FIELD THEORY WITH A *V*-SHAPED POTENTIAL

H. Arodź

Institute of Physics, Jagiellonian University

Cracow School of Theoretical Physics
Zakopane, 2005

Collaborators: P. Klimas, T. Tyranowski.
Based on: Acta Phys. Pol. **B33** (2002) 1241 (nlin.ps/0201001); **B35** (2004) 625 (hep-th/0312036); **B36** (2005) 787 (cond-mat/0501112).

Plan

I.The system of bouncing pendulums

II. Static configurations

III. Periodic waves

IV. Finite size system

V. Summary and remarks

I.The system of bouncing pendulums

II. Static configurations

III. Periodic waves

IV. Finite size system

V. Summary and remarks

Ia. The pendulums

Ib. The pendulums - equations of motion

- Pendulums at the points $x_i = ia$, $i = -N, -N+1, ..., N-1, N$ (N=24.5 in the picture)
- Arm of length R, and a mass m at the free end.
- One degree of freedom per pendulum: the angle $\Phi(x_i, t)$ between the vertical direction and the arm. $\Phi(x_i, t) = 0$ corresponds to the upward position of the i-th pendulum
- Pendulums are connected by elastic strings, κ characterizes the elasticity of the string
- $|\Phi_i| \leq \Phi_0 < \pi$ due to the bounding rods (lines)

Equations of motion when $N = \infty$ and $\Phi(x_i, t) < \Phi_0$:

$$mR^2 \frac{d^2\Phi(x_i, t)}{dt^2} = mgR \sin\Phi(x_i, t) + \kappa \frac{\Phi(x_i - a, t) + \Phi(x_i + a, t) - 2\Phi(x_i, t)}{a}. \tag{1}$$

The gravitational force acting on the mass m, and the torque due to the elastic force from the strings.

Ib. The pendulums - continuum limit

$\Phi(x,t)$: interpolating function of continuous variables x, t.

The identity

$$\Phi(x_i - a, t) + \Phi(x_i + a, t) - 2\Phi(x_i, t) = \int_0^a ds_1 \int_{-a}^0 ds_2 \frac{\partial^2 \Phi(s_1 + s_2 + x, t)}{\partial x^2}\bigg|_{x=x_i}$$

The limit

$$a \to 0, \quad \kappa \to \infty, \quad \kappa a = constans, \quad N \to \infty$$

Eqs.(1) are replaced by

$$mR^2 \frac{d^2\Phi(x,t)}{dt^2} = mgR \sin\Phi(x,t) + \kappa a \frac{\partial^2 \Phi(x,t)}{\partial x^2}$$

Ic. The pendulums - the continuum limit, ctd.

Dimensionless variables:

$$\tau = \sqrt{\frac{g}{R}}\, t, \quad \xi = \sqrt{\frac{mgR}{\kappa a}}\, x$$

$$\frac{\partial^2 \Phi(\xi,\tau)}{\partial \tau^2} - \frac{\partial^2 \Phi(\xi,\tau)}{\partial \xi^2} - \sin \Phi(\xi,\tau) = 0$$

when

$$|\Phi(\xi,\tau)| < \Phi_0.$$

Elastic bouncing from the bounding rods:

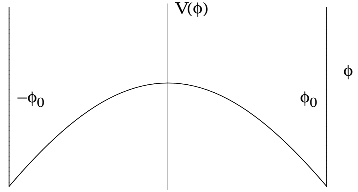
$$\frac{\partial \Phi(\xi,\tau)}{\partial \tau} \rightarrow -\frac{\partial \Phi(\xi,\tau)}{\partial \tau} \quad \text{when} \quad \Phi(\xi,\tau) = \pm\Phi_0.$$

A. C. Scott (1969): a system of elastically coupled pendulums to demonstrate sinus-Gordon solitons.
Our system has very different properties due to the bounding rods.

