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Interaction of Reggeized Gluons in the Baxter-Sklyanin Representation

H. J. de Vega[(a)], **L. N. Lipatov**[(b,⋄)]

(a)LPTHE,* Université Pierre et Marie Curie (Paris VI)
et Denis Diderot (Paris VII), Tour 16, 1er. étage,
4, Place Jussieu 75252 Paris, Cedex 05, France
(b)Petersburg Nuclear Physics Institute,
Gatchina, 188300, St.Petersburg, Russia and
LPMT Université Montpellier 2,
Place Eugène Bataillon, 34095 Montpellier Cedex 05, France

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Abstract

We investigate the Baxter equation for the Heisenberg spin model corresponding to a generalized BFKL equation describing composite states of n Reggeized gluons in the multi-color limit of QCD. The Sklyanin approach is used to find an unitary transformation from the impact parameter representation to the representation in which the wave function factorizes as a product of Baxter functions and a pseudo-vacuum state. We show that the solution of the Baxter equation is a meromorphic function with poles $(\lambda - i\,r)^{-(n-1)}$ $(r = 0, 1, ...)$ and that the intercept for the composite Reggeon states is expressed through the behavior of the Baxter function around the pole at $\lambda = i$. The absence of pole singularities in the two complex dimensional $\overrightarrow{\lambda}$-plane for the bilinear combination of holomorphic and anti-holomorphic Baxter functions leads to the quantization of the integrals of motion.

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1 Introduction

In the leading logarithmic approximation (LLA) of perturbative QCD the Reggeons (reggeized gluons) move in the two-dimensional impact parameter plane $\overrightarrow{\rho}$ and interact pairwise [1, 2].

*Laboratoire Associé au CNRS UMR 7589.

To unitarize the QCD scattering amplitudes at high energies one should take into account the multi-Reggeon exchanges in the t-channel. The composite states of the reggeized gluons satisfy a Schrödinger-like equation [3].

The Reggeon Hamiltonian in the infinite color limit $N_c \to \infty$ takes a simple form and can be written as follows [4],

$$H = \frac{1}{2}(h + h^*) \quad , \quad [h, h^*] = 0\,, \tag{1}$$

where the holomorphic and anti-holomorphic Hamiltonians

$$h = \sum_{k=1}^{n} h_{k,k+1} \quad , \quad h^* = \sum_{k=1}^{n} h^*_{k,k+1}\ , \tag{2}$$

are expressed in terms of the pair BFKL operator [1, 4]:

$$h_{k,k+1} = \log p_k + \log p_{k+1} + \frac{1}{p_k}\,(\log \rho_{k,k+1})\, p_k + \frac{1}{p_{k+1}}\,(\log \rho_{k,k+1})\, p_{k+1} + 2\,\gamma\,. \tag{3}$$

Here $\rho_{k,k+1} = \rho_k - \rho_{k+1}$, $p_k = i\,\frac{\partial}{\partial \rho_k}$, $p^*_k = i\,\frac{\partial}{\partial \rho^*_k}$, and $\gamma = -\psi(1)$ is the Euler-Mascheroni constant.

In this context the Pomeron is a compound state of two reggeized gluons and the Odderon is constructed from three reggeized gluons.

The operator h is invariant under the Möbius transformations [2] with generators:

$$\vec{M} = \sum_{k=1}^{n} \vec{M}_k\,;\ \ M^3_k = \rho_k\,\partial_k\,,\ \ M^-_k = \partial_k\,,\ \ M^+_k = -\rho^2_k\,\partial_k\,.$$

The Casimir operator of this group is

$$\vec{M}^2 = -\sum_{l<r}^{n} \rho^2_{lr}\ \partial_l \partial_r\,. \tag{4}$$

The Hamiltonian h describes the integrable XXX spin model with the spins being the generators $\vec{M}_k$ of the Möbius group [5]. The integrals of motion of this model are generated by the transfer matrix which is the trace of the monodromy matrix satisfying the Yang-Baxter equation [5]. Therefore, the Quantum Inverse Scattering Method [6, 7] can be applied to find an algebraic solution of the Schrödinger equation.

The pair hamiltonian (3) can be obtained from the fundamental monodromy matrix associated to the XXX Heisenberg spin model [5, 8, 9]. Notice that the local operators p_k, ρ_k act in an infinite dimensional Hilbert space whereas the spin operators in the usual Heisenberg model are finite dimensional matrices both for integer or half plus an integer spin.

The auxiliary L-operator for the Heisenberg spin model with $s = -1$ is given below [8, 9, 10]

$$L_k(u) = \begin{pmatrix} u + p_k\,\rho_{k0} & -p_k \\ p_k\,\rho^2_{k0} & u - p_k\,\rho_{k0} \end{pmatrix}\,. \tag{5}$$

where ρ_0 is the coordinate of the composite state.

The auxiliary monodromy matrix for this model can be parametrized as follows

$$T(u) = L_n(u)L_{n-1}(u)\dots L_1(u) = \left(\begin{array}{cc} A(u) & B(u) \\ C(u) & D(u) \end{array}\right) , \tag{6}$$

where n is the number of the reggeized gluons. The transfer matrix is the trace of the monodromy matrix

$$t(u) = A(u) + D(u) = \sum_{j=0}^{n} Q_j \, u^{n-j} ,$$

where

$$Q_j = \sum_{i_1 > i_2 > \dots > i_j} p_{i_1} p_{i_2} \dots p_{i_j} \, \rho_{i_1 i_2} \dots \rho_{i_{j-1} i_j} \rho_{i_j i_1} . \tag{7}$$

The eigenvalues $\Lambda(u)$ of $t(u)$ take the form

$$\Lambda(u) = 2\, u^n + q_2 \, u^{n-2} + q_3 \, u^{n-3} + \dots + q_n .$$

where q_j, $2 \leq j \leq n$ are eigenvalues of the integrals of motion Q_j [5]. In particular, $q_2 = -m(m-1)$ is the eigenvalue of the holomorphic Casimir operator (4) and m is the conformal weight.

The operator $C(u)$ annihilates the pseudovacuum state for the $s = -1$ model [9],

$$C(u) \, \Omega_0 = 0 \quad , \quad \Omega_0 = \prod_{r=1}^{n} \rho_{r0}^{-2} . \tag{8}$$

The operators $B(u)$ can be obtained directly from eq.(6). We find for $n = 2$ and $n = 3$,

$$\begin{aligned} B^{(n=2)}(u) &= -u\,(p_1 + p_2) + p_1 \, p_2 \, \rho_{12} \qquad (9) \\ B^{(n=3)}(u) &= -u^2\,(p_1 + p_2 + p_3) + u\,(p_1 \, p_2 \, \rho_{12} + p_1 \, p_3 \, \rho_{13} + p_2 \, p_3 \, \rho_{23}) - p_1 \, p_2 \, p_3 \, \rho_{12} \, \rho_{23} \end{aligned}$$

For arbitrary n one obtains,

$$\begin{aligned} B^{(n)}(u) &= -\sum_{k=0}^{n-1} b_k \, u^{n-1-k} \quad \text{where} \quad b_0 = P \equiv \sum_{i=1}^{n} p_i \quad , \quad b_1 = -\sum_{1 \leq i < j \leq n} p_i \; p_j \; \rho_{ij} \; , \\ b_2 &= \sum_{1 \leq i_1 < i_2 < i_3 \leq n} p_{i_1} \; p_{i_2} \; p_{i_3} \; \rho_{i_1 i_2} \; \rho_{i_2 i_3} \quad , \\ & \dots , \\ b_l &= (-1)^l \sum_{1 \leq i_1 < i_2 < \dots < i_{l+1} \leq n} p_{i_1} \; p_{i_2} \; \dots \; p_{i_l} \; p_{i_{l+1}} \; \rho_{i_1 i_2} \; \rho_{i_2 i_3} \; \dots \; \rho_{i_l i_{l+1}} \; , \\ & \dots , \end{aligned}$$

$$b_{n-1} \quad = \quad (-1)^{n-1} p_1 \, p_2 \, \ldots \, p_n \, \rho_{12} \, \rho_{23} \, \ldots \, \rho_{n-1,n} \; . \tag{10}$$

The operators B with different spectral parameters commute

$$[B(u) \, , \, B(v)] = 0$$

and therefore, one can write them in factorized form as a product of the operator zeros $\widehat{\lambda}_k$ of $B(u)$:

$$B(u) = -P \prod_{k=1}^{n-1} \left(u - \widehat{\lambda}_k \right) \quad , \quad \left[\widehat{\lambda}_{k_1}, \widehat{\lambda}_{k_2} \right] = \left[\widehat{\lambda}_k, P \right] = 0 \, ,$$

following E. K. Sklyanin [12].

The wave function describing composite states of reggeized gluons in the holomorphic impact parameter space ρ can be written as follows [12] (see also [9])

$$\psi(\rho_1, \rho_2, \ldots, \rho_n; \rho_0) = Q\left(\widehat{\lambda}_1\right) Q\left(\widehat{\lambda}_2\right) \ldots \, Q\left(\widehat{\lambda}_n\right) \prod_{r=1}^{n} \rho_{r0}^{-2} \quad , \quad \rho_{r0} = \rho_r - \rho_0 \; . \tag{11}$$

The function $Q\,(\lambda)$ satisfies the Baxter equation [13]

$$\Lambda(\lambda) \, Q\,(\lambda) = (\lambda + i)^n \, Q\,(\lambda + i) + (\lambda - i)^n \, Q\,(\lambda - i) \; . \tag{12}$$

For the Odderon case, the dependence of the energy from the eigenvalues of the integrals of motion has been found with the use of the Baxter equation [14] and of the duality symmetry [11].

In this paper we systematically develop the construction of composite Reggeon states using the Baxter-Sklyanin (BS) representation, in which the operator zeros $\widehat{\lambda}_k$ of $B(u)$ are diagonal. The matrix elements relating the momentum and BS representations obey solvable ordinary differential equations for $n = 2, 3$. These matrix elements are elementary functions for the pomeron case and hypergeometric functions for the odderon case. In the BS representation the wave function of the composite state is written as a product of the Baxter functions and the pseudo-vacuum state.

For the pomeron, we provide general formulas for the Baxter function valid in the whole complex λ plane and study its analytic properties. It turns out that the most efficient way to solve the Baxter equation in the present context is to use the pole expansions (Mittag-Löffler).

We show that the Pomeron wave function has no singularities on the real axis as a function of $\sigma = Re\,\lambda$ and hence it can be normalized. This corresponds to the single-valuedness condition in the coordinate representation.

We derive also the analytic Bethe Ansatz equations and construct the Baxter function as an infinite product of Bethe Ansatz roots.

The solution $Q(u)$ of the Baxter equation for the general n-reggeon case is constructed as an infinite sum over poles from order 1 up to $n-1$. Their residues satisfy simple recurence relations. It is shown, that the quantization condition for the integrals of motion follows from the condition of the cancellation of the pole singularities in the two complex dimensional

$\overrightarrow{\lambda}$-plane for the bilinear combination of holomorphic and anti-holomorphic Baxter functions $Q(\overrightarrow{\lambda})$.

For the odderon, we explicitly construct the BS representation and investigate the properties of the odderon wave functions in this representation. The completeness and orthogonality relations for these functions are discussed.

We derive new formulas for the eigenvalues of the Reggeon hamiltonian written through the Baxter function. These formulas generalize the result for the Pomeron to any number of Reggeons. The energy turns to be expressed in terms of the behavior of the Baxter function near its poles at $\lambda = i$ which are present for arbitrary n.

The BS representation promises to be an appropriate starting point to find new composite Reggeon states for $n > 3$. In particular, it will be interesting to generalize the Odderon solution constructed in ref.[15] to the case of many Reggeons.

2 BS representation for the wave function

In order to solve the Baxter equation, one should fix the class of functions in which the solution is searched. The case of integer conformal weight m has been considered in refs.[9, 10, 18]. It was assumed there that the solutions were entire functions with the asymptotics

$$Q(\lambda) \sim \lambda^{m-n} \,,$$

but such functions do not exist for physical values of the conformal weights m.

We want to find the conditions which should be satisfied by the solutions of the Baxter equation from the known information about the eigenfunctions Φ of the Schrödinger equation in the two-dimensional impact parameter space $\overrightarrow{\rho}$ [2, 16]. For this purpose we transform the wave function to the BS representation in which the operator $B(u)$ is diagonal by an unitary transformation $\Phi(\overrightarrow{\rho_1}, \ldots, \overrightarrow{\rho_n}; \overrightarrow{\rho_0})$.

To begin with, let us go to the momentum representation (with removed gluon propagators):

$$\Psi_{m,\widetilde{m}}(\overrightarrow{p_1},\, \overrightarrow{p_2},\, \ldots, \overrightarrow{p_n}) =$$
$$\prod_{r=1}^{n} (\overrightarrow{p_r})^2 \int \prod_{k=1}^{n} \left[\frac{d^2\rho_k}{2\pi} \, \exp(i\overrightarrow{p_k} \cdot \overrightarrow{\rho_{k0}}) \right] \Phi_{m,\widetilde{m}}(\overrightarrow{\rho_1},\, \overrightarrow{\rho_2},\, \ldots, \overrightarrow{\rho_n}; \overrightarrow{\rho_0}) \,. \tag{13}$$

Here $\Phi_{m,\widetilde{m}}(\overrightarrow{\rho_1},\, \overrightarrow{\rho_2},\, \ldots, \overrightarrow{\rho_n}; \overrightarrow{\rho_0})$ is the wave function of the composite state in the two-dimensional impact parameter space $\overrightarrow{\rho}$. It belongs to the principal series of the unitary representations of the Möbius group and is an eigenfunction of its Casimir operators

$$\vec{M}^2 \Phi_{m,\widetilde{m}} = m(m-1)\Phi_{m,\widetilde{m}} \quad , \quad \left(\vec{M}^*\right)^2 \Phi_{m,\widetilde{m}} = \widetilde{m}(\widetilde{m}-1)\Phi_{m,\widetilde{m}} \,.$$

Here

$$m = \frac{1}{2} + i\nu + \frac{n}{2} \quad , \quad \widetilde{m} = \frac{1}{2} + i\nu - \frac{n}{2}$$

are conformal weights (the quantities ν and n are correspondingly real and integer numbers for the principal series of the unitary representations). The Casimir operators of the Möbius group are given by eq.(4).

For example, for the pomeron and odderon we have respectively [2, 4],

$$\Phi^{(2)}_{m,\widetilde{m}}(\overrightarrow{\rho_1},\, \overrightarrow{\rho_2}\,;\, \overrightarrow{\rho}_0) = \left(\frac{\rho_{12}}{\rho_{10}\rho_{20}}\right)^m \left(\frac{\rho^*_{12}}{\rho^*_{10}\rho^*_{20}}\right)^{\widetilde{m}} ,$$

$$\Phi^{(3)}_{m,\widetilde{m}}(\overrightarrow{\rho_1},\, \overrightarrow{\rho_2},\, \overrightarrow{\rho_3}\,;\, \overrightarrow{\rho}_0) = \left(\frac{\rho_{23}}{\rho_{20}\rho_{30}}\right)^m \left(\frac{\rho^*_{23}}{\rho^*_{20}\rho^*_{30}}\right)^{\widetilde{m}} \phi_{m,\widetilde{m}}(x\,,x^*)\,,$$

where ϕ is a function of the anharmonic ratio,

$$x = \frac{\rho_{12}\rho_{30}}{\rho_{10}\rho_{32}}\,.$$

Due to the identity

$$\sum_{k=1}^n p_k\,\rho_k = \frac{P}{n}\sum_{k=1}^n \rho_k + \sum_{k=1}^{n-1}\rho_{k,k+1}\sum_{r=1}^k\left(p_r - \frac{P}{n}\right)\,,\;\; P = \sum_{k=1}^n p_k\,,$$

we can express the quantity $B(u)$ in momentum representation in terms of P, $p_k - \frac{P}{n}$ and the operators

$$\rho_{k,k+1} = -i\left(\frac{\partial}{\partial p_k} - \frac{\partial}{\partial p_{k+1}}\right) = -i\frac{\partial}{\partial\left(\sum_{r=1}^k p_r\right)}\,,\quad k = 1,\,2,\,\ldots,\,n-1\,.$$

It is convenient to introduce the new independent variables

$$P = \sum_{k=1}^n p_k\,,\;\; t_1 = \ln\frac{p_1}{P-p_1}\,,\;\; t_2 = \ln\frac{p_1+p_2}{P-p_1-p_2}\,,\ldots,\; t_{n-1} = \ln\frac{P-p_n}{p_n}\,. \tag{14}$$

The quantities t_k take their values in a strip of the complex plane

$$-\infty < Re\, t_k < \infty\,,\quad -\pi < Im\, t_k < \pi\,.$$

There is a helpful representation for the operators b_l besides that given by eq. (10). Namely,

$$b_l = (-1)^l \sum_{1\le i_1<i_2<\ldots<i_l\le n}\left(\sum_{r_1=1}^{i_1} p_{r_1}\right)\left(\sum_{r_2=i_1+1}^{i_2} p_{r_2}\right)\ldots\left(\sum_{r_{l+1}=i_l+1}^{n} p_{r_{l+1}}\right)\prod_{s=1}^l \rho_{i_s,i_s+1}\,, \tag{15}$$

which is related with the duality transformation [11] consisting of the cyclic permutation

$$p_k \to \rho_{k,k+1} \to p_{k+1}$$

and the transposition of the operator multiplication. Note, that with the use of the duality symmetry the wave function in the momentum space (13) for $P = 0$ is proportional to the same function in the coordinate space [11]. Furthermore, the function for arbitrary P can be obtained by an appropriate Möbius transformation.

