $\bar{\alpha}_s \ln(1/x)$ for arbitrary n, m
- Renormalization group in rapidity (from the beginning)
- $H_{\rm eff}$ expected to involve both $V^\dagger$ and W
- $H_{\rm eff}$ expected to be dual under $V^\dagger \leftrightarrow W$

Full Hamiltonian (2/2)



- $$H_{\rm eff} = \frac{1}{2\pi g^2 N_c} \int_{\boldsymbol{x}} {\rm Tr}\left[V^\dagger_\infty (\partial^i W_{-\infty})(\partial^i V_{-\infty}) W^\dagger_\infty\right] \;+\; \text{permutations}$$
- Three independent Wilson lines: $V^\dagger_\infty W_{-\infty} V_{-\infty} W^\dagger_\infty = 1$
- Expand W's to order g^2 $\rightsquigarrow$ JIMWLK
  Expand V's to order g^2 $\rightsquigarrow$ dual of JIMWLK

The Saturation Momentum Reloaded (1/4)

- Splittings in target in dilute region $\to$ Merging in projectile $\to$ UV boundary
- Confirmed by analogy to statistical physics
  Hierarchy $\rightsquigarrow$ Langevin $\rightsquigarrow$ sFKPP $\rightsquigarrow$ cuttof at α_s^2
- $\Delta = 1/(1-\gamma_s)\ln(1/\alpha_s^2)$ = separation of boundaries
  Within Δ, amplitude drops from $\mathcal{O}(1)$ to $\mathcal{O}(\alpha_s^2)$
- Look for a Y-independent BFKL solution

$$\left[\chi_0\left(1+\frac{\partial}{\partial z}\right) - \lambda_s \frac{\partial}{\partial z}\right] T = 0, \qquad z = \ln(Q^2/Q_s^2)$$

- Only real combination satisfying boundary conditions

$$T \sim \exp[-(1-\gamma_{\rm r})\, z] \sin\frac{\pi z}{\Delta}, \qquad \gamma_{\rm i} = \frac{\pi}{\Delta}$$

The Saturation Momentum Reloaded (2/4)

- Real part $\gamma_{\rm r}$ uniquely fixed in terms of $\gamma_{\rm i}$ or Δ or α_s

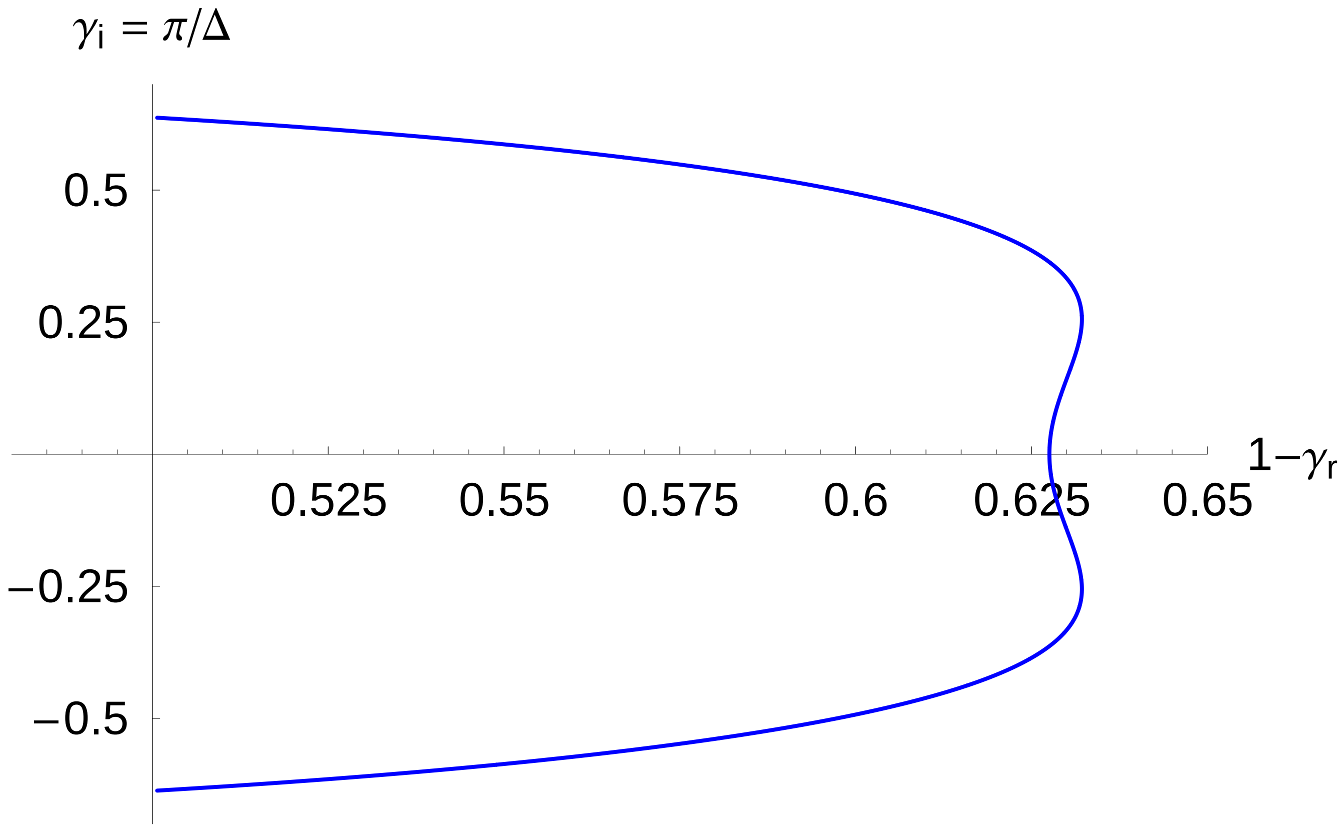
$$\lambda_s = \frac{\chi_0(\gamma)}{1-\gamma} \qquad \text{with} \qquad \mathrm{Im}(\lambda_s) = 0$$