Id. The pendulums - potential and ground states

The corresponding Lagrangian: $L = \frac{1}{2}(\partial_\tau \Phi)^2 - \frac{1}{2}(\partial_\xi \Phi)^2 - V(\Phi)$,

$$V(\Phi) = \begin{cases} \cos\Phi - 1 & \text{for} \quad |\Phi| \le \Phi_0 \\ \infty & \text{for} \quad |\Phi| > \Phi_0. \end{cases}$$



- Two degenerate ground states: $\Phi = \pm\Phi_0$
- Spontaneously broken Z_2 symmetry: $\Phi \to -\Phi$
- Topological sectors
- $V'(\pm\Phi_0) \neq 0!$ - V-shaped potential

I.The system of bouncing pendulums

II. Static configurations

III. Periodic waves

IV. Finite size system

V. Summary and remarks

IIa. The compacton

- The ground states $\Phi = \pm\Phi_0$.
- Static topological defect?

Assumption: $\Phi_0 \ll 1$. Then $\sin\Phi \cong \Phi$, and

$$\frac{\partial^2\Phi(\xi,\tau)}{\partial\tau^2} - \frac{\partial^2\Phi(\xi,\tau)}{\partial\xi^2} - \Phi(\xi,\tau) = 0$$

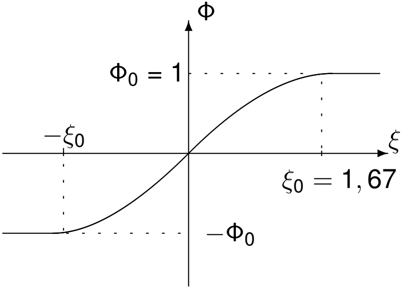
(when $|\Phi| < \Phi_0$)

$$\Phi_c(\xi) = \begin{cases} -\Phi_0 & \text{if} \quad \xi \leq -\xi_0 \\ \Phi_0 \sin\xi & \text{if} \quad -\xi_0 \leq \xi \leq \xi_0 \\ +\Phi_0 & \text{if} \quad \xi \geq \xi_0. \end{cases}$$

In general case: an elliptic function instead of $\sin\xi$.
Lack of exponential tails! Compacton.
One can combine compactons and anti-compactons ($-\Phi_c(\xi)$) into a static chain (because of the zero-range forces)

IIb. The compacton and anti-compacton

IIc. The length of the compacton



The length of the compacton at rest:

$$L \cong \pi \sqrt{\frac{\kappa a}{mgR}} \left(1 + \frac{\Phi_0^2}{16} + \dots\right)$$

when $\Phi_0 \ll 1$, or

$$L \cong 2\sqrt{\frac{\kappa a}{mgR}} \ln \frac{4}{\pi - \Phi_0}$$

when $\Phi_0 \to \pi -$.

IId. The lack of tails and $V' \neq 0$

$$\partial_\xi^2 \Phi - V'(\Phi) = 0,$$

$$\partial_\xi \Phi = \sqrt{2\,(V(\Phi) - V(\Phi_0))}$$

Φ approaches Φ_0 :

$$V(\Phi) - V(\Phi_0) = V'(\Phi_0)(\Phi - \Phi_0) \\ + \frac{1}{2} V''(\Phi_0)(\Phi - \Phi_0)^2 + \frac{1}{3!} V'''(\Phi_0)(\Phi - \Phi_0)^3 + \ldots .$$

The first term is dominating when $\Phi \to \Phi_0 -$.

$$\Phi(\xi) = \Phi_0 - \delta\Phi(\xi),$$

where $\delta\Phi \geq 0$.

$$\partial_\xi \delta\Phi = -\sqrt{2|V'(\Phi_0)|}\sqrt{\delta\Phi}.$$

($V'(\Phi_0)$ is defined as the limit from the side of $\Phi < \Phi_0$).

IId. The lack of tails and $V' \neq 0$, ctd.

General solution:

$$\delta\Phi(\xi) \cong \frac{1}{2}|V'(\Phi_0)|(\xi_0 - \xi)^2,$$

where ξ_0 is an arbitrary constant.
The parabolic approach to the ground state value of the field Φ. This value is reached at $\xi = \xi_0$ exactly.
The parabolic approach is due to the fact that $V'(\Phi_0) \neq 0$.
$V' < 0$ at $\Phi = \Phi_0$ implies a threshold for a force which could move pendulum from the bounding line upward - it has to be strong enough.