In the variables $t_1, \ldots, t_{n-1}$ the matrix element $B(u)$ takes the following form

$$B(u) = -\sum_{k=0}^{n-1} b_k \, u^{n-1-k} \,,$$

where the operators b_k are given by

$$b_k = P \sum_{1 \le l_1 < l_2 < \ldots < l_k \le n-1} \prod_{r=1}^{k-1} \left(1 - e^{t_{l_r} - t_{l_{r+1}}}\right) \prod_{s=1}^{k} i \, \frac{\partial}{\partial t_{l_s}} \,.$$

This representation can be obtained from eq.(15) for b_l taking into account the formulas:

$$i \frac{\partial}{\partial t_k} = i \frac{\left(\sum_{r=1}^{k} p_r\right)\left(P - \sum_{r=1}^{k} p_r\right)}{P} \frac{\partial}{\partial\left(\sum_{r=1}^{k} p_r\right)} = -\frac{\left(\sum_{r=1}^{k} p_r\right)\left(P - \sum_{r=1}^{k} p_r\right)}{P} \, \rho_{k,k+1} \,,$$

$$1 - e^{t_{l_r} - t_{l_{r+1}}} = \frac{P \left(\sum_{s=1}^{l_{r+1}} p_s\right)}{\left(\sum_{s=1}^{l_{r+1}} p_s\right)\left(P - \sum_{s=1}^{l_r} p_s\right)} \,.$$

In particular, for $n = 2$, 3, 4 we obtain

$$B^{(2)}(u) = -P \left(u + i \frac{\partial}{\partial t_1}\right) ,$$

$$B^{(3)}(u) = -P \left[u^2 + iu \left(\frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2}\right) - \left(1 - e^{t_1 - t_2}\right) \frac{\partial}{\partial t_1} \frac{\partial}{\partial t_2}\right] ,$$

$$B^{(4)}(u) = -P \left[u^3 + i \, u^2 \sum_{r=1}^{3} \partial_r - u \sum_{1 \le l_1 < l_2 \le 3} \left(1 - e^{t_{l_1 l_2}}\right) \partial_{l_1} \partial_{l_2} - i \prod_{k=1}^{2} \left(1 - e^{t_{k,k+1}}\right) \partial_1 \partial_2 \partial_3\right] , \quad (16)$$

where

$$t_{l_1 l_2} = t_{l_1} - t_{l_2} \,, \;\; \partial_r = \frac{\partial}{\partial \, t_r} \,.$$

Since in the momentum representation the norm of the wave function is given by

$$\left\| \Psi_{m,\widetilde{m}} \right\|^2 = \int \prod_{r=1}^{n} \frac{d^2 p_r}{\left|p_r\right|^2} \left| \Psi_{m,\widetilde{m}}(\overrightarrow{p_1}, \, \overrightarrow{p_2}, \, \ldots, \overrightarrow{p_n}) \right|^2 ,$$

we obtain after extracting the factor $\delta^2(P - \sum_{k=1}^{n} p_k)$ from $\Psi_{m,\widetilde{m}}$ in the new variables $t_1, t_1^*; \ldots; t_{n-1}, t_{n-1}^*$

$$\left\|\Psi_{m,\widetilde{m}}\right\|^2 = \int \prod_{r=1}^{n-1} d^2 t_r \prod_{s=1}^{n-2} \left|1 - e^{t_s - t_{s+1}}\right|^{-2} \left|\Psi_{m,\widetilde{m}}\right|^2 .$$

The operators $B(u)$ are symmetric $B = B^t$ with respect to this norm with the weight

$$\prod_{s=1}^{n-2} \left|1 - e^{t_s - t_{s+1}}\right|^{-2} .$$

The eigenvalues of the operator zeroes $\widehat{\lambda}_k$ and $\widehat{\lambda^*}_k$ of $B(u)$ in the holomorphic and anti-holomorphic space have the form,

$$\lambda_k = \sigma_k + i\frac{N_k}{2} \quad , \quad \lambda_k^* = \sigma_k - i\frac{N_k}{2}$$

where σ_k is real and N_k is integer. The unitary transformation between t- and λ- representations conserves the norm of the wave function

$$\left\|\Psi_{m,\widetilde{m}}\right\|^2 = \prod_{r=1}^{n-1} \left(\int_{-\infty}^{+\infty} d\sigma_r \sum_{N_r=-\infty}^{+\infty} \right) \left|\Psi_{m,\widetilde{m}}\right|^2 .$$

Let us introduce the kernel $U_{\overrightarrow{\lambda_1},\ldots,\overrightarrow{\lambda_{n-1}}}(\overrightarrow{t_1}, \overrightarrow{t_2}, \ldots, \overrightarrow{t_{n-1}})$ for the unitary transformation between the t- and λ- representations. It satisfies the eigenvalue equations

$$\begin{aligned} B(u)\, U_{\overrightarrow{\lambda_1},\ldots,\overrightarrow{\lambda_{n-1}}}(\overrightarrow{t_1}, \ldots, \overrightarrow{t_{n-1}}) &= -P \prod_{k=1}^{n-1} (u - \lambda_k^*)\, U_{\overrightarrow{\lambda_1},\ldots,\overrightarrow{\lambda_{n-1}}}(\overrightarrow{t_1}, \ldots, \overrightarrow{t_{n-1}}) , \\ B(u)^*\, U_{\overrightarrow{\lambda_1},\ldots,\overrightarrow{\lambda_{n-1}}}(\overrightarrow{t_1}, \ldots, \overrightarrow{t_{n-1}}) &= -P \prod_{k=1}^{n-1} (u - \lambda_k)\, U_{\overrightarrow{\lambda_1},\ldots,\overrightarrow{\lambda_{n-1}}}(\overrightarrow{t_1}, \ldots, \overrightarrow{t_{n-1}}) \end{aligned} \tag{17}$$

and the orthogonality relations

$$\begin{aligned} &\int \prod_{k=1}^{n-1} \frac{d^2\, t_k}{(2\pi)^2} \prod_{r=1}^{n-2} |1 - e^{t_r - t_{r+1}}|^{-2}\, U_{\overrightarrow{\lambda_1},\ldots,\overrightarrow{\lambda_{n-1}}}\, U^*_{\overrightarrow{\lambda'_1},\ldots,\overrightarrow{\lambda'_{n-1}}} = \\ &\Sigma_P \prod_{t=1}^{n-1} \left[\delta(\sigma_k - \sigma'_{r_k})\, \delta_{N_k, N'_{r_k}} \right] . \end{aligned} \tag{18}$$

Here, λ_k^*, λ_k stands for the eigenvalues of the operators $\widehat{\lambda}_k$ and $\widehat{\lambda}_k^*$, respectively and the symbol Σ_P means the sum over all possible permutations $r_1, ..., r_{n-1}$ of the indices $1, 2, ..., n-1$.

To construct the kernel with the correct normalization, let us take into account, that the Kronecker and Dirac δ-functions appears in the right hand side of the above equation as a result of the integration over the region

$$t_2 - t_1 >> 1,\ t_3 - t_2 >> 1,\ \ldots,\ t_{n-1} - t_{n-2} >> 1 ,$$

corresponding to one of the two possibilities

$$p_1 << p_2 << \ldots << p_n$$

or

$$p_1 >> p_2 >> \ldots >> p_n \, .$$

In this region the operator $B(u)$ simplifies as

$$B(u) \to -P \prod_{k=1}^{n-1} (u + i\partial_k) \tag{19}$$

and therefore the kernel for the unitary transformation corresponds to the Fourier transformation

$$U_{\overrightarrow{\lambda_1}, \ldots, \overrightarrow{\lambda_{n-1}}}(\overrightarrow{t_1}, \overrightarrow{t_2}, \ldots, \overrightarrow{t_{n-1}}) \to 2^{\frac{n-1}{2}} \Sigma_P \, e^{i\Phi(\overrightarrow{\lambda_{r_1}}, \ldots, \overrightarrow{\lambda_{r_{n-1}}})} \, \exp[i \sum_{k=1}^{n-1} (t_k \lambda^*_{r_k} + t^*_k \lambda_{r_k})] \, ,$$

where Φ are some phases.

3 BS representation for the pseudo-vacuum state

The wave function of the pseudo-vacuum state in the momentum representation is

$$\Phi^0_{m=n, \widetilde{m}=n}(\overrightarrow{p_1}, \overrightarrow{p_2}, \ldots, \overrightarrow{p_n}) = \int \prod_{k=1}^{n} \left[\frac{4 \, d^2\rho_k}{\pi \, |\rho_{k0}|^4} \, \exp(i\overrightarrow{p_k} \cdot \overrightarrow{\rho_{k0}}) \right] = \prod_{k=1}^{n} |p_k|^2 \ln |p_k|^2 \, . \tag{20}$$

where we subtracted from the distribution $|\rho|^{-4}$ its singular part [19]. This function is an eigenfunction of the transfer matrix with the eigenvalue

$$\Lambda(u) = (u+i)^n + (u-i)^n .$$

In particular, its Möbius conformal weights are

$$m = \widetilde{m} = n$$

and $q_k = 0$ for odd values of k. Notice that the Baxter function for the pseudo-vacuum state is u-independent.

It is important to know the wave function of the pseudo-vacuum state in the BS representation. We see from eq.(16) that $\widehat{\lambda}_1$ in the new variables for the Pomeron state takes the simple form

$$\widehat{\lambda}_1 = -i \frac{\partial}{\partial t_1}$$

and the change of the basis results in

$$< p_1 \, p_2 | P, \lambda_1, \lambda_1^* >= \left| \frac{P}{p_1 \, p_2} \right|^2 \left(\frac{p_1}{p_2} \right)^{i\lambda_1^*} \left(\frac{p_1^*}{p_2^*} \right)^{i\lambda_1} \delta^{(2)}(P - p_1 - p_2) \; . \tag{21}$$

One can obtain from eqs. (20) and (21) the Pomeron pseudo-vacuum wave function in the new variables as

$$\Phi^0(\overrightarrow{P},\overrightarrow{\lambda_1}) = |P|^4 \int d^2p \left(\frac{p}{1-p}\right)^{-i\lambda_1^*} \left(\frac{p^*}{1-p^*}\right)^{-i\lambda_1} \ln|p|^2 \ln|1-p|^2 .$$

It can be written as follows

$$\Phi^0(\overrightarrow{P},\overrightarrow{\lambda_1}) = |P|^4 \lim_{\mu\to\sigma} \frac{\partial}{\partial\sigma}\frac{\partial}{\partial\mu} \int d^2p\, |p|^{-2i\sigma} |1-p|^{2i\mu} \left[\frac{p^*(1-p)}{p(1-p^*)}\right]^{\frac{N}{2}} ,$$

where $\lambda_1 = \sigma + iN/2$. The integral is calculated in a closed form with the use of the anti-Wick rotation (see next section):

$$\Phi^0(\overrightarrow{P},\overrightarrow{\lambda}) = 4\pi(-1)^N \ |\lambda|^2 \ \lim_{\mu\to\sigma} \frac{1}{(\mu-\sigma)^3}. \tag{22}$$

The fact that the wave function for the pomeron pseudo-vacuum state turns out to be divergent is connected with the fact that such state having the weights $m=\widetilde{m}=2$ is outside the space of physical states. However it is important to normalize correctly the Baxter function. The natural regularization of the Heisenberg model can be provided by changing the spin representation $s\to -1+\epsilon$ without losing its integrability. It would lead in particular to the modification of the pseudo-vacuum state (20) and to the convergence of integrals after their analytic continuation.

The above result for $\Phi^0(\overrightarrow{P},\overrightarrow{\lambda})$ can be obtained in a simpler way by taking into account, that in the integral transformation to the λ-representation the large momenta p dominate

$$\Phi^0(\overrightarrow{P},\overrightarrow{\lambda}) \simeq (-1)^N |P|^4 \int d^2p\, e^{-i\frac{\lambda^*}{p}} e^{-i\frac{\lambda}{p^*}} \ln^2|p|^2 = (-1)^N |P|^4 |\lambda|^2 c,$$

where the leading divergent contribution to c does not depend on λ. It corresponds to the following simplification of the operator $B(u)$ for $p \gg P$,

$$B^{(2)}(u) = -P\left(u - \frac{p^2}{P}\, i\,\frac{\partial}{\partial p}\right) .$$

In the case of three particles we have for large $p_1,\ p_2,\ p_3$ with fixed P,

$$\begin{aligned} B^{(3)}(u) &= -P\left[u^2 + 2iu\frac{\partial}{\partial t} - y\left(\frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial t^2}\right)\right] \\ t &= t_1+t_2 = P\left(p_1^{-1} - p_3^{-1}\right) \ll 1 \quad , \quad y = t_1 - t_2 = P\left(p_1^{-1} + p_3^{-1}\right) \ll 1 . \end{aligned} \tag{23}$$

Its eigenvalues and eigenfunctions for small t and y are

$$B^{(3)}(u) = -P(u-\lambda_1^*)(u-\lambda_2^*) ,$$

$$\varphi^1_{\lambda_1^*\lambda_2^*}(t,y) = e^{\frac{i}{2}(\lambda_1^*+\lambda_2^*)t}\left[1-\lambda_1^*\ \lambda_2^*\ y\ \ln\, y\right] ,\ \varphi^2_{\lambda_1^*\lambda_2^*}(t,y) = e^{\frac{i}{2}(\lambda_1^*+\lambda_2^*)t}\, y . \tag{24}$$

Imposing the property of the single-valuedness we can write the transition amplitude in the two-dimensional space:

$$U_{\overrightarrow{\lambda_1},\overrightarrow{\lambda_2}}(\overrightarrow{t_1},\,\overrightarrow{t_2}) = c_{\overrightarrow{\lambda_1},\overrightarrow{\lambda_2}}\, e^{\frac{i}{2}\left[(\lambda_1^*+\lambda_2^*)t+(\lambda_1+\lambda_2)t^*\right]}\left(\frac{y}{\lambda_1\lambda_2}+\frac{y^*}{\lambda_1^*\lambda_2^*}-\ln|y|^2\right)\,,$$

where the constant $c_{\overrightarrow{\lambda_1},\overrightarrow{\lambda_2}}$ for the normalized function U is calculated below (see (116)).

However, the pseudo-vacuum state and other solutions with integer conformal weights do not belong to the space of the physical states. Therefore their unitary transformation to the BS representation should be special and the normalization of the unitary transformation could include only the integration over the large momenta. In this case it is more natural to write for the kernel of this transformation the following expression

$$U_{\overrightarrow{\lambda_1},\overrightarrow{\lambda_2}}(\overrightarrow{t_1},\,\overrightarrow{t_2}) = c^{ps}_{\overrightarrow{\lambda_1},\overrightarrow{\lambda_2}}\, e^{\frac{i}{2}\left[(\lambda_1^*+\lambda_2^*)t+(\lambda_1+\lambda_2)t^*\right]}\,,$$

where the constant $c^{ps}_{\overrightarrow{\lambda_1},\overrightarrow{\lambda_2}}$ does not depend on $\lambda_{1,2}$.

Let us now consider the n-Reggeon case. Again, the large momenta $\overrightarrow{p_k}$, $1\le k\le n$ presumably dominate the pseudo-vacuum wave function $\Phi^0(\overrightarrow{P},\,\overrightarrow{\lambda})$ when expressed as integral transform of eq.(20). This large momenta regime corresponds to small t_k, $1\le k\le n-1$ according to eq.(14),

$$Re\,t_k = \frac{P}{\sum_{r=1}^k p_r} + \mathcal{O}\left(\frac{P}{\sum_{r=1}^k p_r}\right)^2 \quad , \quad Im\,t_k = \pi\,.$$

Since the operator $B^{(n)}(u)$ for small $Re\,t_k$ and $Im\,t_k = \pi$ contains more derivatives ∂_k than factors $t_{r_1r_2}$ compensating them, we obtain in this regime for an arbitrary number n of reggeons

$$B^{(n)}(u) \simeq -P\left[u^{n-1}+u^{n-2}i(n-1)\partial_t+\ldots\right] \simeq -P\left[u^{n-1}-u^{n-2}\sum_{j=1}^{n-1}\lambda_j^*+\ldots\right]$$

where $t = t_1+\ldots+t_{n-1}$.

Therefore, the transformation kernels for n reggeons have the large-p behavior similar to the case of $n=2$ and 3,

$$U_{\overrightarrow{\lambda_1},\ldots,\overrightarrow{\lambda_{n-1}}}(\overrightarrow{t_1},\ldots,\overrightarrow{t_{n-1}}) = c^{ps}_{\overrightarrow{\lambda_1},\ldots,\overrightarrow{\lambda_{n-1}}}\prod_{k=1}^{n-1}\left(e^{i\frac{\sum_{l=1}^{n-1}\lambda_l^*}{n-1}t_k}\, e^{i\frac{\sum_{s=1}^{n-1}\lambda_s}{n-1}t_k^*}\right)$$

where the constant $c_{\overrightarrow{\lambda_1},\ldots,\overrightarrow{\lambda_{n-1}}}^{ps}$ is fixed by the normalization condition with the integration over the region of large momenta.

Thus, the pseudo-vacuum state in the BS representation can be written as follows,

$$\Phi^0(\overrightarrow{P},,\overrightarrow{\lambda_1},\ldots,\overrightarrow{\lambda_{n-1}}) \sim |P|^{2n}\, c^{ps}_{\overrightarrow{\lambda_1},\ldots,\overrightarrow{\lambda_{n-1}}}\times$$

$$\int \prod_{k=1}^{n-1} \left(d^2 t_k \, e^{i \frac{\sum_{l=1}^{n-1} \lambda_l^*}{n-1} t_k} \, e^{i \frac{\sum_{s=1}^{n-1} \lambda_s}{n-1} t_k^*} \right) \prod_{r=1}^{n-2} |t_r - t_{r+1}|^{-2} \prod_{m=1}^{n} |p_m|^2 \ln |p_m|^2 \sim$$

$$|P|^{2n} \, c_{\overrightarrow{\lambda_1}, \ldots, \overrightarrow{\lambda_{n-1}}} \int \prod_{k=1}^{n-1} \left[d^2 p_k \, e^{\frac{i}{n-1} \left(\frac{\sum_{l=1}^{n-1} \lambda_l^*}{\sum_{r=1}^{k} p_r} + \frac{\sum_{s=1}^{n-1} \lambda_s}{\sum_{r=1}^{k} p_r^*} \right)} \right] \prod_{m=1}^{n} \ln |p_m|^2 .$$

Using dimensional arguments we can write the result of the integration for the pseudo-vacuum wave function as

$$\Phi^0(\overrightarrow{P}, , \overrightarrow{\lambda_1}, \ldots, \overrightarrow{\lambda_{n-1}}) \sim |P|^{2n} \, c^{ps}_{\overrightarrow{\lambda_1}, \ldots, \overrightarrow{\lambda_{n-1}}} \left| \sum_{s=1}^{n-1} \lambda_s \right|^{2(n-1)} \tag{25}$$

up to a divergent λ-independent factor which can be regularized by changing the value of the Heisenberg spins $s \to -1 + \epsilon$.

