Kähler geometry and supersymmetry

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Supersymmetry and geometry

$D = 4$ supersymmetric field theories with chiral fermions

$\rightarrow N = 1$ supersymmetry

Kähler geometry

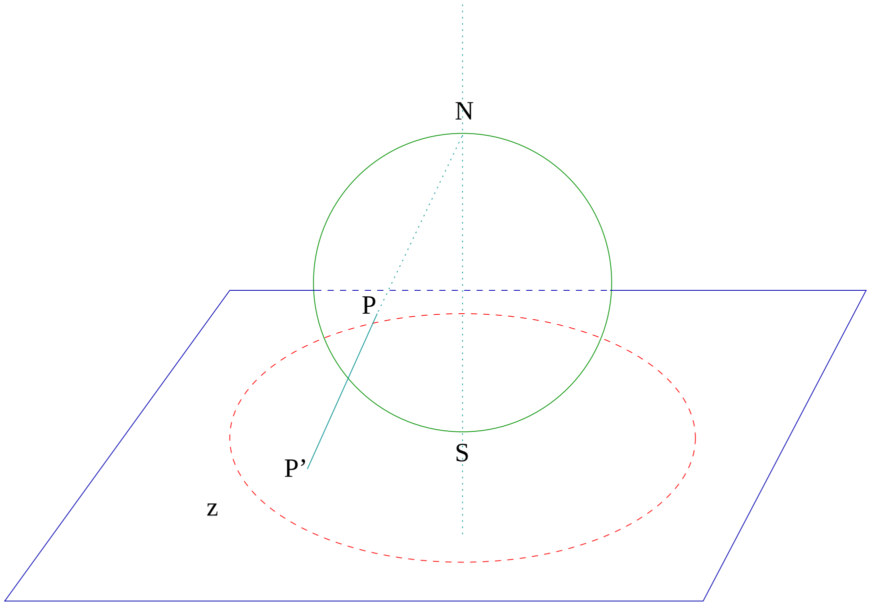
Lecture 1: The supersymmetric sphere and its generalizations

Lecture 2: Applications to theories of quarks and leptons

Earlier work by:
- Bagger and Witten
- Ong
- Buchmüller, Lerche and Yanagida
- Buras and Słominski
- Achiman, Aoyama and van Holten
- Ellwanger
- ...

This lecture based on recent work by
Groot Nibbelink, Nyawelo, Riccioni and van Holten

The sphere and the complex plane



Map of the sphere with diameter $2R = 1$ to the complex plane; associates a complex number z with any $P \neq N$ on the sphere. Any line element ds^2 on the sphere can be expressed in the complex plane as

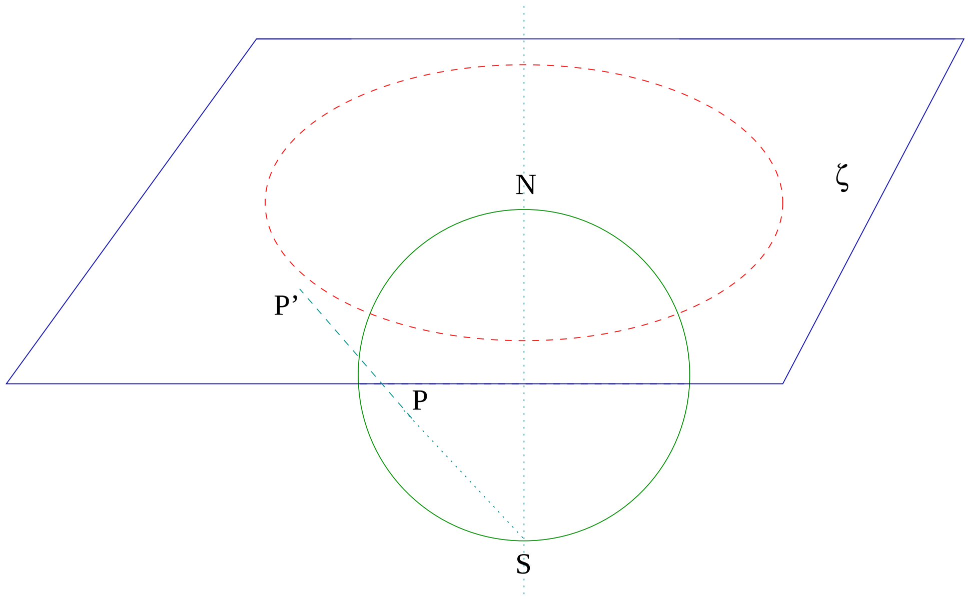
$$ds^2 = R^2 \left(d\theta^2 + \sin^2\theta \, d\varphi^2 \right)$$