- For large separation of boundaries $\Delta \gg 1 \Leftrightarrow \alpha_s \ll 1$

$$\frac{\lambda_s}{\bar{\alpha}_s} = \frac{\chi_0(\gamma_s)}{1-\gamma_s} - \frac{\pi^2(1-\gamma_s)\chi_0''(\gamma_s)}{2\ln^2(\alpha_s^2)} = 4.88 - \frac{150}{\ln^2(\alpha_s^2)}$$

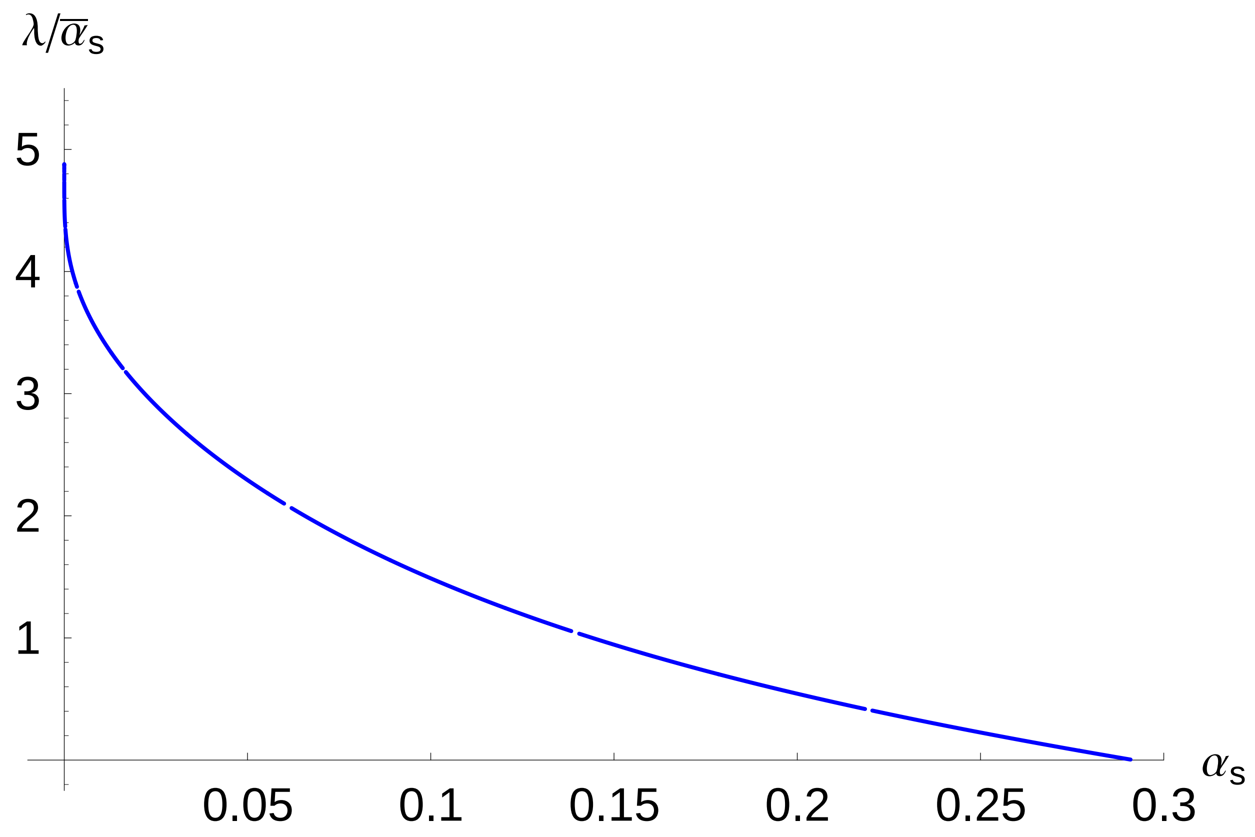
- Correction: parametrically suppressed, but coefficient huge
- Denominator : “Effective” transverse space (same in single boundary)
- Boundaries in BFKL too sharp
  Full equation has well-defined solution for reasonable values of α_s

The Saturation Momentum Reloaded (3/4)



- Real part of anomalous dimension: no significant change
- Reduces to $\gamma_s = 0.372$ when $\Delta \to \infty$

The Saturation Momentum Reloaded (4/4)



- “Speed” of saturation momentum: significant change
- Positive for reasonable values of coupling: $\lambda_s(\alpha_s \lesssim 0.3) > 0$
  But not really under control: Can change $\alpha_s \to \kappa\, \alpha_s$
- Reduces to $\lambda_s = 4.88\, \bar{\alpha}_s$ when $\Delta \to \infty$

Conclusion-Perspectives

- Evolution equations at high-density and low-density (large-N_c)
- Pomeron Loops:
  - Basic "building block" to reach unitarity
  - Free of divergencies
  - NO diffusion to IR, NO diffusion to UV (good for numerics)
  - More important than Next to Leading-BFKL corrections
- Go beyond multicolor limit (done)
- Self-dual effective theory
- Arbitrary density (semi-done)
- Phenomenology may change even at qualitative level