Thus, providing that only large momenta are essential in the unitary transformation, the pseudovacuum state in the BS representation is expressed in terms of the normalization constant $c^{ps}_{\overrightarrow{\lambda_1}, \ldots, \overrightarrow{\lambda_{n-1}}}$ times $\left|\sum_{s=1}^{n-1} \lambda_s\right|^{2(n-1)}$. For the pomeron $c_{\overrightarrow{\lambda}}$ does not depend on λ [see eq.(21)]. As it was argued above, for the kernel $U_{\overrightarrow{\lambda_1}, \ldots, \overrightarrow{\lambda_{n-1}}}(\overrightarrow{t_1}, \ldots, \overrightarrow{t_{n-1}})$, describing the transition between the momentum and BS representations for the pseudo-vacuum wave function, it is natural to take into account in the normalization condition only the contribution from large momenta (presumably this is valid also for all states with integer conformal weights m and $\widetilde{m}$). We obtain in this way,

$$c^{ps}_{\overrightarrow{\lambda_1}, \ldots, \overrightarrow{\lambda_{n-1}}} = 1 \tag{26}$$

up to some λ_k independent factor.

The above arguments are in accordance with the Sklyanin theory in which the pseudo-vacuum state in the wave function is considered as a multiplier allowing to write the other multiplier as a product of the Baxter functions. Because, as it will be shown below, the Baxter function for the n-reggeon composite state contains the poles $Q(\lambda) \sim (\lambda - i\,r)^{-(n-1)}$ for $r = 0, 1, \ldots$, it is natural to expect that the wave function of the pseudo-vacuum state cancels some of these poles. Moreover, since from each solution of the Baxter equation we can obtain other solutions multiplying it by factors $\sinh^k(2\pi\,\lambda)$, this symmetry should appear as a possibility to multiply the pseudo-vacuum state by such factors. Generally, the pseudo-vacuum state is not symmetric under the permutation of the parameters λ_1, λ_2, ..., λ_{n-1}. In order to find the Hamiltonian in the Baxter-Sklyanin representation we consider below the region where $\lambda_{n-1} \to i$, $\lambda_s \to 0$ $(s = 1, 2, \ldots, n-2)$ and we show that the wave function has only a single pole at $\lambda_{n-1} = i$. Therefore, in order to agree with the Baxter representation we should put

$$\lim_{\lambda_{n-1} \to i} c^{ps}_{0, \ldots, 0, \overrightarrow{\lambda_{n-1}}} \to \sinh^{n-2}(2\pi\,\lambda) \, \sinh^{n-2}(2\pi\,\lambda^*) \, . \tag{27}$$

4 BS wave function for the Pomeron

The wave function of the Pomeron in the momentum representation is given by,

$$\Psi_{m,\widetilde{m}}(\overrightarrow{p_1},\,\overrightarrow{p_2}) = \prod_{r=1}^{2}(\overrightarrow{p_r})^2 \int \prod_{k=1}^{2}\left[\frac{d^2\rho_k}{2\pi}\,\exp(i\overrightarrow{p_k}\overrightarrow{\rho_{k0}})\right]\,\left(\frac{\rho_{12}}{\rho_{10}\rho_{20}}\right)^{m}\left(\frac{\rho_{12}^*}{\rho_{10}^*\rho_{20}^*}\right)^{\widetilde{m}} \tag{28}$$

and corresponds to the contribution of a triangle diagram. According to Appendix A the matrix element for this triangle diagram in the momentum space is

$$\Psi_{m,\widetilde{m}}(\overrightarrow{p_1},\,\overrightarrow{p_2}) = C_{m,\widetilde{m}}\quad\int d^2k\,\left[\frac{k\,(P-k)}{p_1-k}\right]^{\widetilde{m}-1}\,\left[\frac{k^*(P^*-k^*)}{p_1^*-k^*}\right]^{m-1}\,, \tag{29}$$

where

$$C_{m,\widetilde{m}} = -(-1)^n\frac{i^{\widetilde{m}-m}\;m\;\widetilde{m}}{2^{m+\widetilde{m}+4}\;\pi^3}\,\frac{\Gamma(1-m)}{\Gamma(\widetilde{m})}\,.$$

Note, that the above integral over $\overrightarrow{k}$ is convergent at the singular points of the integrand and at the infinity providing that m and $\widetilde{m}$ correspond to the principal series of the unitary representations

$$m = \frac{1}{2}+i\nu+\frac{n}{2}\,,\;\widetilde{m} = \frac{1}{2}+i\nu-\frac{n}{2}\,.$$

Terms proportional to $\delta^{(2)}(\overrightarrow{p_1})$ and $\delta^{(2)}(\overrightarrow{p_2})$ [17] can be neglected here since they are multiplied by $\overrightarrow{p_1}^2\;\overrightarrow{p_2}^2$.

One can analytically continue it for other values of m and $\widetilde{m}$. In particular, for $m=\widetilde{m}\to 2$, corresponding to the pseudo-vacuum state, we obtain that the leading contributions from two regions: small $|p_1-k|$

$$\int d^2k\,\left(\frac{k^*(p_1^*+p_2^*-k^*)}{p_1^*-k^*}\right)^{m-1}\,\left(\frac{k(p_1+p_2-k)}{p_1-k}\right)^{\widetilde{m}-1}\simeq -\frac{2\pi\,|p_1|^2\,|p_2|^2}{m+\widetilde{m}-4}$$

and large $|k|$

$$|p_1|^2\,|p_2|^2\int d^2k\,\frac{[k^*(p_1^*+p_2^*-k^*)]^{m-1}}{(p_1^*-k^*)^{m+1}}\,\frac{(k(p_1+p_2-k))^{\widetilde{m}-1}}{(p_1-k)^{\widetilde{m}+1}}\simeq\frac{2\pi\,|p_1|^2\,|p_2|^2}{m+\widetilde{m}-4}$$

cancel. The final result turns out to be proportional to $|p_1|^2\,|p_2|^2\ln|p_1|^2\ln|p_2|^2$.

Thus, the Pomeron wave function $\Phi_{m,\widetilde{m}}(\overrightarrow{P},\,\overrightarrow{\lambda})$ in the BS representation is (for $\lambda=-\lambda_1$)

$$\frac{\Phi_{m,\widetilde{m}}(\overrightarrow{P},\,\overrightarrow{\lambda})}{P^{\widetilde{m}}\;(P^*)^m} = \int\frac{d^2p}{|p(1-p)|^2}\,\left(\frac{p}{1-p}\right)^{i\lambda^*}\left(\frac{p^*}{1-p^*}\right)^{i\lambda}\,\Psi_{m,\widetilde{m}}(\overrightarrow{p},\,\overrightarrow{1}-\overrightarrow{p})$$

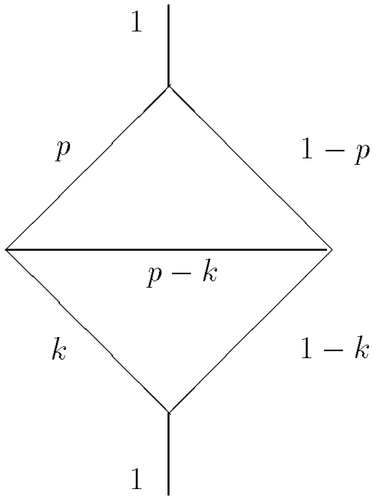
$$= C_{m,\widetilde{m}} \int \frac{d^2 p}{|p(1-p)|^2} \left(\frac{p}{1-p}\right)^{i\lambda^*} \left(\frac{p^*}{1-p^*}\right)^{i\lambda} \int d^2 k \left[\frac{k^*(1-k^*)}{p^*-k^*}\right]^{m-1} \left[\frac{k(1-k)}{p-k}\right]^{\widetilde{m}-1} \quad (30)$$

where $p = p_1/P, p^* = p_1^*/P^*$, and k and k^* were also rescaled by P and P^*, respectively. The integrand is a single-valued function only for the following values of the variable λ:

$$\lambda_1 = \sigma + i\,\frac{N}{2} \quad , \quad \lambda_1^* = \sigma - i\,\frac{N}{2}$$

where σ and N are correspondingly real and integer numbers.

Eq.(30) admits a natural interpretation as the Feynman diagram:



The Feynman rule is as follows: a line where a momentum p flows has associated the (conformal) 'propagator' $p^{i\lambda^*}\,(p^*)^{i\lambda}$.

To calculate $\Psi_{m,\widetilde{m}}(\overrightarrow{P},\,\overrightarrow{\lambda})$ we use the anti-Wick rotation $p_2 \to -ip_0$, $k_2 = -ik_0$ and introduce $i\epsilon$ to keep the singularities off the integration paths:

$$|p|^2 = p\,p^* \to p_1^2 - p_0^2 - i\epsilon\,. \quad (31)$$

Let us concentrate our attention on the integrals over k^* and p^*. The position of the singularities in k^* and p^* in the integrand of eq.(30):

$$(p\,p^* - i\epsilon)^{i\lambda^*-1}\;\left[(1-p)\,(1-p^*) - i\epsilon\right]^{-i\lambda-1}\;\left[(p-k)(p^*-k^*) - i\epsilon\right]^{1-m}$$

depend on the values of k and p. Therefore, the three singularities in p^* (or k^*) may be on one side of the real axis or one of them on one side and two others in the other side. In the first case we can deform the contour on p^* (or k^*) and the integral vanishes. We obtain for the non-zero contributions after enclosing contours of integration over k^* and p^* around the singularities of the integrand

$$\Phi_{m,\widetilde{m}}(\overrightarrow{P},\,\overrightarrow{\lambda}) = P^{\widetilde{m}}\,(P^*)^m\,C_{m,\widetilde{m}}\;i\,\sinh(\pi\lambda)\,\sin(\pi\widetilde{m})\,\Psi_{m,\widetilde{m}}(\overrightarrow{\lambda})\,,$$

where

$$\Psi_{m,\widetilde{m}}(\overrightarrow{\lambda}) = \int_0^1 dp\,\frac{p^{i\lambda^*-1}}{(1-p)^{1+i\lambda^*}} \int_0^p dk\,\frac{(p-k)^{1-\widetilde{m}}}{[k(1-k)]^{1-\widetilde{m}}} \int_1^\infty dp^*\,\frac{(p^*)^{i\lambda-1}}{(p^*-1)^{1+i\lambda}} \int_{-\infty}^0 dk^*\,\frac{(p^*-k^*)^{1-m}}{[k^*(k^*-1)]^{1-m}}$$

$$-(-1)^n \int_0^1 dp\; \frac{p^{i\lambda^*-1}}{(1-p)^{1+i\lambda^*}} \int_p^1 dk\; \frac{(k-p)^{1-\widetilde{m}}}{[k(1-k)]^{1-\widetilde{m}}} \int_{-\infty}^0 dp^*\; \frac{(-p^*)^{i\lambda-1}}{(1-p^*)^{1+i\lambda}} \int_1^\infty dk^*\; \frac{(k^*-p^*)^{1-m}}{[k^*(k^*-1)]^{1-m}} \quad (32)$$

The integrals over p^* and k^* as well as those over p and k can be transformed using relations of the type

$$\begin{aligned}
\Phi^{(1)}_{\widetilde{m}}(p) &= \int_p^1 dk\; \frac{(k-p)^{1-\widetilde{m}}}{[k(1-k)]^{1-\widetilde{m}}} = -\int_{-\infty}^0 dk\; \frac{(p-k)^{1-\widetilde{m}}}{[-k\,(1-k)]^{1-\widetilde{m}}} \,, \\
\Phi^{(2)}_{\widetilde{m}}(p) &= \int_0^p dk\; \frac{(p-k)^{1-\widetilde{m}}}{[k(1-k)]^{1-\widetilde{m}}} = -\int_1^\infty dk\; \frac{(k-p)^{1-\widetilde{m}}}{[k\,(k-1)]^{1-\widetilde{m}}} \,.
\end{aligned} \quad (33)$$

Each of the two terms in eq.(32) factorizes into holomorphic and anti-holomorphic functions. We can thus write $\Psi_{m,\widetilde{m}}(\overrightarrow{\lambda})$ as

$$\Psi_{m,\widetilde{m}}(\overrightarrow{\lambda}) = i\frac{\sinh(\pi\lambda)}{\sin(\pi m)} \left[C^{(2)}_{\widetilde{m}}(\lambda^*)\, C^{(2)}_{m}(\lambda) + (-1)^n\, C^{(1)}_{\widetilde{m}}(\lambda^*)\, C^{(1)}_{m}(\lambda)\right] .$$

where,

$$\begin{aligned}
C^{(1)}_{\widetilde{m}}(\lambda^*) &= \int_0^1 \frac{p^{i\lambda^*-1}\,d\,p}{(1-p)^{1+i\lambda^*}}\, \Phi^{(1)}_{\widetilde{m}}(p) = i\frac{\sin(\pi\widetilde{m})}{\sinh(\pi\lambda^*)} \int_{-\infty}^0 \frac{d\,p\,(-p)^{i\lambda^*-1}}{(1-p)^{1+i\lambda^*}} \Phi^{(2)}_{\widetilde{m}}(p) \,, \\
C^{(2)}_{\widetilde{m}}(\lambda^*) &= \int_0^1 \frac{p^{i\lambda^*-1}\,d\,p}{(1-p)^{1+i\lambda^*}}\, \Phi^{(2)}_{\widetilde{m}}(p) = -i\frac{\sin(\pi\widetilde{m})}{\sinh(\pi\lambda^*)} \int_1^\infty \frac{d\,p\,p^{i\lambda^*-1}}{(p-1)^{1+i\lambda^*}} \Phi^{(1)}_{\widetilde{m}}(p) \,.
\end{aligned} \quad (34)$$

The functions $\Phi^{(1)}_{\widetilde{m}}(p)$ and $\Phi^{(2)}_{\widetilde{m}}(p)$ are related with each other as follows

$$\lim_{\varepsilon\to 0,\, p<0} \left[\Phi^{(1)}_{\widetilde{m}}(p-i\varepsilon) - \Phi^{(1)}_{\widetilde{m}}(p+i\varepsilon)\right] = -i\, \sin(\pi\widetilde{m})\, \Phi^{(2)}_{\widetilde{m}}(p) \,,$$

$$\lim_{\varepsilon\to 0,\, p>1} \left[\Phi^{(2)}_{\widetilde{m}}(p-i\varepsilon) - \Phi^{(2)}_{\widetilde{m}}(p+i\varepsilon)\right] = i\, \sin(\pi\widetilde{m})\, \Phi^{(1)}_{\widetilde{m}}(p) \,.$$

Upon changing the integration variables

$$p \to 1-p,\; k \to 1-k$$

one verifies from eq.(34) that

$$C^{(2)}_{m}(\lambda) = C^{(1)}_{m}(-\lambda)$$

and therefore

$$\frac{\Phi_{m,\widetilde{m}}(\overrightarrow{P},\, \overrightarrow{\lambda})}{P^{\widetilde{m}}\, P^{*m}\, C_{m,\widetilde{m}}} = -\sinh^2(\pi\lambda)\; \left[C^{(1)}_{\widetilde{m}}(\lambda^*)\, C^{(1)}_{m}(\lambda) + (-1)^n\, C^{(1)}_{\widetilde{m}}(-\lambda^*)\, C^{(1)}_{m}(-\lambda)\right] .$$

We use the limiting value for the pseudo-vacuum state $m = \widetilde{m} \to 2$,

$$\Phi^{(1)}_{\widetilde{m}}(p) \to \frac{p(1-p)}{2-\widetilde{m}}\,, C^{(1)}_{\widetilde{m}}(\lambda^*) \to \frac{\pi\,\lambda^*}{\sinh(\pi\lambda^*)(2-\widetilde{m})}\,,\; C_{m,\widetilde{m}} \to \frac{1}{2\pi[2-\widetilde{m}]}$$

to normalize our wave function.

We can write the result in terms of the product of the Baxter functions $Q(\lambda,\, m)$ and the pseudovacuum state:

$$\frac{\Phi_{m,\widetilde{m}}(\overrightarrow{P},\,\overrightarrow{\lambda})}{P^{\widetilde{m}}\,P^{*m}\,C_{m,\widetilde{m}}} = -\frac{(-1)^N\;|\lambda|^2}{m\;\widetilde{m}}\left[Q(\lambda^*,\widetilde{m})\,Q(\lambda,m) + (-1)^n\,Q(-\lambda^*,\widetilde{m})\,Q(-\lambda,m)\right]\,, \tag{35}$$

where $Q(\lambda, m)$ is defined as

$$\begin{aligned} Q(\lambda,m) &= -m\frac{\sinh(\pi\lambda)}{\lambda}\int_0^1 \frac{p^{i\lambda-1}\,d\,p}{(1-p)^{1+i\lambda}}\int_p^1 \frac{(k-p)^{1-m}\,d\,k}{[k(1-k)]^{1-m}} \\ &= -i\sinh(\pi\lambda)\int_0^1 \frac{p^{i\lambda-1}\,d\,p}{(1-p)^{1+i\lambda}}\int_p^1 \frac{(k-p)^{-m}\,d\,k}{[k(1-k)]^{1-m}}\,. \end{aligned} \tag{36}$$

These two equivalent integral forms of the Baxter function are related with integrating by parts in p and using the identity

$$\frac{d}{dk}\left[\frac{k(1-k)}{k-p}\right]^m = -m\,\frac{[k(1-k)]^{m-1}}{(k-p)^{1+m}}\left[p(1-p)+(k-p)^2\right]\;.$$

This corresponds to the fact that the pomeron wave function is an eigenfunction of the Casimir operator of the Möbius group

$$p(1-p)\frac{d^2}{dp^2}\,\Phi^{(1,2)}_{\widetilde{m}}(p) = m(1-m)\,\Phi^{(1,2)}_{\widetilde{m}}(p)\,.$$

Using eq.(36) and the identity

$$\lambda\;p^{i\lambda-1}\;(1-p)^{-i\lambda-1} = -i\;\frac{d}{dp}\left[\;p^{i\lambda}\;(1-p)^{-i\lambda}\right]$$

we obtain that

$$\lambda^2\,Q(\lambda,m) = -i\;m\;\sinh(\pi\lambda)\int_0^1 \frac{p^{i\lambda}\,d\,p}{(1-p)^{i\lambda}}\,\frac{d}{dp}\int_p^1 \frac{(k-p)^{1-m}\,d\,k}{[k(1-k)]^{1-m}}$$

and

$$\left(\frac{p}{1-p}\right)^{i(\lambda+i)} + 2\left(\frac{p}{1-p}\right)^{i\lambda} + \left(\frac{p}{1-p}\right)^{i(\lambda-i)} = \frac{1}{p(1-p)}\left(\frac{p}{1-p}\right)^{i\lambda}\,,$$

The Baxter equation for the pomeron (12) follows using both integral representations (36) for $Q(\lambda, m)$

$$(\lambda+i)^2 Q(\lambda+i,m) - 2\lambda^2\,Q(\lambda,m) + (\lambda-i)^2 Q(\lambda-i,m) = m(1-m)\,Q(\lambda,m)\,. \tag{37}$$

It should be noticed that if $Q(\lambda, m)$ is a solution of the Baxter equation for the Pomeron, then $Q(-\lambda, m)$ is also a solution.