$$\stackrel{R=1/2}{=} \frac{d\bar{z}dz}{(1+\bar{z}z)^2} = g_{z\bar{z}} \, d\bar{z}dz$$

The metric $g_{z\bar{z}}$ can be derived from a real scalar function

$$g_{z\bar{z}} = \frac{1}{(1+\bar{z}z)^2} = \frac{\partial^2 K}{\partial\bar{z}\partial z}$$

where $K(\bar{z}, z)$ is the *Kähler potential*: $K = \ln(1+\bar{z}z)$.



Second map of the sphere to the complex plane tangent in N; includes all points except S. Related to previous one by the holomorphic co-ordinate transformation

$$\zeta = \frac{1}{z}, \qquad \bar{\zeta} = \frac{1}{\bar{z}}$$

Proof:

- $N \leftrightarrow S$ and {interior of unit disk} $\leftrightarrow$ {exterior of unit disk};
- the Kähler potential transforms into

$$\begin{aligned} K(\overline{\zeta}, \zeta) &= \ln(1 + \overline{\zeta}\zeta) = \ln\left(1 + \frac{1}{\bar{z}z}\right) \\ &= \ln(1 + \bar{z}z) + F(z) + \bar{F}(\bar{z}) \end{aligned}$$

where $F(z) = -\ln z + i\pi/2$,

Symmetries of the sphere

The sphere is invariant under rotations around 3 independent axes. An infinitesimal rotation around the (x, y, z)-axes is represented by an inifnitesimal co-ordinate transformation in the complex plane:

$$z' = z + \delta z, \qquad \delta z = i\varphi z + \frac{\theta}{2}\left(1 + z^2\right) + \frac{i\eta}{2}\left(1 - z^2\right)$$

These transformations leave the line element invariant:

1.

$$1 + \bar{z}' z' = (1 + \bar{z} z)\left(1 + \frac{1}{2}(\theta - i\eta)\, z + \frac{1}{2}(\theta + i\eta)\, \bar{z}\right)$$

2.

$$\begin{aligned} K(\bar{z}', z') &= \ln\left(1 + \bar{z}' z'\right) \\ &= K(\bar{z}, z) + f(z) + \bar{f}(\bar{z}) \end{aligned}$$

with

$$f(z) = \frac{i}{2}\varphi + \frac{1}{2}(\theta - i\eta) z$$

The sphere as a coset

Let a be an element of $SU(2)$:

$$aa^\dagger = 1 \quad \to \quad a = \frac{e^{i\alpha\,\tau_3}}{\sqrt{1+\bar{z}z}}\begin{pmatrix} 1 & z \\ -\bar{z} & 1 \end{pmatrix}$$

The class of elements $a(\bar{z}, z; \alpha)$ differing only in the value of α are all related by a $U(1)$ transformation:

$$a(\bar{z}, z; \alpha + \beta) = e^{i\beta\,\tau_3}\, a(\bar{z}, z; \alpha)$$

Identify all elements related by a $U(1)$ transformation
$\to$ coset $SU(2)/U(1)$.

We can pick one representative element from each class by choosing $\alpha = 0$

$$\to \quad \tilde{a}(\bar{z}, z) = a(\bar{z}, z; 0) = \frac{1}{\sqrt{1+\bar{z}z}}\begin{pmatrix} 1 & z \\ -\bar{z} & 1 \end{pmatrix}$$

[Observe that the element

$$\tilde{a}_\infty = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

is missing; it can be recovered by making a second map from coset to complex plane related by $(\zeta = 1/z, \overline{\zeta} = 1/\bar{z})$.]