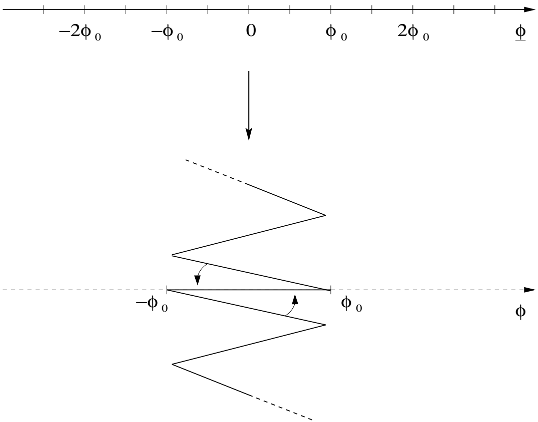
The well-known exponential tails are obtained when $V'(\Phi_0) = 0$ and $V''(\Phi_0) > 0$. In this case

$$\delta\Phi(\xi) \cong c_0 \exp(-\sqrt{V''(\Phi_0)}\xi),$$

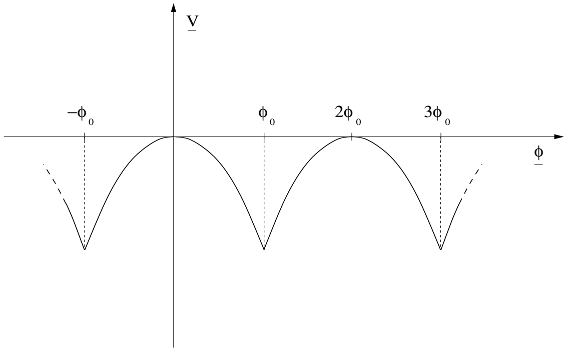
c_0 is a constant.

IIIa. The folding transformation

The bouncing condition $\Rightarrow$ discontinuity of velocities of pendulums.
'Unfolded' model: a new model with a field $\underline{\Phi}(\xi, \tau)$ such that $\partial_\tau \underline{\Phi}$ is continuous in τ. $\underline{\Phi}$ can take arbitrary real values.
The relation between Φ and $\underline{\Phi}$:



IIIb. The unfolded model



$$\underline{V}(\underline{\Phi}) = -\frac{\Phi_0^2}{6}\left[1 - \frac{12}{\pi^2}\cos(\frac{\pi}{\Phi_0}\underline{\Phi}) + \frac{12}{\pi^2}\sum_{n=2}^{\infty}\frac{(-1)^n}{n^2}\cos(\frac{n\pi}{\Phi_0}\underline{\Phi})\right]$$

Non-analytic perturbation of the well-known sinus-Gordon model

IIIc. Spatially homogeneous motions

$$\Phi(\tau): \qquad \frac{d^2\Phi}{d\tau^2} = \Phi \quad \text{when} \quad |\Phi| < \Phi_0 \ll 1$$

In the unfolded model: oscillations around a minimum of $\underline{V}$

$$\underline{\Phi} = \Phi_0 + \Phi_0\, \epsilon(\tau), \qquad |\epsilon| < 1$$

$$\boxed{\ddot{\epsilon} = \epsilon - \text{sign}(\epsilon)}$$

Nonlinear equation for small oscillations around the ground state!
Solutions:

$$\Phi(\tau) = \Phi_m \cosh(\tau - \tau_0), \qquad 0 < \Phi_m < \Phi_0.$$

The reflection from the rod occurs at τ_r such that $\Phi(\tau_r) = \Phi_0$.
Patching such solutions together $\rightarrow$ solution periodic in τ,
$T_{osc} = 4\, \text{arcosh}(\Phi_0/\Phi_m)$.
'Flights' above the potential hills:

$$\Phi(\tau) = \pm u \sinh(\tau - \tau_0),$$

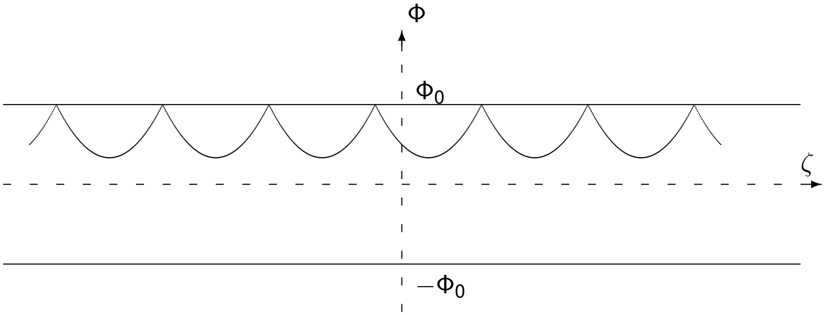
$\tau_0, u > 0$ - constants. The patching yields another periodic (in τ) solution, $T_{fl} = 4\, \tau_q, \quad \tau_q = \text{arsinh}(\Phi_0/u)$.