5 Analytic properties of the Baxter function for the Pomeron

The functions $\Phi_m^{(1)}(p)$ and $\Phi_m^{(2)}(p)$ defined by eq.(33) can be written in terms of the hypergeometric function ${}_2F_1(\alpha,\beta;\gamma;z)$ and the Legendre function $P_{m-1}(z)$ [20]

$$\begin{aligned}
\Phi_m^{(2)}(p) &= \Phi_m^{(1)}(1-p) \\
&= \frac{\pi\,(1-m)}{\sin(\pi\,m)}\,p\,F(1-m,\,m;\,2\,;\,p) \\
&= \frac{\pi}{m\,\sin(\pi\,m)}\,p(1-p)\frac{d}{d\,p}P_{m-1}(1-2p)\;.
\end{aligned} \tag{38}$$

Therefore, we have for the Baxter function

$$Q(\lambda,m) = -i\,\frac{\pi\,\sinh(\pi\lambda)}{\sin(\pi\,m)}\int_0^1 dp\;(1-p)^{i\lambda-1}\;p^{-i\lambda-1}\;P_{m-1}(1-2p) \tag{39}$$

$$= -\frac{\pi^2\,m\,(1-m)}{\sin\pi\,m}\;{}_3F_2(-i\lambda+1,2-m,\,1+m;\,2,\,2;\,1)\;, \tag{40}$$

where the generalized hypergeometric function ${}_3F_2$ is defined as follows [20]

$${}_3F_2(\alpha_1,\,\alpha_2,\,\alpha_3\,;\,\beta_1,\,\beta_2;\,z) = \sum_{k=0}^{\infty}\frac{(\alpha_1)_k\;(\alpha_2)_k\;(\alpha_3)_k}{(\beta_1)_k\,(\beta_2)_k}\,\frac{z^k}{k!}\,,\;\;(\alpha)_k = \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)}\,.$$

One can find for the pseudo-vacuum state at $m\to 2$,

$$Q(\lambda,m) = \frac{2\pi}{m-2} + \mathcal{O}(1)\;. \tag{41}$$

We thus have the following series representation for the Baxter function

$$Q(\lambda,m) = -\pi\sum_{k=1}^{\infty}\frac{k\,\Gamma(m+k)\;\Gamma(1-m+k)}{(k!)^3}\;\frac{\Gamma(k-i\lambda)}{\Gamma(1-i\lambda)}\;. \tag{42}$$

The late terms in this series behave as $k^{-1-i\lambda}$. Hence, this is a convergent series for $Im\,\lambda < 0$.

In order to analytically continue the Baxter function to the upper λ plane we insert in eq.(39) the series representation of the Legendre function [21]

$$\begin{aligned}
P_{m-1}(1-2p) &= \frac{\sin^2\pi m}{\pi^2}\sum_{k=0}^{\infty}\frac{\Gamma(m+k)\;\Gamma(1-m+k)}{(k!)^2}\,[2\,\psi(k+1) \\
&- \psi(k+1-m)-\psi(k+m)-\ln(1-p)]\,(1-p)^k
\end{aligned} \tag{43}$$

Integrating term by term in eq.(39) yields the series,

$$Q(\lambda,m) = \frac{i\,\pi}{\lambda} + \frac{\sin\pi m}{\Gamma(1+i\lambda)}\;\sum_{k=1}^{\infty}\frac{k\;\Gamma(m+k)\;\Gamma(1-m+k)}{(k!)^3}\;\Gamma(k+i\lambda)\times$$

$$[2\,\psi(k+1)+\psi(k)-\psi(k+1-m)-\psi(k+m)-\psi(i\lambda+k)] \tag{44}$$

The late terms in this series behave as $k^{-2+i\lambda}$. Hence, this is a convergent series for $Im\,\lambda > -1$. Eq.(44) explicitly display simple poles at

$$\lambda = 0, i, 2i, 3i, \ldots, l\,i, \ldots\ .$$

Actually, the poles at $\lambda = +il$ $(l = 1, 2, \ldots)$ arise from the logarithmic singularity of $P_{m-1}(1-2p)$ near $p = 1$ in the integral (39) [see eq.(43)].

Direct calculation yields from eq.(44) for the residue at these points

$$\begin{aligned} r_l(m) &\equiv \lim_{\lambda\to il}\,[\lambda - il]\; Q(\lambda, m) \\ r_0(m) &= i\,\pi \\ r_l(m) &= -i\,\pi\, m(1-m)\,{}_3F_2(-l+1, 2-m,\, 1+m;\, 2,\, 2;\, 1) = -\frac{\sin\pi m}{i\,\pi}\, Q(-il, m) \end{aligned} \tag{45}$$

for $l = 1,\, 2, \ldots$. It is interesting to notice that the residues of $Q(\lambda, m)$ at $\lambda = +il$ $(l = 1, 2, \ldots)$ are expressed in terms of $Q(\lambda, m)$ at $\lambda = -il$ $(l = 1, 2, \ldots)$.

In summary, eqs.(42) and (44) explicitly show that $Q(\lambda, m)$ is a meromorphic function of λ with simple poles at $\lambda = +il$ $(l = 0, 1, 2, \ldots)$. The appearance of these poles is related with the logarithmic singularities of the wave function at $p_1 = 0$ and $p_2 = 0$.

The pomeron wave function considered as a function of real σ for integer N may have in principle singularities [Recall that $\lambda = \sigma + i\frac{N}{2}$]. We find from eq.(35),

$$\begin{aligned} &\text{Pole at } \sigma = 0 \text{ of } [Q(\lambda^*, \widetilde{m})\, Q(\lambda, m) + (-1)^n\, Q(-\lambda^*, \widetilde{m})\, Q(-\lambda, m)] = \\ &= \frac{1}{\sigma}\,[r_l(m)\; Q(-il, \widetilde{m}) - (-1)^n Q(-il, m) r_l(\widetilde{m})] = 0 \end{aligned} \tag{46}$$

where N is even, $l = N/2$, we used eq.(45) and the relationship

$$\sin\pi m = (-1)^n\ \sin\pi\widetilde{m}\ .$$

It is important to notice that the poles at $\sigma = 0$ cancel, thus allowing to normalize the pomeron eigenfunctions. Analogous cancellations of poles of the Baxter function at real σ_k for a higher number of Reggeons should lead to the quantization of the integrals of motion q_r, $r > 2$. In coordinate space this quantization appears as a consequence of the single-valuedness condition (see [14]).

Notice that both eqs.(42) and (44) exhibit the $m \Longleftrightarrow 1-m$ symmetry:

$$Q(\lambda, m) = Q(\lambda, 1-m)\ .$$

Therefore, $Q(\lambda, m)$ depends on m through the invariant combination $m(1-m)$ as we check explicitly below [see eqs.(48)-(49)]. Note, that this combination for the principal series takes the form

$$m(1-m) = \frac{1}{4} - \left(i\nu + \frac{n}{2}\right)^2 .$$

We see from eq.(39) and (44) that the function $Q(\lambda, m)$ obeys the relation

$$\bar{Q}(\lambda, \bar{m}) = Q(-\bar{\lambda}, m) .$$

That is, $Q(\lambda, m)$ is real for purely imaginary λ and real m. We have for real λ and m,

$$\mathrm{Re}\, Q(-\lambda, m) = +\mathrm{Re}\, Q(\lambda, m) \quad , \quad \mathrm{Im}\, Q(-\lambda, m) = -\mathrm{Im}\, Q(\lambda, m) .$$

The asymptotic behavior of $Q(\lambda, m)$ for large λ is derived in appendix B starting from the integral representation (39),

$$Q(\lambda, m) = 4\sqrt{\pi} \left[(4\,i\,\lambda)^{m-2} \; \frac{\Gamma\left(m - \frac{1}{2}\right)\, \Gamma(2-m)}{\Gamma(m)} + \right. \tag{47}$$

$$\left. + \, (4\,i\,\lambda)^{-1-m} \; \tan \pi m \; \frac{\Gamma(m)\; \Gamma(m+1)}{\Gamma\left(m + \frac{1}{2}\right)} + \mathcal{O}\left(\lambda^{m-4}, \lambda^{-m-3}\right) \right] .$$

The Baxter equation for the pomeron (37) written in the form

$$Q(\lambda, m) = \frac{(\lambda + i)^2 Q(\lambda + i, m) + (\lambda - i)^2 Q(\lambda - i, m)}{2\lambda^2 + m(1-m)}$$

would seem to suggest that $Q(\lambda, m)$ has singularities at the zeros $\pm i\; \eta_m$ of the denominator where

$$\eta_m \equiv \sqrt{\frac{1}{2}\; m(1-m)} = \sqrt{\frac{1}{8} - \left(i\nu + \frac{n}{2}\right)^2} .$$

However, we know that $Q(\lambda, m)$ is analytic there. Therefore, the following relation holds,

$$\frac{Q(i\,\eta_m + i, m)}{Q(i\,\eta_m - i, m)} = -\left(\frac{\eta_m - 1}{\eta_m + 1}\right)^2$$

5.1 Dispersion (Mittag-Löffler) representation of the Baxter function

We obtain from the Baxter equation (37) a recurrence relation for the residues $r_l(m)$

$$(l+1)^2\; r_{l+1}(m) + (l-1)^2\; r_{l-1}(m) = \left[2\, l^2 + m(m-1)\right] r_l(m) \quad \text{for } l \geq 1 \;,$$

$$r_1(m) = m(m-1)\; r_0(m) . \tag{48}$$

All residues are thus determined in terms of the residue at the origin $r_0(m)$. That is,

$$r_1(m) = m(m-1)\; r_0(m) \;,\; r_2(m) = \frac{1}{4}\; m(m-1)\, [2 + m(m-1)]\;\; r_0(m) \;,$$

$$r_3(m) = \frac{1}{9}\, m(m-1)\, \left[3 + \frac{5}{2}\, m(m-1) + \frac{1}{4}\, m^2(m-1)^2\right]\, r_0(m)\,. \tag{49}$$

It is easy to check that eqs.(42) and (45) agree with these results.

The asymptotic behavior of the residues $r_l(m)$ follows from the recursion relation (48). We find, for large l

$$r_l(m) = c(m)\, l^{m-2} + c(1-m)\, l^{-m-1}\,. \tag{50}$$

Therefore, we can write the following Mittag-Löffler (dispersion) representation for the Baxter function

$$\begin{aligned} Q(\lambda,m) &= \sum_{l=0}^{\infty} \frac{r_l(m)}{\lambda - il} \\ &= \frac{i\,\pi}{\lambda} - \frac{\sin \pi m}{i\,\pi} \sum_{l=1}^{\infty} \frac{Q(-il,m)}{\lambda - il}\,. \end{aligned} \tag{51}$$

The asymptotic behavior (50) guarantees the convergence of this sum for $-1 < \mathrm{Re}\, m < 2$.

In general, we have for $m < p+2$ where p is a positive integer or zero[22]

$$Q(\lambda,m) = F_p(\lambda,m) + \sum_{l=0}^{\infty} r_l(m) \left[\frac{1}{\lambda - i\,l} - h_{l,p}(\lambda)\right]\;,$$

where,

$$\begin{aligned} F_p(\lambda,m) &= \sum_{k=0}^{p} \frac{(\lambda+i)^k}{k!}\, Q^{(k)}(-i,m)\,, \\ h_{l,p}(\lambda) &= \sum_{k=0}^{p} \frac{i^{1-k}}{(l+1)^{k+1}}\, (\lambda+i)^k\,. \end{aligned} \tag{52}$$

For example,

$$F_0(\lambda,m) = Q(-i,m) = -\pi^2\, \frac{m(1-m)}{\sin \pi m}\quad,$$

$$F_1(\lambda,m) = Q(-i,m) + (\lambda+i)Q'(-i,m) = -\pi^2\, \frac{m(1-m)}{\sin \pi m} + i\pi(\lambda+i) \sum_{k=2}^{\infty} \frac{\Gamma(m+k)\,\Gamma(1-m+k)}{(k-1)\,(k!)^2}$$

We get from eq.(44) for $Q(\lambda,m)$ in the limit $\lambda \to i$,

$$\lim_{\lambda\to i} Q(\lambda,m) = \pi\; m\,(1-m) \left[-\frac{i}{\lambda - i} + 2 - \psi(m) - \psi(1-m) + 2\psi(1)\right] \tag{53}$$

and therefore

$$-\,i \lim_{\lambda\to i} \frac{d}{d\,\lambda} \ln Q(\lambda,m) = \frac{i}{\lambda - i} + 2 - \psi(m) - \psi(1-m) + 2\psi(1)\,. \tag{54}$$

We obtain for the other (independent) solution $Q(-\lambda, m)$ of the Baxter equation,

$$- i \lim_{\lambda \to -i} \frac{d}{d\,\lambda} \ln Q(-\lambda, m) = \frac{i}{\lambda + i} - 2 + \psi(m) + \psi(1-m) - 2\psi(1) \; . \tag{55}$$

The behaviour of the Baxter function $Q(\lambda, m)$ near its nearest pole $\lambda = i$ can be also computed from the Mittag-Löffler expansion (51) with the result

$$Q(\lambda, m) \stackrel{\lambda \to i}{=} \frac{i\,\pi\,m\,(1-m)}{\lambda - i} - i r_0(m) + i \sum_{l=2}^{\infty} \frac{r_l(m)}{l-1} \tag{56}$$

Equating this result with eq.(53) yields the sum rule,

$$-i r_0(m) + \sum_{l=2}^{\infty} \frac{r_l(m)}{l-1} = i\,\pi\,m\,(1-m)\,[\psi(m) + \psi(1-m) - 2\psi(1) - 2] \; .$$

For large λ the Mittag-Löffler series (51) is dominated by its late terms. Notice that the sum of residues

$$\sum_{l=0}^{\infty} r_l(m) = 0 \tag{57}$$

vanishes. The sum of late terms can be approximated by an integral. Using eq.(50) we find

$$Q(\lambda, m) \stackrel{\lambda \gg 1}{=} c(m)\, \frac{\pi}{\sin \pi m}\, (i\,\lambda)^{m-2} + (m \Longleftrightarrow 1-m)$$

in perfect agreement with eq.(47).

5.2 Infinite Product Representation of the Baxter Function

As we have seen, the Baxter function $Q(\lambda, m)$ is a meromorphic function of λ with simple poles at $\lambda = +il$ $(l = 1, 2, \ldots)$. Therefore, the function

$$\frac{Q(\lambda, m)}{\Gamma(i\lambda)}$$

is an entire function. Entire functions can be represented as infinite products over their zeroes [22].

Numerical study of the Baxter function $Q(\lambda, m)$ in its Mittag-Löffler representation showed that all zeroes of $Q(\lambda, m)$ are in the positive imaginary λ axis for Re $m < 2$. We have in addition p real zeroes for $p + 4 >$ Re $m > p + 2$ where p is a positive integer or zero.

For large λ the imaginary zeroes $\lambda_k,\ k = 1, 2, \ldots$ are equally spaced and follows for large k the law:

$$\lambda_k = i \left[k + 1 - m + \mathcal{O}\left(\frac{1}{k^a}\right)\right] \tag{58}$$

for Re $m < 1/2$ and where $a \sim 0.5$.

We assume the infinite product representation[22],

$$\frac{Q(\lambda,m)}{\Gamma(i\lambda)} = B\; e^{A\,\lambda}\;\prod_{k=1}^{\infty}\left(1-\frac{\lambda}{\lambda_k}\right)e^{\frac{\lambda}{\lambda_k}}\;.$$

The asymptotic behaviors (58) and (121) are consistent provided (see Appendix B)

$$A = -i\;\psi(2-m)\;.$$

In addition, $B=-\pi$ according to eq.(51). In summary, we have,

$$\frac{Q(\lambda,m)}{\Gamma(i\lambda)} = -\pi\; e^{-i\lambda\,\psi(2-m)}\;\prod_{k=1}^{\infty}\left(1-\frac{\lambda}{\lambda_k}\right)e^{\frac{\lambda}{\lambda_k}}\;. \tag{59}$$

where $\psi(z)$ is the digamma function.

The Bethe Ansatz equations are algebraic equations on zeroes of the Baxter function. They follow from the Baxter equation (37) and this infinite product representation. Eq.(37) can be recasted as,

$$\left[2\lambda^2-m(1-m)\right]\frac{Q(\lambda,m)}{\Gamma(i\lambda)} = -i(\lambda+i)\;\frac{Q(\lambda+i,m)}{\Gamma(i[\lambda+i])}+i\lambda(\lambda-i)^2\;\frac{Q(\lambda-i,m)}{\Gamma(i[\lambda-i])}$$

Setting here $\lambda=\lambda_k$ yields,

$$\frac{\lambda_k\;(\lambda_k-i)^2}{\lambda_k+i} = e^{2\psi(2-m)}\prod_{l=1}^{\infty}\left[e^{\frac{2i}{\lambda_l}}\,\frac{\lambda_l-\lambda_k-i}{\lambda_l-\lambda_k+i}\right]\;. \tag{60}$$

We see that the Bethe Ansatz equations are here an infinite system of algebraic equations. In the present Reggeon problem it is more effective to solve the Baxter equations by looking at the poles of the Baxter function [see eq.(48)] rather than to the Bethe Ansatz equations (60). In customary cases the Bethe Ansatz equations are the more effective tool [7].

Since the Baxter function is explicitly known for the Pomeron we can find an infinite number of sum rules for the zeroes λ_k just by matching eqs. (42) or (44) with eq.(59).