Symmetries of the coset

The symmetries of the sphere must also be symmetries of the coset. The rotations of the spehere can be reobtained as follows:

Under an $SU(2)$ transformation $\tilde{a} \to a_g = \tilde{a} g$ the product is again an element of $SU(2)$, generally in a different equivalence class and in a different gauge. The gauge can be restored by a g-dependent and z-dependent $U(1)$ transformation $h_g(\bar{z}, z)$:

$$\tilde{a}_g = h_g a_g = h_g \tilde{a} g = \frac{1}{\sqrt{1 + \bar{z}_g z_g}} \begin{pmatrix} 1 & z_g \\ -\bar{z}_g & 1 \end{pmatrix}$$

where $(\bar{z}_g, z_g) = (\bar{z}', z')$ if

$$g \simeq \begin{pmatrix} 1 - i\varphi/2 & (\theta + i\eta)/2 \\ -(\theta - i\eta)/2 & 1 + i\varphi/2 \end{pmatrix} \in SU(2)$$

and

$$h_g(\bar{z}, z) \simeq \begin{pmatrix} 1 + i\alpha(\bar{z}, z) & 0 \\ 0 & 1 - i\alpha(\bar{z}, z) \end{pmatrix} \in U(1)$$

with

$$\alpha(\bar{z}, z) = (\bar{f}(\bar{z}) - f(z))/2$$

Dressing the sphere

The sphere can be dressed with many different structures. In physics they are usually called fields, in mathematics bundles over the sphere. The various types of fields/bundles are classified by their behaviour under rotations.

Examples

- scalar bundles: $S_g(\bar{z}_g, z_g) = S(\bar{z}, z)$
- covariant vector bundles:

$$V_g\, dz_g = V\, dz, \qquad \bar{V}_g\, d\bar{z}_g = \bar{V}\, d\bar{z}$$

- contravariant vector bundles:

$$W_g \frac{\partial}{\partial z_g} = W \frac{\partial}{\partial z}, \qquad \bar{W}_g \frac{\partial}{\partial \bar{z}_g} = \bar{W} \frac{\partial}{\partial z}$$

- complex line bundles

$$\Omega_g = e^{fg(z)}\Omega, \qquad \bar{\Omega}_g = e^{\bar{f}g(\bar{z})}\,\bar{\Omega}$$

- direct products of all such bundles

E.g.:

$$(e^K)_g = e^{K+fg+\bar{f}g}$$

The supersymmetric sphere

Supersymmetry associates with every complex scalar co-ordinate z an anti-commuting spinor ψ, transforming as a contravariant vector bundle on the sphere:

$$\psi_g = \frac{\partial z_g}{\partial z}\,\psi, \qquad \bar{\psi}_g = \bar{\psi}\,\frac{\partial \bar{z}_g}{\partial \bar{z}}$$

Under infinitesimal rotations

$$\delta\psi = (i\varphi + (\theta - i\eta)z)\,\psi = 2f_g(z)\psi$$

$$\delta\bar{\psi} = \bar{\psi}\,(-i\varphi + (\theta + i\eta)\bar{z}) = 2\bar{f}_g(\bar{z})\bar{\psi}$$

This is consistent with the definition of a superco-ordinate

$$\Phi = z + \Theta\psi + \ldots$$

transforming under linearized rotations as

$$\Phi_g \simeq (1 + i\varphi)\Phi + \frac{\theta}{2}\left(1 + \Phi^2\right) + \frac{i\eta}{2}\left(1 - \Phi^2\right)$$

From sphere to field theory

In mechanics one encounters the problem of a particle moving on the surface of a sphere $\simeq$ particle motion on the coset $SU(2)/U(1)$.

The action for this dynamical problem is

$$S = R^2 \int dt \, \left(\dot{\theta}^2 + \sin^2\theta \, \dot{\varphi}^2\right) \overset{(R=1/2)}{=} \int dt \, \frac{\dot{\bar{z}}\dot{z}}{(1+\bar{z}z)^2}$$

If there exists a degree of freedom $z(x)$ in every point in space, we get a field theory

$$S[\bar{z}, z] = \int d^D x \, \frac{\partial\bar{z}\cdot\partial z}{(1+\bar{z}z)^2}$$

The fields in this theory parametrize the coset $SU(2)/U(1)$.