IIId. Lorentz boost

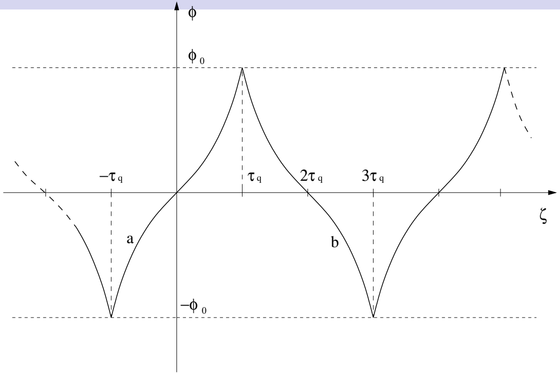
'Lorentz' symmetry of the evolution equation $\Rightarrow$ the substitution

$$\tau \rightarrow \zeta = \frac{v\tau - \xi}{\sqrt{v^2 - 1}}$$

yields another solution: the infinite wave, periodic in ξ and τ.
$v > 1$ - the phase velocity, $1/v < 1$ - the group velocity
The wave based on one rod:



IIIe. Bouncing between two rods



$$\zeta = \frac{v\tau - \xi}{\sqrt{v^2 - 1}}$$

Dispersion relations for the waves:

$$\omega^2 - k^2 = \mu^2, \quad \mu^2 \in (0, \infty).$$

I.The system of bouncing pendulums

II. Static configurations

III. Periodic waves

IV. Finite size system

V. Summary and remarks

IVa. The boundary conditions

The two outermost pendulums are kept in the upward position by an external force: $\Phi(x_{-N}, t) = 0, \;\; \Phi(x_N, t) = 0.$

IVb. Z_2 symmetry breaking transition

When

$$\frac{\pi^2 \kappa a}{4mgR(aN)^2} > 1$$

$\Phi = 0$ is the stable state!

If this condition is not satisfied, e.g., κa is too small, small perturbations of the stated $\Phi = 0$ grow exponentially - the system evolves towards a new stable state.

Then there are two ground states - one just shown, the other follows from it by the Z_2 transformation

$$\Phi \to -\Phi.$$

The condition above means that the system is too short to host the pair 1/2-compacton + 1/2-anticompacton at the boundaries.

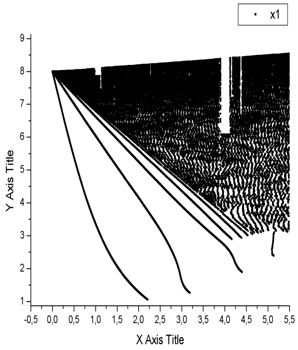
Changing κ or R one can trigger the Z_2 symmetry breaking transition.

The final state may contain several topological defects.

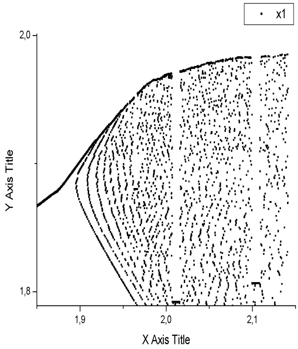
IVc. Example of the final state

IVd. Radiation from expanded (squized) 1/2-compacton

expanded compacton



squized compacton



↑ position, → time

I.The system of bouncing pendulums

II. Static configurations

III. Periodic waves

IV. Finite size system

V. Summary and remarks

Summary and remarks

- V-shaped potential:
  - compactons
  - only nonlinear oscillations around the ground states
  - transfer of energy to large momentum modes

* * *

- Interaction of compacton with anti-compacton with precise initial data (in particular, better fractals)
- Dynamics of Z_2 symmetry breaking phase transition (# of defects)
- Propagation of radiation
- Discrete system of pendulums bouncing from the rods (system with UV cutoff)
- Quantum version of the model (spectrum)