For example, for $\lambda\to i$ we get from eq.(59),

$$-i\lim_{\lambda\to i}\frac{d}{d\,\lambda}\ln Q(\lambda,m) = \frac{i}{\lambda-i}+1-\gamma+\sum_{k=1}^{\infty}\frac{1}{i\,\lambda_k\,(1+i\,\lambda_k)}-\psi(2-m)\;.$$

Equating with eq.(54) yields,

$$\sum_{k=1}^{\infty}\frac{1}{i\,\lambda_k\,(1+i\,\lambda_k)} = 1-\gamma-\psi(1-m)-\frac{1}{m-1}\;.$$

6 Solving the Baxter equation for the n-reggeon state

The Baxter equation for the odderon takes the form

$$\left[2\lambda^3 - m(m-1)\lambda + i\mu\right] Q\,(\lambda;\, m,\mu) =$$

$$(\lambda+i)^3\, Q\,(\lambda+i;\, m,\mu) + (\lambda-i)^3\, Q\,(\lambda-i;\, m,\mu)\ . \tag{61}$$

where

$$q_3 = i\mu\,,\ \ Im\,(\mu) = 0\,. \tag{62}$$

The reality propery of μ is needed to obtain the single-valuedness of the odderon wave function in the coordinate space [16].

Eq.(61) can be solved asymptotically for large λ making the power-like ansatz,

$$Q\,(\lambda;\, m,\mu) \overset{\lambda\to\infty}{=} \lambda^a \left[1 + \frac{b}{\lambda} + \mathcal{O}\left(\frac{1}{\lambda^2}\right)\right] \tag{63}$$

Inserting eq.(63) into eq.(61) yields two solutions: $a_+ = m-3$ and $a_- = -m-2$ connected by the the $m \Longleftrightarrow 1-m$ symmetry. The general solution of eq.(61) will have thus the asymptotic behavior

$$\begin{aligned} Q\,(\lambda;\, m,\mu) &\overset{\lambda\to\infty}{=} A_+\, \lambda^{m-3}\left[1 - i\frac{\mu}{(m-1)(m-2)\lambda} + \mathcal{O}\left(\frac{1}{\lambda^2}\right)\right] + \\ &+ \quad A_-\, \lambda^{-m-2}\left[1 - i\frac{\mu}{m(m+1)\lambda} + \mathcal{O}\left(\frac{1}{\lambda^2}\right)\right] \end{aligned}$$

where $A_\pm$ are constants.

For the pomeron case the Baxter function can be expressed as a sum of simple poles in the upper plane [see eq.(51)]. This is not the case for the odderon. If we try an Ansatz with only simple poles in the upper plane,

$$Q_0\,(\lambda) = \sum_{r=0}^{\infty} \frac{c_r(m,\mu)}{\lambda - r\,i}\,, \tag{64}$$

the asymptotic behaviour of this sum for $\lambda \to \infty$ turns out to be

$$Q_0\,(\lambda) = c/\lambda\,,\ \ c = \sum_{r=0}^{\infty} c_r(m,\mu)\,, \tag{65}$$

where c is not zero. Therefore Q_0 does not satisfy the Baxter equation for the odderon at $\lambda \to \infty$. In particular the behaviour (63) is not fulfilled by $Q_0\,(\lambda)$. Note, that in the case of the Pomeron the simple pole ansatz indeed satisfies the Baxter equation at infinity because the sum of residues vanishes [eq.(57)].

For the odderon, we have to include in the Anzatz double poles. It will be shown below, that in the general case of n reggeons Q contains poles of the order $n-1$. The odderon Baxter function can be written in the form

$$Q\,(\lambda;\, m,\, \mu) = \sum_{r=0}^{\infty} \left[-\frac{a_r(m,\, \mu)}{(\lambda - r\, i)^2} + i\, \frac{b_r(m,\, \mu)}{\lambda - r\, i} \right] . \tag{66}$$

The residues satisfy the recurrence relations

$$\begin{aligned}
& a_1(m,\, \mu) = -\mu\, a_0(m,\, \mu) \quad , \quad 8\, a_2(m,\, \mu) = [2 + m(m-1) - \mu]\ a_1(m,\, \mu)\ , \\
& (r+1)^3\, a_{r+1}(m,\, \mu) = \left[2\, r^3 + m(m-1)\, r - \mu\right]\ a_r(m,\, \mu) - (r-1)^3\, a_{r-1}(m,\, \mu)\ , \\
& b_1(m,\, \mu) = -\mu\, b_0(m,\, \mu) + m(m-1)\, a_0(m,\, \mu) - 3\, a_1(m,\, \mu)\, , \\
& 8\, b_2(m,\, \mu) = [2 + m(m-1) - \mu]\ b_1(m,\, \mu) + [6 + m(m-1)]\ a_1(m,\, \mu) - 12\, a_2(m,\, \mu)\, , \\
& (r+1)^3\, b_{r+1}(m,\, \mu) = \left[2\, r^3 + m(m-1)\, r - \mu\right]\ b_r(m,\, \mu) - (r-1)^3\, b_{r-1}(m,\, \mu) \\
& + \left[6\, r^2 + m(m-1)\right]\ a_r(m,\, \mu) - 3\,(r+1)^2\, a_{r+1}(m,\, \mu) - 3\,(r-1)^2\, a_{r-1}(m,\, \mu)\ .
\end{aligned} \tag{67}$$

We can normalize

$$a_0(m,\, \mu) = 1. \tag{68}$$

In order to fulfill the Baxter equation at infinity and to obtain the asymptotic behaviour (63), we impose

$$\sum_{r=0}^{\infty} b_r(m,\, \mu) = 0\ .$$

This equation fixes the value of $b_0(m,\, \mu)$. Therefore, all coefficients $a_n(m,\, \mu)$ and $b_n(m,\, \mu)$ are univocally determined.

We find for large r that both $a_r(m,\, \mu)$ and $b_r(m,\, \mu)$ decrease as r^{m-3}. As in the pomeron case, the asymptotic behaviour of the Baxter function for large λ is governed by the late terms in the Mittag-Löffler series (66). Evaluating the sum of such late terms we reproduce the asymptotic behaviour (64).

We see from the recurrence relations (67) that for $\mu \to 0$ the double poles with $r > 0$ disappear ($a_r(m,\, 0) = 0$). Only the double pole at the origin remains and the solution of the Baxter equation for the Odderon can be expressed in terms of that for the Pomeron. With our normalization we obtain:

$$Q\,(\lambda;\, m,\, 0) = -\frac{Q\,(\lambda,\, m)}{i\, \pi \lambda}\, , \tag{69}$$

where $Q\,(\lambda,\, m)$ is the pomeron wave function.

Eq.(69) holds because for small λ

$$\lim_{\lambda \to 0} Q\,(\lambda,\, m) = \frac{i\pi}{\lambda} + Q_0(m)\, .$$

To calculate $Q_0(m)$ we use eq.(53),

$$Q\,(-i,\,m) = -\pi^2\,\frac{m(m-1)}{\sin(\pi m)} \tag{70}$$

and the Baxter equation for $Q\,(\lambda,\,m)$ near $\lambda = 0$. We obtain

$$Q_0(m) = \pi\left[\frac{\pi}{\sin(\pi m)} + \psi(m) + \psi(1-m) - 2\psi(1)\right] \tag{71}$$

in agreement with the asymptotics at $m \to 2$ [see eq.(41)]

$$Q\,(\lambda,\,m) \to \frac{2\,\pi}{m-2}\,.$$

Therefore, we obtain the residue $b_0(m,\,\mu)$ of the simple pole in the Odderon solution for $\mu = 0$ as

$$\lim_{\mu\to 0} b_0(m,\,\mu) = \frac{\pi}{\sin(\pi m)} + \psi(m) + \psi(1-m) - 2\psi(1)\,. \tag{72}$$

Another independent solution of the Baxter equation can be written as follows

$$Q\,(-\lambda;\,m,\,-\mu) = \sum_{r=0}^{\infty}\left[-\frac{a_r(m,\,-\mu)}{(\lambda + r\,i)^2} - i\,\frac{b_r(m,\,-\mu)}{\lambda + r\,i}\right]\,. \tag{73}$$

Due to the property of the holomorphic factorization the Baxter function in the two-dimensional complex space has the form

$$Q_{m,\widetilde{m};\,\mu}\left(\overrightarrow{\lambda}\right) =$$

$$C^{(1)}_{m,\widetilde{m};\,\mu}\,Q\,(\lambda;\,m,\,\mu)\;Q\,(\lambda^*;\,\widetilde{m},\,-\mu) + C^{(2)}_{m,\widetilde{m};\,\mu}\,Q\,(-\lambda;\,m,\,-\mu)\;Q\,(-\lambda^*;\,\widetilde{m},\,\mu)\;, \tag{74}$$

where we took into account, that $q_3^* = -i\mu$ and the coefficients $C^{(1)}$, $C^{(2)}$ and the parameter μ are fixed by the condition of the normalizability of $Q_{m,\widetilde{m};\mu}\left(\overrightarrow{\lambda}\right)$. To show, that it is possible to satisfy this condition, let us investigate the holomorphic solution $Q\,(\lambda;\,m,\,\mu)$ near the points $\lambda = -i\,r$ $(r = 1, 2...)$:

$$\lim_{\lambda\to -i\,r} Q\,(\lambda;\,m,\,\mu) = A_r\,(m,\,\mu) + i(\lambda + ir)\,B_r\,(m,\,\mu)\;. \tag{75}$$

It is obvious, that $A_r\,(m,\,\mu)$ and $B_r\,(m,\,\mu)$ satisfy the same recurrence relations as $a_r\,(m,\,-\mu)$ and $b_r\,(m,\,-\mu)$ correspondingly. Therefore A_r should be proportional to a_r (for $r > 0$)

$$A_r\,(m,\,\mu) = \alpha\,(m,\,\mu)\;a_r\,(m,\,-\mu)\;. \tag{76}$$

But B_r is proportional to b_r only if the proportionalyty coefficient is the same

$$B_r\,(m,\,\mu) = \alpha\,(m,\,\mu)\;b_r\,(m,\,-\mu)\;. \tag{77}$$

This coefficient $\alpha\,(m,\,\mu)$ can be calculated with the use of the pole representation for the Baxter function and of the above recurrence relations for the pole residues.

The last condition for B_r is possible only for discreet values of μ and is needed to have a normalized solution in the two-dimensional space $\overrightarrow{\lambda}$. To show it we should take into account, that

$$\lambda = \sigma + i\,\frac{N}{2}\,,$$

where σ is real and N is integer. The possible singularities of $Q_{m,\widetilde{m};\mu}$ for even N are the poles of the first and second order at $\sigma = 0$. The residues of these poles should be zero to have a possibility to normalize the wave function. From the condition of the cancellation of the double pole at $\sigma = 0$ we obtain the following relation:

$$C^{(1)}_{m,\widetilde{m};\,\mu}\ \ \alpha\,(\widetilde{m},\,-\mu) = -C^{(2)}_{m,\widetilde{m};\,\mu}\ \ \alpha\,(m,\,-\mu) \tag{78}$$

and therefore we can chose $Q_{m,\widetilde{m};\,\mu}\left(\overrightarrow{\lambda}\right)$ in the form:

$$Q_{m,\widetilde{m};\,\mu}\left(\overrightarrow{\lambda}\right) =$$

$$\alpha\,(m,\,-\mu)\ Q\,(\lambda;\,m,\,\mu)\ Q\,(\lambda^*;\,\widetilde{m},\,-\mu) - \alpha\,(\widetilde{m},\,-\mu)\ Q\,(-\lambda;\,m,\,-\mu)\ Q\,(-\lambda^*;\,\widetilde{m},\,\mu)\ . \tag{79}$$

The condition of the cancellation of the single pole at $\sigma = 0$ gives the equation

$$\alpha\,(m,\,-\mu)\ (b_r\,(m,\,\mu)\ A_r\,(\widetilde{m},\,-\mu) - a_r\,(m,\,\mu)\ B_r\,(\widetilde{m},\,-\mu)) =$$

$$-\,\alpha\,(\widetilde{m},\,-\mu)\ (b_r\,(\widetilde{m},\,\mu)\ A_r\,(m,\,-\mu) - a_r\,(\widetilde{m},\,\mu)\ B_r\,(m,\,-\mu))\ . \tag{80}$$

One can easily verify, that it is indeed fulfilled, providing that

$$A_r\,(m,\,-\mu) = \alpha\,(m,\,-\mu)\ a_r\,(m,\,\mu)\ ,\ \ B_r\,(m,\,-\mu) = \alpha\,(m,\,-\mu)\ b_r\,(m,\,\mu)\ . \tag{81}$$

It is enough to impose this condition for $r = 1$,

$$A_1\,(m,\,-\mu)\ \ b_1\,(m,\,\mu) - a_1\,(m,\,\mu)\ \ B_1\,(m,\,-\mu) = 0\ ,$$

which leads to the quantization of the integral of motion:

$$\mu = \mu_k(m,\,\widetilde{m})\,. \tag{82}$$

Let us now consider the Baxter equation for the n-reggeon composite state:

$$\Lambda^{(n)}(\lambda;\ \vec{\mu})\,Q\,(\lambda;\,m,\vec{\mu}) = (\lambda+i)^n\ Q\,(\lambda+i;\,m,\vec{\mu}) + (\lambda-i)^n\ Q\,(\lambda-i;\,m,\vec{\mu})\ , \tag{83}$$

where $\Lambda^{(n)}(\lambda)$ is the polynomial

$$\Lambda^{(n)}(\lambda;\ \vec{\mu}) = \sum_{k=0}^{n}(-i)^k\,\mu_k\,\lambda^{n-k}\ ,\ \mu_0 = 2,\ \mu_1 = 0,\ \mu_2 = m(m-1)\,, \tag{84}$$

where we assume, that $\mu_r = i^r\,q_r$ for $r > 2$ are real numbers. It will be shown below, that the last condition is needed in order to have a normalizable wave functions.

We search the solution of this equation in the form of a sum over the poles of the orders from 1 up to $n-1$:

$$Q\,(\lambda;\, m, \vec{\mu}) = \sum_{r=0}^{\infty} \frac{P^{(n-2)}_{r;m,\vec{\mu}}(\lambda)}{(\lambda - i\,r)^{n-1}}\,. \tag{85}$$

Putting this ansatz in the equation, we obtain recurrence relations for polynomials $P^{(n-2)}_{r;m,\vec{\mu}}(\lambda)$ of the order $n-2$ allowing to calculate them providing that $P^{(n-2)}_{0;m,\vec{\mu}}(\lambda)$ is known.

Let us define the expansion of a function $f(\lambda)$ in the power series over $\lambda - i\,r$ up to the order $n-2$:

$$\left(f(\lambda)\right)^{(n-2)}_{r} = (\lambda - i\,r)^{n-2} \lim_{\lambda \to i\,r} \frac{f(\lambda)}{(\lambda - i\,r)^{n-2}}. \tag{86}$$

Then the recurrence relations for the coefficients of polynomials $P^{(n-2)}_{r;m,\vec{\mu}}(\lambda)$ can be written as follows

$$\begin{gathered}\left(\Lambda^{(n-2)}(\lambda,\ \vec{\mu}) P^{(n-2)}_{r;m,\vec{\mu}}(\lambda)\right)^{(n)}_{r} = \\ \left((\lambda + i)^n P^{(n-2)}_{r+1;m,\vec{\mu}}(\lambda + i)\right)^{(n-2)}_{r} + \left((\lambda - i)^n P^{(n-2)}_{r-1;m,\vec{\mu}}(\lambda - i)\right)^{(n-2)}_{r}\,.\end{gathered} \tag{87}$$

We can normalize the solution as follows

$$\lim_{\lambda \to i} P^{(0)}_{0;m,\vec{\mu}}(\lambda) = 1 \tag{88}$$

Then the other independent coefficients of the polynomial $P^{0)}_{0,m,\vec{\mu}}(\lambda)$ are determined from the condition

$$\lim_{\lambda \to \infty} Q\,(\lambda;\, m, \vec{\mu}) \sim \lambda^{-n+m}\,, \tag{89}$$

necessary to provide $Q\,(\lambda;\, m, \vec{\mu})$ to be a solution of the Baxter equation at infinity. According to the Baxter equation this condition is fulfilled if

$$\lim_{\lambda \to \infty} \lambda^{n-2} \sum_{r=0}^{\infty} \frac{P^{(n-2)}_{r;m,\vec{\mu}}(\lambda)}{(\lambda - i\,r)^{n-1}} = 0\,. \tag{90}$$

It gives a set of linear equations to calculate the coefficients of the polynomial $P^{0)}_{0,m,\vec{\mu}}(\lambda)$.

Let us calculate now the behaviour of the Baxter function near the regular points $\lambda = -i\,r$ $(r = 1, 2, ...)$:

$$\lim_{\lambda \to -i\,r} \frac{Q\,(\lambda;\, m,\, \vec{\mu})}{(\lambda + i\,r)^{n-2}} = \frac{S^{(n-2)}_{r;m,\vec{\mu}}(\lambda)}{(\lambda + i\,r)^{n-2}}\,, \tag{91}$$

where $S^{(n-2)}_{r;m,\vec{\mu}}(\lambda)$ are polynomials obeying certain recurrence relations due to the Baxter equation. These recurrence relations are the same as for $P^{(n-2)}_{r;m,\vec{\mu}}(\lambda)$ but with the substitution

$$\vec{\mu} \to \vec{\mu^s}\,, \tag{92}$$

where μ^s is obtained from μ by changing the signs of the parameters μ_k with odd indices:

$$\mu_k \to (-1)^k \, \mu_k\,.$$

This fact is related with the symmetry of the Baxter equation to the transformation $\lambda \to -\lambda$, $\mu \to \mu^s$.

Therefore we can chose the parameters μ_r for $r = 3, 4, ...n$ in such way, that

$$S^{(n-2)}_{r;m,\vec{\mu}}(\lambda) = \alpha^{(n-2)}_{m,\vec{\mu}}\, P^{(n-2)}_{r;m,\vec{\mu}^s}(\lambda)\,. \tag{93}$$

For these values of μ_r one can construct the normalized Baxter function in the two dimensional space $\overrightarrow{\lambda}$, where

$$\lambda = \sigma + i\,\frac{N}{2}\,.$$

Indeed, according to the property of the holomorphic separability we have

$$Q_{m,\,\widetilde{m},\,\vec{\mu}}\left(\overrightarrow{\lambda}\right) =$$

$$C^{(1)}_{m,\widetilde{m};\,\vec{\mu}}\, Q\left(\lambda;\, m,\, \vec{\mu}\right)\, Q\left(\lambda^*;\, \widetilde{m},\, \vec{\mu^s}\right) + C^{(2)}_{m,\widetilde{m};\,\vec{\mu}}\, Q\left(-\lambda;\, m,\, \vec{\mu^s}\right)\, Q\left(-\lambda^*;\, \widetilde{m},\, \vec{\mu}\right)\,, \tag{94}$$

where we took into account, that $q^*_k = -i\mu_k$ for odd k and $q^*_k = i\mu_k$ for even k.