Goldstone theorem If a continuous global symmetry group G is broken spontaneously to a proper subgroup H, then there is a massless scalar field for every broken generator in G. These fields parametrize the coset G/H.

• The field theory on the sphere describes the Goldstone bosons of spontaneous symmetry breaking

$$SU(2) \quad \rightarrow \quad U(1)$$

Supersymmetric Goldstone model

In 4-D space-time the supersymmetric extension of the field theory on the sphere exists and is obtained by extending the complex fields $(\bar{z}, z)$ to chiral superfields

$$\Phi(x, \Theta) = z(x) + \bar{\Theta}_R \psi_L(x) + \ldots$$

The superfield lagrangean

$$\mathcal{L} = \int d^2\Theta_L d^2\Theta_R \, K(\bar{\Phi}, \Phi) = \int d^2\Theta_L d^2\Theta_R \, \ln\left(1 + \bar{\Phi}\Phi\right)$$

has the component expansion

$$\mathcal{L} = \frac{\partial \bar{z} \cdot \partial z}{(1 + \bar{z}z)^2} + \frac{\bar{\psi}_L \overleftrightarrow{\not{D}} \psi_L}{(1 + \bar{z}z)^2} - \frac{(\bar{\psi}_L \gamma_\mu \psi_L)^2}{(1 + \bar{z}z)^4}$$

- The bosonic part is the Goldstone model for $SU(2)/U(1)$.
- The covariant derivative of the fermion field is defined by

$$\not{D}\psi_L = (\not{\partial} + \not{\partial} z \, \Gamma_{zz}^{\ \ z}) \, \psi_L = \left(\not{\partial} - \frac{2\bar{z}\not{\partial} z}{1 + \bar{z}z} \right) \psi_L$$

A simplification and a problem

Redefine the spinor field by

$$\chi_L = e^{-K(\bar{z},z)}\,\psi_L = \frac{\psi_L}{1+\bar{z}z}$$

New lagrangean:

$$\mathcal{L} = \frac{\partial\bar{z}\cdot\partial z}{(1+\bar{z}z)^2} + \bar{\chi}_L \overset{\leftrightarrow}{\not{D}} \chi_L - \left(\bar{\chi}_L\gamma_\mu\chi_L\right)^2$$

Under $SU(2)$ transformations the new spinor field transforms as

$$\begin{aligned} (\chi_L)_g &= e^{f_g(z)-\bar{f}_g(\bar{z})}\,\chi_L \\ &= \text{field dep. } U(1) \text{ transformation} \rightarrow \text{local} \end{aligned}$$

The covariant derivative simplifies to

$$\not{D}\chi_L = (\not{\partial} - i\not{A})\,\chi_L$$

with

$$A_\mu = -i\,\frac{\bar{z}\overset{\leftrightarrow}{\partial}_\mu z}{(1+\bar{z}z)}$$

- Chiral fermion coupled to (composite) $U(1)$ gauge field $\rightarrow U(1)$ anomaly!

Eliminating the anomaly

We can cancel the anomaly by dressing the sphere with an additional matter supermultiplet of which the fermion has opposite $U(1)$-charge.

Introduce a chiral superfield $A = a + \bar{\Theta}_R \phi_L + \ldots$ transforming as a line bundle

$$A_g = e^{-2fg(\Phi)}\, A$$

An invariant action for this superfield can be constructed:

$$\mathcal{L}_m = e^{2K(\bar{\Phi},\Phi)}\, \bar{A}A$$

After a redefinition

$$\hat{\phi} = e^{K(\bar{z},z)} \left(\phi + \frac{2a\bar{z}}{1+\bar{z}z} \psi \right)$$

the new fermion field transforms precisely opposite to the quasi-Goldstone fermion ψ:

$$\hat{\phi}_g = e^{-fg(z)+\bar{f}\bar{g}(\bar{z})}\, \hat{\phi}$$

It appears in the component action in the form

$$\Delta\mathcal{L}_m = \bar{\hat{\phi}}_L \overleftrightarrow{\not{D}}\, \hat{\phi}_L + \ldots$$

with $\not{D}\hat{\phi} = (\not{\partial} + i\not{A})\hat{\phi}$.

Kähler cosets

The sphere is the appropriate manifold to describe the spontaneous breaking of $SU(2) \rightarrow U(1)$. It is also a

Kähler manifold

- it has real dimension $2N$;
- it can be parametrized everywhere by a local set of *complex* co-ordinates $(\bar{z}^{\underline{i}}, z^i)$, $i = 1, \dots, N$;
- co-ordinates in different neighborhoods can be patched together by *holomorphic* co-ordinate transformations;
- local geometry can be described in terms of a real line element

$$ds^2 = g_{i\underline{i}}(\bar{z}, z)\, d\bar{z}^{\underline{i}}\, dz^i, \qquad g_{i\underline{i},j} = g_{j\underline{i},i}$$

Not every coset is Kähler manifold, but there are some large classes with interesting properties:

$$\frac{SU(N+M)}{SU(N) \times SU(M) \times U(1)}, \qquad \frac{SO(2N)}{SU(N) \times U(1)}$$

$$\frac{E_6}{SO(10) \times U(1)}, \qquad \frac{E_8}{SO(10) \times SU(3) \times U(1)}$$

On all these manifolds G/H we can construct supersymmetric field theories with chiral fermions transforming in a linear representation of H.

Quarks and leptons

In the standard models quarks and leptons build representations of the gauge groups $SU(3) \times SU(2) \times U_Y(1)$.

particle	color multiplicity	isospin multiplicity	isospin I_3	hypercharge Y
ν_L	1	2	$+1/2$	$-1/2$
e_L	1	2	$-1/2$	$-1/2$
ν_L^c	1	1	0	0
e_L^c	1	1	0	$+1$
u_L	3	2	$+1/2$	$+1/6$
d_L	3	2	$-1/2$	$+1/6$
u_L^c	3	1	0	$-2/3$
d_L^c	3	1	0	$+1/3$

Beyond the standard model: $SU(3) \times SU(3)$,
or gauge unification: $SU(5)$, $SO(10)$, or E_6;
quarks and leptons in

GUT group	fermion representations
$SU(5)$	5_L, 10_L
$SO(10)$	16_L
E_6	27_L

- Consistency if chiral gauge theories requires absence of chiral anomalies.

Unified chiral gauge theories

Program:
- Start from a Kähler coset G/H
  with Goldstone supermultiplets $\Phi^i = (z^i, \psi^i_L)$
- Eliminate anomalies
- Gauge the symmetries G

$\rightarrow$ the vector bosons of the broken symmetries become massive and the Goldstone bosons disappear from the spectrum.

- In the effective low-energy theory there are (some) quarks and leptons without observable superpartners.

- Supersymmetry is broken upon gauging the internal symmetries.

In the following we focus on the elimination of anomalies and the source of supersymmetry breaking.

General chiral superfield actions

The most general action for N chiral superfields ϕ^i is of the form

$$K[\bar{\phi}^{\underline{i}}, \phi^i]|_D + \mathrm{Re}\, W[\phi^i]_F$$

where $K[\bar{\phi}^{\underline{i}}, \phi^i]$ is a general real superfield function of $(\bar{\phi}_i, \phi^i)$.