The condition of the cancellation of the most singular poles $1/\sigma^{n-1}$ of $Q_{m,\,\widetilde{m},\,\vec{\mu}}\left(\overrightarrow{\lambda}\right)$ at $\sigma = 0$ gives

$$\frac{C^{(1)}_{m,\widetilde{m};\,\vec{\mu}}}{C^{(2)}_{m,\widetilde{m};\,\vec{\mu}}} = -\frac{\alpha^{(n-2)}_{m,\,\vec{\mu}^s}}{\alpha^{(n-2)}_{\widetilde{m},\,\vec{\mu}^s}}. \tag{95}$$

The other poles are canceled providing that

$$S^{(n-2)}_{r;m,\vec{\mu}}(\lambda) = \alpha^{(n-2)}_{m,\vec{\mu}}\, P^{(n-2)}_{r;m,\vec{\mu}^s}(\lambda)\,. \tag{96}$$

It is enough to impose this condition for $r = 1$. Therefore for the corresponding discreet values of the parameters μ_r $(r = 3, 4, ...)$ one can construct the normalized Baxter function in the $\overrightarrow{\lambda}$-space.

We show below that the energy of the composite reggeon states can be calculated in terms of the behavior of the Baxter function near the singularity $\lambda = i$ [see eq.(105)]. Therefore, the energy spectrum of the composite states is obtained from the normalizability condition of $Q_{m,\,\widetilde{m},\,\vec{\mu}}\left(\overrightarrow{\lambda}\right)$

7 The Hamiltonian in the BS representation

The high energy asymptotics of the scattering amplitude corresponding to the contribution related to the t-channel exchange of the composite state of n reggeized gluons in the multi-color QCD has the form

$$A(s,t) \sim i^{n-1}\, s\, s^{\Delta}\,,\ \ \Delta = -\frac{g^2}{8\pi^2} N_c\, E\,,$$

where E is the ground state energy for the Schrödinger equation

$$E\,\Psi_{m,\widetilde{m}}(\overrightarrow{\rho_1},\,\overrightarrow{\rho_2},\,\ldots,\overrightarrow{\rho_n};\overrightarrow{\rho_0}) = H\,\Psi_{m,\widetilde{m}}(\overrightarrow{\rho_1},\,\overrightarrow{\rho_2},\,\ldots,\overrightarrow{\rho_n};\overrightarrow{\rho_0})\,.$$

The Reggeon Hamiltonian is

$$H = \frac{1}{2}\sum_{k=1}^{n} H_{k,k+1}\,.$$

Here $1/2$ is the ration of the color factors for the adjoint and singlet representations of the color group and the pair BFKL Hamiltonian is given by

$$\begin{aligned} H_{1,2} &= \ln|p_1|^2 + \ln|p_2|^2 + \\ &+ \frac{p_1 p_2^*}{|p_1|^2\,|p_2|^2}\,\ln|\rho_{12}|^2\,p_1^* p_2 + \frac{p_2 p_1^*}{|p_1|^2\,|p_2|^2}\,\ln|\rho_{12}|^2\,p_2^* p_1 - 4\,\psi(1)\,. \end{aligned} \tag{97}$$

It enjoys the property of holomorphic separability

$$H_{1,2} = h_{12} + h_{12}^*\,,$$

where

$$h_{12} = \ln(p_1 p_2) + \frac{1}{p_1}\,(\ln\rho_{12})\,p_1 + \frac{1}{p_2}\,(\ln\rho_{12})\,p_2 - 2\,\psi(1)\,.$$

We now perform the unitary transformation of the hamiltonian to the BS representation, where P and the roots $\widehat{\lambda}_1,\,\widehat{\lambda}_2,\ldots,\widehat{\lambda}_{n-1}$ of the equation $B(u)=0$ are diagonal operators. In this new representation both the integrals of motion and the hamiltonian should have simple separability properties.

Let us start with the case of the Pomeron, where

$$H = H_{1,2}\,.$$

We now calculate the action of the hermitially conjugated Hamiltonian on the eigenfunctions of the operator $B(u)$ given by eq.(21) (with $\lambda = -\lambda_1 = \sigma + iN/2$,

$$\begin{aligned} H_{1,2}^{+}\left(\frac{p}{1-p}\right)^{-i\lambda^*}\left(\frac{p^*}{1-p^*}\right)^{-i\lambda} &= \ln\left[\frac{|p|^2\,|1-p|^2}{m^4}\right]\left(\frac{p}{1-p}\right)^{-i\lambda^*}\left(\frac{p^*}{1-p^*}\right)^{-i\lambda} - \\ -\frac{1}{\pi}\int d^2k\,\frac{[k\,p^*(1-k^*)(1-p) + k^*\,p\,(1-k)(1-p^*)]}{\left(|k-p|^2+m^2\right)\,|k|^2\,|1-k|^2}&\left(\frac{k}{1-k}\right)^{-i\lambda^*}\left(\frac{k^*}{1-k^*}\right)^{-i\lambda}\,, \end{aligned}$$

where $P=1$ and $m\to 0$ is an infrared regulator, corresponding to the vector boson mass rescaled by P (cf. [1]).

Using the anti-Wick rotation of momenta $k_2 = -ik_0$ and $p_2 = -ip_0$ as in eq.(31) after some transformations we can write the result in holomorphically separable form

$$H_{1,2}^{+}\left(\frac{p}{1-p}\right)^{-i\lambda^*}\left(\frac{p^*}{1-p^*}\right)^{-i\lambda}$$

$$= \left(\frac{p^*}{1-p^*}\right)^{-i\lambda} h_{12} \left(\frac{p}{1-p}\right)^{-i\lambda^*} + \left(\frac{p}{1-p}\right)^{-i\lambda^*} h_{12}^* \left(\frac{p^*}{1-p^*}\right)^{-i\lambda} ,$$

where

$$h_{12} \left(\frac{p}{1-p}\right)^{-i\lambda^*}$$

$$= \left[\ln \frac{p(1-p)}{\varepsilon^2} + \pi\, i\, \coth(\pi\lambda^*)\right] \left(\frac{p}{1-p}\right)^{-i\lambda^*} - \int_{p+\varepsilon}^{1} dk\, \frac{(p+k-2\,k\,p)\; k^{-1-i\lambda^*}}{(k-p)\,(1-k)^{1-i\lambda^*}}$$

and

$$h_{12}^* \left(\frac{p^*}{1-p^*}\right)^{-i\lambda}$$

$$= \left[\ln \frac{p^*(1-p^*)}{\varepsilon^2} + \pi\, i\, \coth(\pi\lambda)\right] \left(\frac{p^*}{1-p^*}\right)^{-i\lambda} - \int_{p^*+\varepsilon}^{1} d\,k^*\, \frac{(p^*+k^*-2\,k^*\,p^*)\; (k^*)^{-1-i\lambda}}{(k^*-p^*)\,(1-k^*)^{1-i\lambda^*}} .$$

Here $\varepsilon \to 0$ is an intermediate infrared cut-off. The hamiltonians h_{12} and h_{12}^* have branch point singularities at $p = 0,\, 1,\, \infty$ and $p^* = 0,\, 1,\, \infty$, respectively, but the total hamiltonian $H_{1,2}$ is single-valued.

We obtain for $|p| \to 0$:

$$\lim_{|p|\to 0} H_{1,2}\; (p)^{-i\lambda^*}\, (p^*)^{-i\lambda}$$

$$= \left[-\ln|p|^2 + \psi(1+i\lambda^*) + \psi(1-i\lambda^*) + \psi(1+i\lambda) + \psi(1-i\lambda) - 4\psi(1)\right] (p)^{-i\lambda^*} (p^*)^{-i\lambda} .$$

Taking into account that in the integral

$$\Psi_{m,\widetilde{m}}(\overrightarrow{p},\, \overrightarrow{1} - \overrightarrow{p}) = \frac{1}{2\pi^2} \int_{-\infty}^{+\infty} d\,\sigma \sum_{N=-\infty}^{+\infty} \left(\frac{p}{1-p}\right)^{-i\lambda^*} \left(\frac{p^*}{1-p^*}\right)^{-i\lambda} \Phi_{m,\widetilde{m}}(\overrightarrow{1},\, \overrightarrow{\lambda}) ,\quad \lambda = \sigma + i\frac{N}{2}$$

for $|p| \to 0$ the leading asymptotics corresponds to $N = 0$, and shifting the integration contour in σ in the upper half-plane up to the first singularity of $\Psi_{m,\widetilde{m}}(\overrightarrow{1},\, \overrightarrow{\lambda})$, corresponding to the poles at $\lambda, \lambda^* = i$, we obtain for the hamiltonian near these singularities

$$\lim_{\lambda,\lambda^*\to i} H_{1,2}\, \Phi_{m,\widetilde{m}}(\overrightarrow{1},\, \overrightarrow{\lambda}) = \lim_{\lambda,\lambda^*\to i} \left[i\left(\frac{\partial}{\partial\lambda} + \frac{\partial}{\partial\lambda^*} + \frac{1}{\lambda - i} + \frac{1}{\lambda^* - i}\right) + 2\right] \Phi_{m,\widetilde{m}}(\overrightarrow{1},\, \overrightarrow{\lambda}) .$$

Using eq.(35) for the Pomeron wave function in this limit,

$$\Phi_{m,\widetilde{m}}(\overrightarrow{1},\, \overrightarrow{\lambda}) \sim Q(\lambda, m)\, Q(\lambda^*, \widetilde{m})\; |\lambda|^2 ,$$

the Pomeron energy is given as follows

$$E_{12} = i \lim_{\lambda,\lambda^*\to i} \left\{ \frac{\partial}{\partial\lambda} \ln\left[(\lambda - i)\lambda^2\, Q(\lambda, m)\right] + \frac{\partial}{\partial\lambda^*} \ln\left[(\lambda^* - i)(\lambda^*)^2\, Q(\lambda^*, \widetilde{m})\right] \right\} .$$

We obtain for the Pomeron energy using the behavior for $Q(\lambda, m)$ near $\lambda = i$ [eq.(53)]

$$E_{12} = \psi(m) + \psi(1-m) + \psi(\widetilde{m}) + \psi(1-\widetilde{m}) - 4\psi(1) \tag{98}$$

in agreement with the known result [1].

8 Energy for multi-reggeon composite states

Let us investigate the behavior of the wave functions for the composite states in the region where the values of gluon momenta are strictly ordered:

$$|p_1| << |p_2| << \ldots << |p_n|\,.$$

To begin with, we consider the Odderon case, where the wave function is given by

$$\Psi_{m,\widetilde{m}}(\overrightarrow{p_1},\,\overrightarrow{p_2},\,\overrightarrow{p_3}) =$$

$$\prod_{r=1}^{3}(\overrightarrow{p_r})^2\int\prod_{k=1}^{3}\left[\frac{d^2\rho_k}{2\pi}\,\exp(i\overrightarrow{p_k}\cdot\overrightarrow{\rho_{k0}})\right]\left(\frac{\rho_{23}}{\rho_{20}\rho_{30}}\right)^m\left(\frac{\rho^*_{23}}{\rho^*_{20}\rho^*_{30}}\right)^{\widetilde{m}}\phi_{m,\widetilde{m}}(x\,,x^*),$$

where

$$x=\frac{\rho_{12}\rho_{30}}{\rho_{10}\rho_{32}}$$

and the function $\phi_{m,\widetilde{m}}(x\,,x^*)$ has the property of the holomorphic factorization

$$\phi_{m,\widetilde{m}}(x\,,x^*)=\sum_{i,k}c_{ik}\,\phi^i_m(x\,)\,\phi^k_{\widetilde{m}}(x^*)\,+(q_3\leftrightarrow -q_3).$$

The functions $\phi^i_m(x)$, $\phi^k_{\widetilde{m}}(x^*)$ are independent eigenfunctions of the integral of motion [4]

$$A_m\phi^i_m(x\,)=a_{1-m}\,a_m\,\phi^i_m(x\,)\;=q_3\phi^i_m(x\,)\,,$$

where A_m is given by

$$A_m=i^3\,x(1-x)\,\Big[x(1-x)\,\partial^2+(2-m)\,(1-2x)\,\partial-(2-m)\,(1-m)\Big]\,\partial\,.$$

The operators

$$a_m=x\,(1-x)\,(i\partial)^{m+1}\,,\;\;a_{1-m}=x\,(1-x)\,(i\partial)^{2-m}$$

perform the duality transformation [11].

The three independent eigenfunctions $\phi^i_m(x)$ have the following small-x asymptotics [16]:

$$\phi^1_m(x\,)\simeq x+O(x^2)\quad,\quad\phi^2_m(x\,)\simeq 1+O(x\ln x)\quad,\quad\phi^3_m(x\,)\simeq x^m\,[1+O(x)]\;,$$

which correspond to the following asymptotics of $\phi_{m,\widetilde{m}}(x\,,x^*)$ enjoying single-valuedness at the singular points,

$$\lim_{x\to 0}\phi_{m,\widetilde{m}}(x\,,x^*)\simeq x^m\;x^{*\widehat{m}}\;+\;c\;|x|^2\;\ln|x|^2\quad,\quad\lim_{x\to\infty}\phi_{m,\widetilde{m}}(x\,,x^*)\simeq 1+\;c\,x^{m-1}\;x^{*\widehat{m}-1}\;\ln|x|^{-2}$$

and

$$\lim_{x\to 1}\phi_{m,\widetilde{m}}(x\,,x^*)\simeq(1-x)^m\;(1-x^*)^{\widehat{m}}+c\,|1-x|^2\;\ln|1-x|^2\;.$$

For the function $\Psi_{m,\widetilde{m}}(\overrightarrow{p_1},\, \overrightarrow{p_2},\, \overrightarrow{p_3})$ in the limit $|p_1| \to 0$ the region of large $|\rho_{10}|$ is essential. Taking into account only the singular terms in this limit, we obtain

$$\Psi_{m,\widetilde{m}}(\overrightarrow{p_1},\, \overrightarrow{p_2},\, \overrightarrow{p_3}) \simeq \Psi_{m,\widetilde{m}}(\overrightarrow{p_2},\, \overrightarrow{p_3})\, \frac{1}{2}\, |p_1|^2 \ln \frac{|P|^2}{|p_1|^2}$$

where

$$\Psi_{m,\widetilde{m}}(\overrightarrow{p_2},\, \overrightarrow{p_3}) =$$

$$\prod_{r=2}^{3} |p_r|^2 \int \prod_{k=2}^{3} \left[\frac{d^2\rho_k}{2\pi}\, \exp(i\overrightarrow{p_k} \cdot \overrightarrow{\rho_{k0}}) \right] \left(\frac{\rho_{23}}{\rho_{20}\rho_{30}} \right)^{m-1} \left(\frac{\rho_{23}^*}{\rho_{20}^*\rho_{30}^*} \right)^{\widetilde{m}-1} \partial\partial^* \phi_{m,\widetilde{m}}(x\,, x^*)$$

and

$$x = \frac{\rho_{30}}{\rho_{32}}\,.$$

The last function can be simplified in the limit $p_2 \to 0$, corresponding to $\rho_{20} \to \infty$ and $x \to 0$. Indeed, we can use the expansion

$$\partial\partial^* \phi_{m,\widetilde{m}}(x\,, x^*) \simeq m\widetilde{m}\; x^{m-1} (x^*)^{\widehat{m}-1} + c\, \ln |x|^2 + \dots$$

and verify, that the dependence from ρ_{30} is canceled in the contribution to the integrand from the first term in the right hand side, leading to a vanishing result after integration. Therefore taking into account only the second term, we obtain,

$$\Psi_{m,\widetilde{m}}(\overrightarrow{p_2},\, \overrightarrow{p_3}) \sim c_3\,.$$

Hence, the resulting behavior for the Odderon wave function at $|p_1| << |p_2| << |p_3|$ is

$$\Psi_{m,\widetilde{m}}(\overrightarrow{p_1},\, \overrightarrow{p_2},\, \overrightarrow{p_3}) \sim c_3\; |p_1|^2\; \ln \frac{|P|^2}{|p_1|^2}\,. \tag{99}$$

This is in agreement with the fact that it is an eigenfunction of the integrals of motion q_2 and q_3 provided that we take into account in $\Psi_{m,\widetilde{m}}$ also the regular terms proportional to p_1 and p_1^*.

It is natural to expect a similar behavior

$$\Psi_{m,\widetilde{m}}(\overrightarrow{p_1}, \overrightarrow{p_2}, \dots, \overrightarrow{p_n}) \sim c_n\; |p_1|^2 \ln \frac{|P|^2}{|p_1|^2} \tag{100}$$

for the case of n reggeized gluons in the limit $|p_1| << |p_2| << \dots << |p_n|$. Indeed, there are two independent solutions for the eigenfunctions of the integrals of motion Q_j (7) which behave at small p_1 correspondingly as $f + g\, p_1 \ln p_1$ and g, where f and g are some functions of p_k analytic near $p_1 = 0$. For the single-valuednes property we should multiply two such functions depending on the holomorphic and anti-holomorphic variables. Further, because the operators Q_j have more derivatives over p_k $(k = 2, 3, ..., n)$ than the momenta compensating them, $|g|^2$ should be a constant for small values of these momenta.

In the opposite limit

$$|p_n| << |p_{n-1}| << ... << |p_1|$$

we obtain correspondingly

$$\Psi_{m,\widetilde{m}}(\overrightarrow{p_1}, \overrightarrow{p_2}, \ldots, \overrightarrow{p_n}) \sim c_n \; |p_n|^2 \ln \frac{|P|^2}{|p_n|^2} \,. \tag{101}$$

The fact that the behavior of $\Psi_{m,\widetilde{m}}$ at $|p_n| \to 0$ for the composite state of n reggeized gluons is the same as in the Pomeron case implies the existence of a pole in

$$\Psi_{m,\widetilde{m}}(\overrightarrow{\lambda_1}, \overrightarrow{\lambda_2}, \ldots, \overrightarrow{\lambda}_{n-1})$$

at $\lambda_{n-1} = i$ and at $\lambda^*_{n-1} = i$.

Indeed, for $1 = |p_1| >> |p_2| >> \ldots >> |p_n|$ we have

$$\Psi_{m,\widetilde{m}}(\overrightarrow{p_1}, \overrightarrow{p_2}, \ldots, \overrightarrow{p_n}) = \tag{102}$$

$$2^{\frac{n-1}{2}} \prod_{k=1}^{n-1} \left(\int_{-\infty}^{+\infty} d\sigma_k \sum_{N_k=-\infty}^{+\infty} \exp[i(t_k \lambda^*_k + t^*_k \lambda_k)] \right) \Psi_{m,\widetilde{m}}(\overrightarrow{\lambda_1}, \overrightarrow{\lambda_2}, \ldots, \overrightarrow{\lambda}_{n-1})$$

and therefore the contour of the integration in σ_{n-1} for $N_{n-1} = 0$ should be shifted in the complex plane up to the pole $(\sigma_{n-1} - i)^{-2}$.