In components after elimination of the auxiliary fields:

$$\begin{aligned}\mathcal{L} \;=\;& -g_{i\underline{i}}\left(\partial \bar{z}^{\underline{i}} \cdot \partial z^i + \frac{1}{2}\bar{\psi}_+^{\underline{i}} \overleftrightarrow{\not{D}}\, \psi_-^i\right) - g^{\underline{i}i}\bar{W}_{;\underline{i}} W_{;i} \\ &+ W_{;ij}\bar{\psi}_-^i\psi_-^j + \bar{W}_{;\underline{i}\,\underline{j}}\bar{\psi}_+^{\underline{i}}\psi_+^{\underline{j}} + \frac{1}{8}R_{i\underline{i}\,j\underline{j}}\,\bar{\psi}_+^{\underline{i}}\gamma^\mu\psi_-^i\,\bar{\psi}_+^{\underline{j}}\gamma_\mu\psi_-^j .\end{aligned}$$

Here we use the notation

$$g_{i\underline{i}}(\bar{z}, z) = K_{,i\underline{i}}(\bar{z}, z), \qquad R_{i\underline{i}\,j\underline{j}} = g_{i\underline{i},j\underline{j}} - g^{\underline{k}k} g_{i\underline{k},j} g_{k\underline{i},\underline{j}}$$

and covariant derivatives

$$\not{D}\psi_-^i = \not{\partial}\psi_-^i + \not{\partial} z^j\, \Gamma_{jk}{}^i\, \psi_-^k, \qquad \Gamma_{kj}{}^i = g^{\underline{i}i} g_{k\underline{i},j}$$

(and complex conjugates)

Kähler geometry

1. Kähler potential

By construction the metric of the scalar fields is of Kähler type:

$$\frac{\partial g_{i\underline{i}}}{\partial z^j} = \frac{\partial g_{j\underline{i}}}{\partial z^i}, \qquad \frac{\partial g_{i\underline{i}}}{\partial \bar{z}^{\underline{j}}} = \frac{\partial g_{i\underline{j}}}{\partial \bar{z}^{\underline{i}}}$$

$\leftrightarrow$ locally the metric is a gradient w.r.t. both holomorphic and anti-holomorphic co-ordinates

$$g_{i\underline{i}} = \frac{\partial K}{\partial \bar{z}^{\underline{i}} \partial z^i} = K_{,i\underline{i}}$$

Example: $CP^N = SU(N+1)/SU(N) \times U(1)$

$$K(\bar{z}, z) = \ln(1 + \bar{z} \cdot z) \quad \rightarrow \quad g_{i\underline{i}} = K_{,i\underline{i}} = \frac{\delta_{i\underline{i}}}{(1 + \bar{z} \cdot z)^2}$$

2. Tangent space: $\mathbf{C}^N$

$\rightarrow$ rotation group in tangent space is $U(N)$.

Examples: for the sphere the tangent space rotations is $U(1)$
for CP^N it is $U(N)$.

- Chiral fermions in reps. of $U(N)$

3. Holomorphic co-ordinate transformations

$$z'^{\,i} = z^{i} - \xi^{i}(z), \qquad \bar{z}'^{\,\underline{i}} = \bar{z}^{i} - \bar{\xi}^{\,\underline{i}}(\bar{z})$$

Fields on the Kähler manifold transform as

a. Scalars: $\delta S(\bar{z}, z) = \xi^{i} S_{,i} + \bar{\xi}^{\underline{i}} S_{,\underline{i}}$

b. Vectors: $\delta V_i = \xi^{j}_{\ ,i} V_j + \xi^{j} V_{i,j} + \bar{\xi}^{\underline{j}} V_{i,\underline{j}}$

c. Tensors:

$$\delta g_{i\underline{i}} = \xi^{k}_{\ ,i}\, g_{k\underline{i}} + \bar{\xi}^{\underline{k}}_{\ ,\underline{i}}\, g_{i\underline{k}} + \xi^{k} g_{i\underline{i},k} + \bar{\xi}^{\underline{k}} g_{i\underline{i},\underline{k}}$$

d. Line bundles:

$$\delta S = \lambda F_{\xi}\, S$$

(to be explained later.)

3. Holomorphic co-ordinate transformations

$$z'^{\,i} = z^{i} - \xi^{i}(z), \qquad \bar{z}'^{\,\underline{i}} = \bar{z}^{i} - \bar{\xi}^{\,\underline{i}}(\bar{z})$$

Fields on the Kähler manifold transform as

a. Scalars: $\delta S(\bar{z}, z) = \xi^{i} S_{,i} + \bar{\xi}^{\underline{i}} S_{,\underline{i}}$

b. Vectors: $\delta V_i = \xi^{j}_{\;,i} V_j + \xi^{j} V_{i,j} + \bar{\xi}^{\underline{j}} V_{i,\underline{j}}$

c. Tensors:

$$\delta g_{i\underline{i}} = \xi^{k}_{\;,i}\, g_{k\underline{i}} + \bar{\xi}^{\underline{k}}_{\;,\underline{i}}\, g_{i\underline{k}} + \xi^{k} g_{i\underline{i},k} + \bar{\xi}^{\underline{k}}\, g_{i\underline{i},\underline{k}}$$

N.B.:
1. the r.h.s. of these equations are the *Lie derivatives* of the fields w.r.t. the vector fields $(\xi, \bar{\xi})$;

2. the Lie derivative of the metric tensor can be written as

$$\delta g_{i\underline{i}} = \frac{\partial \xi_{\underline{i}}}{\partial z^{i}} + \frac{\partial \bar{\xi}_{i}}{\partial \bar{z}^{\underline{i}}} - \bar{\Gamma}_{\underline{i}\,\underline{j}}^{\;\;\underline{k}} \xi_{\underline{k}} - \Gamma_{ij}^{\;\;k} \bar{\xi}_{k} = \xi_{\underline{i};i} + \bar{\xi}_{i;\underline{i}}$$

3. Any solution of the equation

$$\xi_{\underline{i};i} + \bar{\xi}_{i;\underline{i}} = 0$$

is a co-ordinate transformation which leaves the metric invariant: a Killing vector.

Symmetries of Kähler manifolds

- If a Killing vector field exists everywhere on the Kähler manifold, it defines a global symmetry of the manifold.

- If the manifold possesses r global symmetries, the Killing vectors define an r-dimensional Lie algebra $\left\{R_A(z), \bar{R}_A(\bar{z})\right\}_{A=1}^{r}$:

$$\delta_A R_B^i = R_A^j R_{B,j}^i - R_B^j R_{A,j}^i = f_{AB}^{\ \ C} R_C^i$$

$$\Leftrightarrow \qquad [\delta_A, \delta_B]\, z^i = f_{AB}^{\ \ C}\, \delta_C z^i$$

- Even though the metric is invariant, this is not necessarily true for the Kähler potential:

$$\delta_A K(\bar{z}, z) = R_A^i K_{,i} + \bar{R}_A^{\underline{i}} K_{\underline{i}} = F_A(z) + \bar{F}_A(\bar{z})$$

leaves the metric invariant. $\rightarrow$ there exist real scalars $M_A(\bar{z}, z)$ s.t.

$$R_A^i K_{,i} - F_A = -\bar{R}_A^{\underline{i}} K_{,\underline{i}} + \bar{F}_A = -iM_A$$

As the $R_A^i(z)$ and $F_A(z)$ are holomorphic:

$$K_{,i\underline{i}} R_A^i = -iM_{A,\underline{i}} \quad \rightarrow \quad R_A^i = -iM_{A,\underline{i}}\, g^{\underline{i}i}$$

- Holomorphic Killing vectors are gradients of real Killing potentials M_A.

Example: Sphere

$$M(\varphi,\theta,\eta)=\frac{\varphi}{2}\left(\frac{1-\bar{z}z}{1+\bar{z}z}\right)-\frac{i\theta}{2}\left(\frac{z-\bar{z}}{1+\bar{z}z}\right)-\frac{\eta}{2}\left(\frac{z+\bar{z}}{1+\bar{z}z}\right)$$

$$\rightarrow\ -ig^{\bar{z}z}M_{,\bar{z}}=i\varphi z+\frac{\theta}{2}\left(1+z^2\right)+\frac{i\eta}{2}\left(1-z^2\right)=\delta z$$

From this we construct the holomorphic variations of the Kähler potential:

$$F(z;\varphi,\theta,\eta)=\delta zK_{,z}+iM=\frac{i}{2}\varphi+\frac{1}{2}\left(\theta-i\eta\right)z$$