We can find the singular part of the Hamiltonian in the BS representation for small $|p_n|$ similarly to the Pomeron case:

$$\frac{1}{2} \left(H_{n,n-1} + H_{1n}\right) p_n^{-i\lambda^*_{n-1}} p_n^{*-i\lambda_{n-1}} = \frac{1}{2} \ln |p_1 p_{n-1}|^2 \, p_n^{-i\lambda^*_{n-1}} p_n^{*-i\lambda_{n-1}} + \tag{103}$$

$$\Big[-\ln |p_n|^2 + \psi(1 + i\lambda^*_{n-1}) + \psi(1 - i\lambda^*_{n-1}) + \psi(1 + i\lambda_{n-1}) + \psi(1 - i\lambda_{n-1}) - 4\psi(1) \Big] \; p_n^{i\lambda^*_{n-1}} p_n^{*-i\lambda_{n-1}} \,.$$

The first term in the right hand side can be combined with the other pair hamiltonians. After that, we obtain for $1 = |p_1| >> |p_2| >> \ldots >> |p_n|$:

$$\left(\frac{1}{2} \ln |p_1 p_{n-1}|^2 + \frac{1}{2} \sum_{r=1}^{n-2} H_{r,r+1} \right) \prod_{k=2}^{n-1} p_k^{-i\lambda^*_k} p_k^{*-i\lambda_k} =$$

$$\frac{1}{2} \sum_{r=2}^{n-1} \left[\psi(1 + i\lambda^*_r) + \psi(1 - i\lambda^*_r) + \psi(1 + i\lambda_r) + \psi(1 - i\lambda_r) - 4\psi(1) \right] \prod_{k=2}^{n-1} p_k^{-i\lambda^*_k} p_k^{*-i\lambda_k} \,.$$

Thus, for the constant behavior $\Psi_{m,\widetilde{m}}(\overrightarrow{p_1}, \ldots, \overrightarrow{p_{n-1}}) \sim c$ at $|p_2| >> \ldots >> |p_n|$ (corresponding to $\lambda_1 = \lambda^*_1 = \ldots = \lambda_{n-2} = \lambda^*_{n-2} = 0$) the last contribution vanishes and therefore we obtain for the composite state energy the result similar to the pomeron case

$$E = i \lim_{\lambda, \lambda^* \to i} \left\{ \frac{\partial}{\partial \lambda} \ln \left[(\lambda - i)\, \lambda \, \Psi_m(\lambda) \right] + \frac{\partial}{\partial \lambda^*} \ln \left[(\lambda^* - i)\, \lambda^* \; \Psi_{\widetilde{m}}(\lambda^*) \right] \right\} \,, \tag{104}$$

where $\Psi_m(\lambda_{n-1})$ and $\Psi_{\widetilde{m}}(\lambda^*_{n-1})$ are correspondingly holomorphic and anti-holomorphic factors of the wave function at $\lambda_k = \lambda^*_k = 0,\ 1 \leq k \leq n-2$,

$$\Psi_{m,\widetilde{m}}(0,0,...,\overrightarrow{\lambda_{n-1}}) \Longrightarrow \Psi_m(\lambda_{n-1})\,\Psi_{\widetilde{m}}(\lambda^*_{n-1})\,.$$

This quantity can be related with the Baxter function and the normalization factor for the pseudovacuum state [see eq. (25)]:

$$\Psi_{m,\widetilde{m}}(0,0,...,\overrightarrow{\lambda_{n-1}}) \Longrightarrow c^{ps}_{0,0,...,\overrightarrow{\lambda_{n-1}}}\ |\lambda_{n-1}|^{2(n-1)}\ Q(\lambda_{n-1},m)\ Q(\lambda^*_{n-1},\widetilde{m})\,.$$

As it was argued above, for the pseudovacuum state it looks plausible, that the correct normalization of the kernel for the transition between momentum and BS representations corresponds to $c^{ps}_{0,0,...,\overrightarrow{\lambda_{n-1}}} = \sinh^{n-2} 2\pi\lambda$ [see eq.(27)]. We obtain in this case for the energy,

$$E = i \lim_{\lambda,\lambda^*\to i} \left\{ \frac{\partial}{\partial\lambda} \ln\left[\sinh^{n-1}(2\pi\,\lambda)\,\lambda^n\,Q(\lambda,m)\right] + \frac{\partial}{\partial\lambda^*} \ln\left[\sinh^{n-1}(2\pi\,\lambda^*)\,\lambda^{*n}\,Q(\lambda^*,\widetilde{m})\right] \right\}\,, \tag{105}$$

Thus, the energy is expressed in terms of the behavior of the Baxter function $Q(\lambda,m)$ near $\lambda = +i$.

In the customary case of spin chains the Baxter function is a polynomial of degree L,

$$Q_{XXX}(\lambda) = \prod_{k=1}^{L}(\lambda - \lambda_k)$$

where the λ_k are solutions of the Bethe Ansatz equations[6, 7]. The energy of the XXX chain is given by,

$$E_{XXX} = -2\sum_{k=1}^{L} \frac{1}{\lambda_k^2+1} = i\frac{d}{d\lambda}\log \left.\frac{Q_{XXX}(\lambda+i)}{Q_{XXX}(\lambda-i)}\right|_{\lambda=0} \tag{106}$$

In the present case the Baxter function is a meromorphic function with an infinite number of poles and zeroes as discussed in sec. 5.2. It can be expressed as an infinite product of the Bethe Ansatz solutions [see eq.(59)]. The eigenvalue expression here is not given by eq.(106) but by eq.(98)

$$E_{12} = 1 + \gamma + \psi(2-m) - \sum_{k=1}^{\infty} \frac{1}{i\,\lambda_k\,(1+i\,\lambda_k)} + [m \to \widetilde{m}]\,,$$

in the pomeron case.

9 BS representation for the Odderon wave function

Here we consider the λ-representation for the Odderon, where the operator

$$B^{(3)}(u) = -P\left[u^2 + iu\left(\frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2}\right) - \left(1 - e^{t_1-t_2}\right)\frac{\partial}{\partial t_1}\frac{\partial}{\partial t_2}\right]$$

is diagonal. Introducing the new variables,

$$t = t_1 + t_2 = \ln \left[\frac{p_1\,(p_1 + p_2)}{(p_2 + p_3)\,p_3} \right] \quad , \quad z = e^y = e^{t_1 - t_2} = \frac{p_1\,p_3}{(p_2 + p_3)(p_1 + p_2)} \; ,$$

we obtain,

$$B^{(3)}(u) = -P \left[u^2 + 2iu\frac{\partial}{\partial t} + (1-z)\; z\frac{\partial}{\partial z}\, z\frac{\partial}{\partial z} - (1-z) \left(\frac{\partial}{\partial t} \right)^2 \right] .$$

To diagonalize $B^{(3)}$ one should find the eigenvalues and eigenfunctions of the differential operators

$$i\frac{\partial}{\partial t}\varphi = -\frac{\lambda_1^* + \lambda_2^*}{2}\,\varphi\,, \quad \left[(1-z)\; z\frac{\partial}{\partial z}\, z\frac{\partial}{\partial z} - z\, \left(\frac{\lambda_1^* + \lambda_2^*}{2} \right)^2 \right] \varphi = - \left(\frac{\lambda_1^* - \lambda_2^*}{2} \right)^2 \varphi\,,$$

where λ_k are the eigenvalues of the zeroes of $B^{(3)}(u)$:

$$B^{(3)}(u) = -P \prod_{k=1}^{2} \left(u - \widehat{\lambda}_k \right) .$$

The solution of the above equations can be written in terms of hypergeometric functions

$$\varphi_{\lambda_1^*\lambda_2^*}(t\,,\, z) = e^{i\,\frac{\lambda_1^*+\lambda_2^*}{2}\,t}\, z^{i\,\frac{\lambda_1^*-\lambda_2^*}{2}}\, F(-i\lambda_2^*\,,\, i\lambda_1^*\,;\, 1 + i(\lambda_1^* - \lambda_2^*)\,;\, z) =$$

$$e^{i\,\frac{\lambda_1^*+\lambda_2^*}{2}\,t}\, \frac{\Gamma(1 + i(\lambda_1^* - \lambda_2^*))\; z^{i\frac{\lambda_1^*-\lambda_2^*}{2}}}{\Gamma(-i\lambda_2^* + 1)\,\Gamma(i\lambda_1^*)} \int_1^\infty \left(\frac{x-1}{x-z} \right)^{-i\lambda_2^*} x^{-i\lambda_1^*-1}\, d\,x\,.$$

An independent solution follows by interchanging here λ_1^* and λ_2^*.

Therefore, we can write the following relation between the wave functions in momentum and BS representations,

$$\Psi_{m,\widetilde{m}}(\overrightarrow{p_1},\, \overrightarrow{p_2},\, \overrightarrow{p_3}) =$$

$$P^{\widetilde{m}}(P^*)^m \prod_{k=1}^{2} \left(\sum_{N_k=-\infty}^{+\infty} \int_{-\infty}^{+\infty} d\sigma_k \right) U_{\overrightarrow{\lambda_1},\overrightarrow{\lambda_2}}(\overrightarrow{t}\,,\, \overrightarrow{z})\; \Psi_{m,\widetilde{m}}(\overrightarrow{\lambda_1},\, \overrightarrow{\lambda_2})\,,$$

where

$$\lambda_k = \sigma_k + i\,\frac{N_k}{2}\;,\;\; \lambda_k^* = \sigma_k - i\,\frac{N_k}{2}\,. \tag{107}$$

and

$$U_{\overrightarrow{\lambda_1},\overrightarrow{\lambda_2}}(\overrightarrow{t}\,,\, \overrightarrow{z}) = C_{\overrightarrow{\lambda_1},\overrightarrow{\lambda_2}}\;\; e^{it\frac{\lambda_1^*+\lambda_2^*}{2}}\;\, e^{it^*\frac{\lambda_1+\lambda_2}{2}}\;\; U_{\overrightarrow{\lambda_1},\overrightarrow{\lambda_2}}(\overrightarrow{z})\,.$$

Here the function $U_{\overrightarrow{\lambda_1},\overrightarrow{\lambda_2}}(\overrightarrow{z})$ is defined as follows

$$U_{\overrightarrow{\lambda_1},\overrightarrow{\lambda_2}}(\overrightarrow{z}) \;= z^{i\frac{\lambda_1^*-\lambda_2^*}{2}}\, (z^*)^{i\frac{\lambda_1-\lambda_2}{2}} \int \frac{d^2x}{|x|^2}\; x^{-i\lambda_1^*}\; (x^*)^{-i\lambda_1}\; \left(\frac{x-1}{x-z} \right)^{-i\lambda_2^*} \left(\frac{x^*-1}{x^*-z^*} \right)^{-i\lambda_2} \tag{108}$$

and the normalization constant $C_{\overrightarrow{\lambda_1},\overrightarrow{\lambda_2}}$ can be found from the orthogonality condition

$$\int \frac{d^2t\, d^2z}{|z(1-z)|^2}\, U_{\overrightarrow{\lambda_1},\overrightarrow{\lambda_2}}(\overrightarrow{t},\overrightarrow{z})\, U^*_{\overrightarrow{\lambda_1'},\overrightarrow{\lambda_2'}}(\overrightarrow{t},\overrightarrow{z}) = \sum_P \prod_{k=1}^{2} \delta(\sigma_k - \sigma'_{i_k})\, \delta_{N_k, N'_{i_k}}. \tag{109}$$

It should be taken into account, that due to the symmetry properties of $U_{\overrightarrow{\lambda_1},\overrightarrow{\lambda_2}}(\overrightarrow{z})$ under $\lambda_1 \leftrightarrow \lambda_2$ two terms appear in the right hand side of the orthogonality equation.

Let us show that the kernel of the unitary transformation $U_{\overrightarrow{\lambda_1},\overrightarrow{\lambda_2}}(\overrightarrow{t},\overrightarrow{z})$ has an interpretation in terms of the Feynman diagram as it was in the case of the Pomeron wave function (30). After changing the integration variable x into k as follows,

$$x = \frac{p_1}{1-p_1}\,\frac{k}{1-k}$$

eq.(108) takes the form

$$\frac{U_{\overrightarrow{\lambda_1},\overrightarrow{\lambda_2}}(\overrightarrow{t},\overrightarrow{z})}{C_{\overrightarrow{\lambda_1},\overrightarrow{\lambda_2}}} = \left(\frac{p_1}{p_3}\right)^{i\lambda_2^*} \left(\frac{p_1^*}{p_3^*}\right)^{i\lambda_2} \int \frac{d^2k}{|k(1-k)|^2}$$
$$\left(\frac{k}{1-k}\right)^{-i\lambda_1^*} \left(\frac{k^*}{1-k^*}\right)^{-i\lambda_1} \left(\frac{k+p_1-1}{k-p_3}\right)^{-i\lambda_2^*} \left(\frac{k^*+p_1^*-1}{k^*-p_3^*}\right)^{-i\lambda_2}. \tag{110}$$

The wave function in the λ-representation has the form

$$\Psi_{m,\widetilde{m}}(\overrightarrow{\lambda_1},\overrightarrow{\lambda_2}) =$$

$$\int \frac{d^2p_1 d^2p_3}{|p_1|^2|1-p_1-p_3|^2|p_3|^2}\, U^*_{\overrightarrow{\lambda_1},\overrightarrow{\lambda_2}}(\overrightarrow{t},\overrightarrow{z})\; \Psi_{m,\widetilde{m}}(\overrightarrow{p_1},\overrightarrow{1}-\overrightarrow{p_1}-\overrightarrow{p_3},\overrightarrow{p_3})$$

The associated Feynman diagram is depicted in fig. 1.

10 Properties of the Odderon wave function

The wave function $U_{\overrightarrow{\lambda_1},\overrightarrow{\lambda_2}}(\overrightarrow{z})$ is a solution of the differential equations of hypergeometric type both as a function of z and z^*,

$$\left[(1-z)\, z\frac{d}{dz}\, z\frac{d}{dz} - z\left(\frac{\lambda_1^*+\lambda_2^*}{2}\right)^2 + \left(\frac{\lambda_1^*-\lambda_2^*}{2}\right)^2\right] U = 0\,,$$
$$\left[(1-z^*)\, z^*\frac{d}{dz^*}\, z^*\frac{d}{dz^*} - z^*\left(\frac{\lambda_1+\lambda_2}{2}\right)^2 + \left(\frac{\lambda_1-\lambda_2}{2}\right)^2\right] U = 0\,,$$

Therefore, $U_{\overrightarrow{\lambda_1},\overrightarrow{\lambda_2}}(\overrightarrow{z})$ is a bilinear combination of independent solutions being functions of z and z^*. In addition, $\Phi_{\overrightarrow{\alpha},\overrightarrow{\beta}}(\overrightarrow{z})$ should be a single valued function of $\overrightarrow{z}$ near the singularities

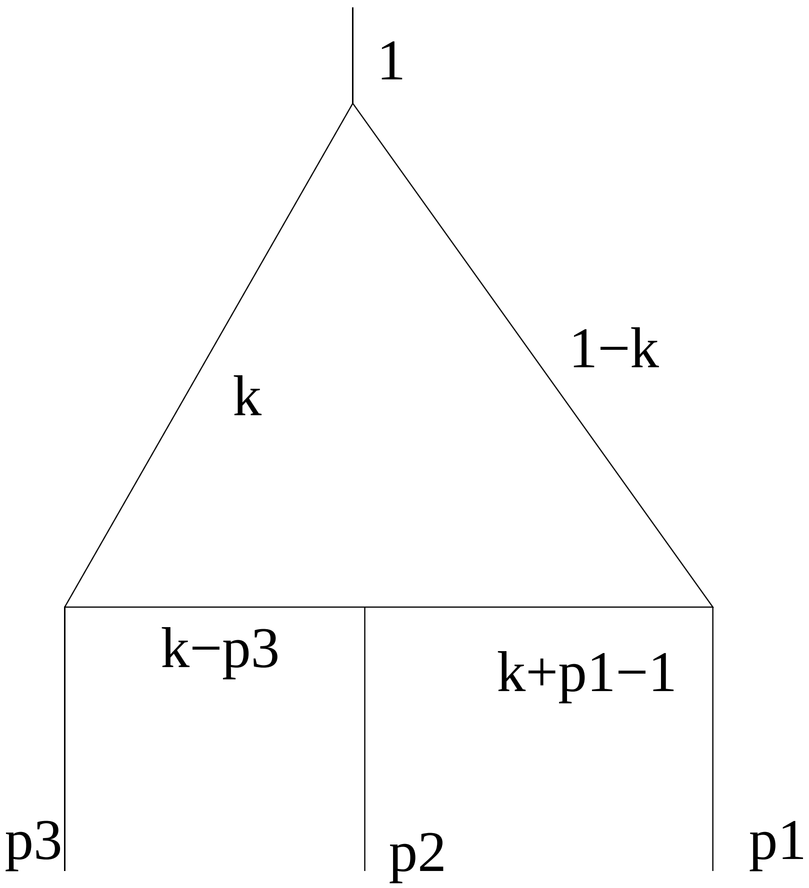


Figure 1: Odderon Feynman diagram

$|z| = 0\ , |z-1| = 0\ , |z| = \infty$. The effective way to satisfy these requirements [23] is to use the monodromy properties [20] of the two independent solutions:

$$z^{i\frac{\lambda_1-\lambda_2}{2}} F(-i\lambda_2\, , \, i\lambda_1\, ; \, 1+i(\lambda_1-\lambda_2)\, ; \, z)\ , \quad z^{-i\frac{\lambda_1-\lambda_2}{2}} F(-i\lambda_1\, , \, i\lambda_2\, ; \, 1-i(\lambda_1-\lambda_2)\, ; \, z)$$

and analogous expressions in z^*. Thus, we obtain,

$$U_{\overrightarrow{\lambda_1},\overrightarrow{\lambda_2}}(\overrightarrow{z}) = K_{\overrightarrow{\lambda_1},\overrightarrow{\lambda_2}} \left[\chi_{\lambda_1\lambda_2}(z^*)\chi_{\lambda_1^*\lambda_2^*}(z) - \chi_{\lambda_2\lambda_1}(z^*)\chi_{\lambda_2^*\lambda_1^*}(z)\right]\ , \tag{111}$$

where

$$\begin{aligned} \chi_{\lambda_1\lambda_2}(z^*) &\equiv a_{\lambda_1\lambda_2}\ (z^*)^{i\frac{\lambda_1-\lambda_2}{2}} F(-i\lambda_2\, , \, i\lambda_1\, ; \, 1+i(\lambda_1-\lambda_2)\, ; \, z^*) \quad , \\ a_{\lambda_1\lambda_2} &\equiv \frac{\Gamma(i\lambda_1)\Gamma(-i\lambda_2)}{\Gamma(1+i(\lambda_1-\lambda_2))}\ , \end{aligned} \tag{112}$$

and an analogous expression for $\chi_{\lambda_1^*,\lambda_2^*}(z)$. This result can be verified by the direct calculation of the integral (108). In such way we obtain for the constant $K_{\overrightarrow{\lambda_1},\overrightarrow{\lambda_2}}$ in eq.(111):

$$K_{\overrightarrow{\lambda_1},\overrightarrow{\lambda_2}} = i\,|\lambda_2|^2\ \frac{\sinh(\pi\lambda_2)\ \sinh(\pi\lambda_1)}{\sinh(\pi(\lambda_1-\lambda_2))}\,,$$

where we used [see eqs.(107)]

$$\frac{\sinh(\pi\lambda_2)\ \sinh(\pi\lambda_1)}{\sinh(\pi(\lambda_1-\lambda_2))} = \frac{\sinh(\pi\lambda_2^*)\ \sinh(\pi\lambda_1^*)}{\sinh(\pi(\lambda_1^*-\lambda_2^*))}.$$

In summary, collecting all factors we find for the matrix elements of the unitary transformation relating momentum and BS representations,

$$\frac{U_{\overrightarrow{\lambda_1},\overrightarrow{\lambda_2}}(\overrightarrow{t},\overrightarrow{z})}{C_{\overrightarrow{\lambda_1},\overrightarrow{\lambda_2}}\ K_{\overrightarrow{\lambda_1},\overrightarrow{\lambda_2}}} =$$

$$e^{i(t\frac{\lambda_1^*+\lambda_2^*}{2}+t^*\frac{\lambda_1+\lambda_2}{2})}\ \left[\chi_{\lambda_1\lambda_2}(z^*)\chi_{\lambda_1^*\lambda_2^*}(z) - \chi_{\lambda_2\lambda_1}(z^*)\chi_{\lambda_2^*\lambda_1^*}(z)\right]\ , \tag{113}$$

where $\chi_{\lambda_1\lambda_2}(z^*)$ and $\chi_{\lambda_1^*\lambda_2^*}(z)$ are given by eq.(112).