Action of symmetries on the potentials

The action of isometries on the Kähler potential implies

$$\begin{aligned} [\delta_A, \delta_B]\, K &= \delta_{[A} F_{B]} + \delta_{[A} \bar{F}_{B]} \\ &= f_{AB}^{\ \ C}\, \delta_C K \end{aligned}$$

from which it follows that *(cocycle condition)*

$$\delta_A F_B - \delta_B F_A = f_{AB}^{\ \ C}\, F_C$$

Then under isometries the Killing potentials transform in the adjoint representation:

$$\delta_A M_B = f_{AB}^{\ \ C} M_C \quad \rightarrow \quad M_A^2 = \text{invariant}$$

The same is also true for holomorphic line bundles, in particular

$$\delta_A S = \lambda F_A\, S \quad \rightarrow \quad [\delta_A, \delta_B] S = \lambda \delta_{[A} F_{B]}\, S = f_{AB}^{\ \ C}\, \delta_C S$$

For any complex line bundle one can construct an invariant

$$\bar{S} S\, e^{-\lambda K(\bar{z}, z)}$$

N.B.: on compact manifolds λ is quantized.

Anomalies and their elimination

All pure Kähler cosets suffer from anomalies; in particular, there always is a chiral $U(1)$.

Example:

$$\frac{SU(5)}{SU(3) \times SU(2) \times U(1)}$$

$\longrightarrow$ Goldstone supermultiplets

$$\Phi_{\alpha}^{\ r} = (z_{\alpha}^{\ r}, \psi_{L\,\alpha}^{\ \ r}), \quad \alpha = (1,2), \quad r = (1,2,3)$$

The chiral fermions in this representation represent a quark doublet.

Remarks

- With this fermion content the triangle diagrams for $SU(3) \times SU(2) \times U(1)$ currents produce anomalies.
- As in the standard model we can eliminate the anomalies by dressing the coset manifold with other fields, containing chiral fermions in quark singlet, lepton doublet and lepton singlet representations of $SU(3) \times SU(2) \times U(1)$

$\longrightarrow$ complete the fermions to a full family

- This requires line bundle representations to fix the $U_Y(1)$ hypercharges to come out correctly.

Gauging the isometries

Gauging $SU(5)$ requires introduction of a vector supermultiplet $(A_\mu^a, \lambda^a, D^a)$ carrying the adjoint representation of $SU(5)$.

- The gauge fields splits into *massless* vector bosons for $H = SU(3) \times SU(2) \times U(1)$, and *massive* vector bosons for the broken generators of $SU(5)$.
- In this process the Goldstone scalars z disappear from the physical spectrum.
- The gaugino's split into representations of H as follows:

$$24 \rightarrow (8, 1) + (1, 3) + (1, 1) + (3, 2)_R + (3, 2)_L$$

  The $(3, 2)_R$ combines with the quasi-Goldstone fermions ψ_L to form massive Dirac partners of the massive gauge bosons.
- The $(3, 2)_L$ remains massless and now represents the quark doublet; they have no observable superpartners (squarks).
- The auxiliary fields D^a can be eliminated to produce the scalar potential

$$V_D = \frac{g^2}{2} M_a^2$$

  where M_a are the $SU(5)$ Killing potentials; they can be evaluated at $z = \bar{z} = 0$ (unitary gauge).
- Without an additional Fayet-Iliopoulos term
  $V_D|_{z=0} > 0 \rightarrow$ spontaneous supersymmetry breaking.

Conclusions and outlook

- Supersymmetric coset models can be made into consistent anomaly-free 4-D field theories.

- In these models some or all quarks and leptons are quasi-Goldstone fermions or an equivalent set of unpaired gaugino's.

- In the gauged models these quasi-Goldstone fermions have no scalar superpartners $\rightarrow$ these models are clearly distinguishable from the MSSM or standard super GUTs.

- In most cases supersymmetry is automatically broken by 'internal' Fayet-Iliopoulos term.

- Hierarchy problem: ??

- In some models V_D drives the theory to singular points of the Kähler metric $g_{i\underline{i}}$
  $\rightarrow$ some scalars decouple, symmetries restored;
  strong coupling regime?