We find for $z \to 0$ the asymptotic behavior,

$$\frac{U_{\overrightarrow{\lambda_1},\overrightarrow{\lambda_2}}(\overrightarrow{t},\overrightarrow{z})}{C_{\overrightarrow{\lambda_1},\overrightarrow{\lambda_2}}\ K_{\overrightarrow{\lambda_1},\overrightarrow{\lambda_2}}} =$$

$$e^{i(t\frac{\lambda_1^*+\lambda_2^*}{2}+t^*\frac{\lambda_1+\lambda_2}{2})}\ \left[a_{\lambda_1\lambda_2}a_{\lambda_1^*\lambda_2^*}\, z^{*i\frac{\lambda_1-\lambda_2}{2}}\ z^{i\frac{\lambda_1^*-\lambda_2^*}{2}} - a_{\lambda_2\lambda_1}a_{\lambda_2^*\lambda_1^*}\, z^{*-i\frac{\lambda_1-\lambda_2}{2}}\ z^{-i\frac{\lambda_1^*-\lambda_2^*}{2}}\right]\ .$$

While the phases of the constant factors in eq.(113) are special functions, the squared modulus is quite simple. Indeed, we find

$$\left|a_{\lambda_1,\lambda_2}\ a_{\lambda_1^*,\lambda_2^*}\ K_{\overrightarrow{\lambda_1},\overrightarrow{\lambda_2}}\right| = \frac{\pi}{\sqrt{2}}\left|\frac{\lambda_2}{\lambda_1(\lambda_1-\lambda_2)}\right|\ .$$

The normalization condition (109) then yields,

$$C_{\overrightarrow{\lambda_1},\overrightarrow{\lambda_2}} \sim \left|\frac{\lambda_1}{\lambda_2}(\lambda_1-\lambda_2)\right| \tag{114}$$

up to a numerical constant.

Using the relation between hypergeometric functions with mutually inversed arguments [20] we obtain for $z \to \infty$

$$\chi_{\lambda_1,\lambda_2}(z^*) \overset{|z|\to\infty}{=} \frac{\Gamma(i(\lambda_1+\lambda_2))\,\Gamma(-i\lambda_2)}{\Gamma(1+i\lambda_1)}\,(z^*)^{i\frac{\lambda_1+\lambda_2}{2}}\ e^{i\pi\lambda_2\frac{Im(z)}{|Im(z)|}} +$$
$$+\frac{\Gamma(-i(\lambda_1+\lambda_2))\Gamma(i\lambda_1)}{\Gamma(1-i\lambda_2)}\,(z^*)^{-i\frac{\lambda_1+\lambda_2}{2}}\ e^{-i\pi\lambda_1\frac{Im(z)}{|Im(z)|}}\ .$$

As a consequence of this asymptotic behavior, the interference terms in $\Phi_{\overrightarrow{\lambda_1},\overrightarrow{\lambda_2}}(\overrightarrow{z})$ cancel using the relation

$$\sinh(\pi\lambda_1)\sinh(\pi\lambda_2^*) = \sinh(\pi\lambda_2)\sinh(\pi\lambda_1^*)\ .$$

We obtain with the use of the equality

$$\sinh(\pi\lambda_1)\sinh(\pi\lambda_1^*) - \sinh(\pi\lambda_2)\sinh(\pi\lambda_2^*) =$$

$$e^{\pi(\lambda_2-\lambda_2^*)}\sinh(\pi(\lambda_1-\lambda_2))\sinh(\pi(\lambda_1+\lambda_2))$$

the following asymptotics for $U_{\overrightarrow{\lambda_1},\overrightarrow{\lambda_2}}(\overrightarrow{z})$ for large $|z|$

$$U_{\overrightarrow{\lambda_1},\overrightarrow{\lambda_2}}(\overrightarrow{z}) = b^{(1)}_{\overrightarrow{\lambda_1},\overrightarrow{\lambda_2}}\,(z^*)^{i\frac{\lambda_1+\lambda_2}{2}}\,z^{i\frac{\lambda_1^*+\lambda_2^*}{2}} + b^{(2)}_{\overrightarrow{\lambda_1},\overrightarrow{\lambda_2}}\,(z^*)^{-i\frac{\lambda_1+\lambda_2}{2}}\,z^{-i\frac{\lambda_1^*+\lambda_2^*}{2}}$$

where

$$b^{(1)}_{\overrightarrow{\lambda_1},\overrightarrow{\lambda_2}} = \frac{\pi\,\Gamma(-i\lambda_1)\Gamma(1-i\lambda_2)\Gamma(i(\lambda_1^*+\lambda_2^*))}{\Gamma(1-i(\lambda_1+\lambda_2))\Gamma(i\lambda_2^*)\Gamma(1+i\lambda_1^*)}\,,$$

$$b^{(2)}_{\overrightarrow{\lambda_1},\overrightarrow{\lambda_2}} = -\frac{\pi\,\Gamma(i\lambda_2^*)\Gamma(1+i\lambda_1^*)\Gamma(-i(\lambda_1+\lambda_2))}{\Gamma(1+i(\lambda_1^*+\lambda_2^*))\Gamma(-i\lambda_1)\Gamma(1-i\lambda_2)}\,.$$

This can also be obtained from the integral (108) by direct calculation.

We analogously find using the series expansion for the hypergeometric function when $c = a+b+1$ [24]

$$\chi_{\lambda_1,\lambda_2}(z^*) = (z^*)^{i\frac{\lambda_1-\lambda_2}{2}}\left\{\frac{1}{\lambda_1\lambda_2} + \frac{1-z^*}{\Gamma(1-i\lambda_2)\Gamma(1+i\lambda_1)}\times\right.$$
$$\sum_{n=0}^{\infty}\frac{\Gamma(n+1-i\lambda_2)\Gamma(n+1+i\lambda_1)}{n!(n+1)!}(1-z^*)^n\times$$
$$\left.[\log(1-z^*)-\psi(n+1)-\psi(n+2)+\psi(n+1-i\lambda_2)+\psi(n+1+i\lambda_1)]\right\}\ .$$

We obtain in the limit $z^*\to 1$:

$$\lim_{z^*\to 1}\chi_{\lambda_1,\lambda_2}(z^*) = \frac{1}{\lambda_1\lambda_2}+$$
$$+(1-z^*)\left[\ln(1-z^*)+\psi(1-i\lambda_2)+\psi(1+i\lambda_1)-\psi(1)-\psi(2)+\frac{i}{2\lambda_1}-\frac{i}{2\lambda_2}\right]$$

in agreement with eq.(24). Thus we obtain,

$$\lim_{z\to 1}\frac{U_{\overrightarrow{\lambda_1},\overrightarrow{\lambda_2}}(\overrightarrow{z})}{K_{\overrightarrow{\lambda_1},\overrightarrow{\lambda_2}}} = -i\pi\frac{\sinh(\pi(\lambda_1-\lambda_2))}{\sinh(\pi\lambda_1)\sinh(\pi\lambda_2)}\left(\frac{1-z}{\lambda_1\lambda_2}+\frac{1-z^*}{\lambda_1^*\lambda_2^*}+|1-z|^2\ln|1-z|^2\right)\ , \qquad (115)$$

where we used the relations

$$\psi(1-i\lambda_2)+\psi(1+i\lambda_1)-\psi(1-i\lambda_1)-\psi(1+i\lambda_2)+\frac{i}{\lambda_1}-\frac{i}{\lambda_2} = -i\pi\frac{\sinh(\pi(\lambda_1-\lambda_2))}{\sinh(\pi\lambda_1)\sinh(\pi\lambda_2)}\,.$$

Again, the result (115) can be obtained directly from the integral representation (108) for $\Phi_{\overrightarrow{\alpha},\overrightarrow{\beta}}(\overrightarrow{z})$. Taking into account the found above value of the normalization constant we have for large momenta p_k and fixed P

$$\frac{U_{\overrightarrow{\lambda_1},\overrightarrow{\lambda_2}}(\overrightarrow{t},\,\overrightarrow{z})}{\pi|\lambda_1\lambda_2(\lambda_1-\lambda_2)|}=e^{i(t\frac{\lambda_1^*+\lambda_2^*}{2}+t^*\frac{\lambda_1+\lambda_2}{2})}\left(\frac{1-z}{\lambda_1\lambda_2}+\frac{1-z^*}{\lambda_1^*\lambda_2^*}+|1-z|^2\ln|1-z|^2\right). \tag{116}$$

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12 Appendix A

We compute in this Appendix the integral in eq.(28).

To start we need the Fourier transform[20],

$$\int\frac{d^2z}{2\pi}\,e^{i\vec{q}\cdot\vec{z}}\,z^m\,(z^*)^{\widetilde{m}}=\frac{i^{\widetilde{m}-m}}{2^{-m-\widetilde{m}-1}}\,q^{-\widetilde{m}-1}\,(q^*)^{-m-1}\,\frac{\Gamma(1+\widetilde{m})}{\Gamma(-m)} \tag{117}$$

Entering the factor $(\overrightarrow{p_1})^2\,(\overrightarrow{p_2})^2$ inside the integral in eq.(28) as $\nabla_1^2\,\nabla_2^2$ we find after partial integration,

$$\Psi_{m,\widetilde{m}}(\overrightarrow{p_1},\,\overrightarrow{p_2})=|m(m-1)|^2\int\frac{d^2z_1}{2\pi}\frac{d^2z_2}{2\pi}\,e^{i(\vec{p}_1\cdot\vec{z}_1+\vec{p}_2\cdot\vec{z}_2)}\,\frac{(z_1-z_2)^{m-2}}{(z_1\,z_2)^m}\frac{(z_1^*-z_2^*)^{\widetilde{m}-2}}{(z_1^*\,z_2^*)^{\widetilde{m}}} \tag{118}$$

Now, we replace the z-factors in the integrand by the integral representation (117):

$$z^{-m}\,(z^*)^{-\widetilde{m}}=\frac{i^{m-\widetilde{m}}}{2^{m+\widetilde{m}-1}}\frac{\Gamma(1-m)}{\Gamma(\widetilde{m})}\int\frac{d^2k}{2\pi}\,e^{i\vec{q}\cdot\vec{k}}\,(k^*)^{m-1}\,k^{\widetilde{m}-1}\,.$$

The z-integrals in eq.(118) give now Dirac delta functions and we obtain eq.(29).

13 Appendix B

We derive here the asymptotic behavior of the Baxter function for the Pomeron starting from the integral representation (39). We change the integration variable to

$$y\equiv 2\text{ArgTanh}\,(1-2p)\quad,\quad p=\frac{1}{2}\left(1-\tanh\frac{y}{2}\right)$$

and obtain,

$$Q(\lambda,m) = i\,\frac{\pi\,\sinh(\pi\lambda)}{\sin(\pi\,m)}\int_{-\infty}^{+\infty} dy\; e^{i\lambda\,y}\;P_{m-1}\left(\tanh\frac{y}{2}\right)$$

The function $P_{m-1}(z)$ has a cut running from $z=-1$ till $z=-\infty$. Therefore, $P_{m-1}\left(\tanh\frac{y}{2}\right)$ has cuts in the y-plane from $y=i(2n+1)\,\pi$ till $y=-\infty$ where n is an integer. We now deform the integration path around the cut from $y=i\,\pi$ till $y=-\infty$ and we find,

$$Q(\lambda,m) = i\,\frac{\pi\,\sinh(\pi\lambda)}{\sin(\pi\,m)}\,e^{-\pi\,\lambda}\int_0^{+\infty} dx\; e^{i\lambda\,x}\left[P_{m-1}\left(-\coth\frac{x}{2}+i0\right)-P_{m-1}\left(-\coth\frac{x}{2}-i0\right)\right] \tag{119}$$

where we changed the integration variable as $y=i\,\pi-x$. The integral (119) is dominated for large λ by the end-point $x=0$. Therefore, we insert in eq.(119) the representation of $P_{m-1}(z)$ appropriate for large $z=-\coth\frac{x}{2}\pm i0$[20]

$$\begin{aligned} P_{m-1}(z) &= \frac{\tan\pi m\;\Gamma(m)}{2^m\,\sqrt{\pi}\,\Gamma\left(m+\frac12\right)}\;z^{-m-2}\;{}_2F_1\left(\frac{m+1}{2},\frac{m}{2};m+\frac12;\frac{1}{z^2}\right)+\\ &+\;\frac{2^{m-1}\,\Gamma\left(m-\frac12\right)}{\sqrt{\pi}\,\Gamma\left(m\right)}\;z^{m-1}\;{}_2F_1\left(\frac{1-m}{2},1-\frac{m}{2};-m+\frac32;\frac{1}{z^2}\right) \end{aligned} \tag{120}$$

Keeping here the dominant terms for large z and using the relation

$$\left(-\coth\frac{x}{2}+i0\right)^{-m}-\left(-\coth\frac{x}{2}-i0\right)^{-m} = -2i\,\sin\pi m\left(\coth\frac{x}{2}\right)^{-m}\quad,\quad x>0\quad,$$

we get for $\lambda\gg 1$ (for Re $m>1/2$)

$$\begin{aligned} Q(\lambda,m) &= 4\,\sqrt{\pi}\;(4\,i\,\lambda)^{m-2}\;\frac{\Gamma\left(m-\frac12\right)\;\Gamma\left(2-m\right)}{\Gamma\left(m\right)}\left\{1+\mathcal{O}\left(\frac{1}{\lambda^2}\right)+\right.\\ &+\left.\;(4\,i\,\lambda)^{1-2m}\;\tan\pi m\;\frac{\Gamma^2\left(m\right)\;\Gamma\left(m+1\right)}{\Gamma\left(m+\frac12\right)\;\Gamma\left(m-\frac12\right)\;\Gamma\left(2-m\right)}\left[1+\mathcal{O}\left(\frac{1}{\lambda^2}\right)\right]\right\}. \end{aligned} \tag{121}$$

For Re $m<1/2$ one should just replace $m\Rightarrow 1-m$.

The case $m=1/2$ follows by taking the limit $m\to 1/2$ in eq.(121) with the result for $\lambda\gg 1$

$$Q(\lambda,\frac12) = \frac{\sqrt{\pi}}{2\;(i\;\lambda)^{3/2}}\left\{\left[\log\left(\frac{i\,\lambda}{4}\right)+2-3\gamma\right]+\mathcal{O}\left(\frac{1}{\lambda^2}\right)\right\}.$$

Let us now derive the asymptotic behavior of $Q(\lambda,m)$ starting from their infinite product representation (59) and the asymptotic distribution of their zeros (58).

For $\lambda\gg 1$ the product will be dominated by zeros of the order $\sim\lambda$. We can then write,

$$\prod_{k=1}^{\infty}\left(1-\frac{\lambda}{\lambda_k}\right)e^{\frac{\lambda}{\lambda_k}} \simeq \prod_{k=1}^{M}\left[\left(\frac{1-\frac{\lambda}{\lambda_k}}{1+i\frac{\lambda}{k+1-m}}\right)e^{\frac{\lambda}{\lambda_k}+\frac{i\lambda}{k+1-m}}\right]\prod_{k=1}^{\infty}\left(1+\frac{i\lambda}{k+1-m}\right)e^{-\frac{i\lambda}{k+1-m}}$$

where M is a cutoff $1 \ll M \ll |\lambda|$. We obtain for $\lambda \gg 1$ using the formulae [20]

$$\prod_{k=1}^{\infty} \left(1 + \frac{iy}{k+x}\right) e^{-\frac{iy}{k}} = e^{-i\gamma\, y} \frac{\Gamma(1+x)}{\Gamma(1+x+iy)}$$

$$\psi(x+1) + \gamma = \sum_{k=1}^{\infty} \frac{x}{k(x+k)} \; ,$$

$$Q(\lambda, m) = \text{constant } \lambda^{m-2} \tag{122}$$

in perfect agreement with eq.(121).

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