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Workshop on Low x Physics, Kraków

“Solving the BFKL equation with running coupling”

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1. Freezing the coupling in the infrared
2. Eigenfunctions
3. Solution in a simple case
4. Future work

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BFKL cross sections

$$\frac{d\sigma_{AB}(s)}{d\mathbf{k}_1 d\mathbf{k}_2} = \int \frac{d\omega}{2\pi i}\left(\frac{s}{s_0}\right)^{\omega} \phi_A(\mathbf{k}_1)\underline{\mathcal{G}_\omega(\mathbf{k}_1,\mathbf{k}_2)}\phi_B(\mathbf{k}_2)$$

With the variable $t \equiv \ln\left(\frac{k^2}{\Lambda^2_{QCD}}\right)$ BFKL is

$$\omega\, \mathcal{G}_\omega(t_1,t_2) = \delta(t_1-t_2)+\int_{-\infty}^{+\infty} dt\, \underline{\mathcal{K}(t_1,t)}\, \mathcal{G}_\omega(t,t_2)$$

Our choice to run the coupling is

$$\mathcal{K}(t,t') = \alpha_s(\max\{t,t'\})\mathrm{K}(t,t')$$

We make it constant below a scale t_0

$$\alpha_s(t) = \underbrace{\frac{1}{b\,t_0}\theta(t_0-t)}_{IR\ \text{fixed}} + \underbrace{\frac{1}{b\,t}\theta(t-t_0)}_{UV\ \text{runs}}$$

with $b \sim \beta_0$

The kernel $\mathcal{K}(t,t')$ is now

$$\left[\frac{1}{bt_0}\mathcal{K}^{<}(t,t')\theta(t_0-t')+\frac{1}{bt}\mathcal{K}^{<}(t,t')\theta(t'-t_0)\right]\theta(t-t')$$

$$+\left[\frac{1}{bt_0}\mathcal{K}^{>}(t,t')\theta(t_0-t')+\frac{1}{bt'}\mathcal{K}^{>}(t,t')\theta(t'-t_0)\right]\theta(t'-t)$$

At **Leading Order**

$$\begin{aligned}\mathcal{K}^{<}(t,t') &= \lim_{N\to\infty}\mathcal{K}_N^{<}(t,t')\\ &= \lim_{N\to\infty}\sum_{r=0}^{N}\left\{e^{(r+1/2)(t'-t)}-\frac{\delta(t-t')}{r+1}\right\}\end{aligned}$$

$$\begin{aligned}\mathcal{K}^{>}(t,t') &= \lim_{N\to\infty}\mathcal{K}_N^{>}(t,t')\\ &= \lim_{N\to\infty}\sum_{r=0}^{N}\left\{e^{(r+1/2)(t-t')}-\frac{\delta(t-t')}{r+1}\right\}\end{aligned}$$

$$\begin{aligned}
\int_{-\infty}^{t} \mathcal{K}_N^{<}(t,t')e^{\gamma t'}dt' &= \chi_N^{<}(\gamma)e^{\gamma t} \\
&= \sum_{r=0}^{N} \left\{ \frac{1}{r+1/2+\gamma} - \frac{1}{r+1} \right\} e^{\gamma t}
\end{aligned}$$

$$\begin{aligned}
\int_{t}^{\infty} \mathcal{K}_N^{>}(t,t')e^{\gamma t'}dt' &= \chi_N^{>}(\gamma)e^{\gamma t} \\
&= \sum_{r=0}^{N} \left\{ \frac{1}{r+1/2-\gamma} - \frac{1}{r+1} \right\} e^{\gamma t}
\end{aligned}$$

The **LO** kernel is recovered for

$$\begin{aligned}
\chi_{\infty}(\gamma) &= \chi_{\infty}^{<}(\gamma) + \chi_{\infty}^{>}(\gamma) \\
&= 2\gamma_{\mathrm{E}} - \psi\left(\frac{1}{2}+\gamma\right) - \psi\left(\frac{1}{2}-\gamma\right)
\end{aligned}$$

The gluon Green's function

$$\mathcal{G}_\omega(t_1, t_2) = \int_{\mathcal{C}} d\mu \frac{f_\mu(t_1) f_\mu^*(t_2)}{\omega - \mu}$$

In terms of the eigenfunctions $f_\mu(t)$

$$\boxed{\int_{-\infty}^{+\infty} dt' \ \mathcal{K}(t, t') \ f_\mu(t') = \mu \ f_\mu(t)}$$

$f_\mu^*(t)$ can be found using

Completeness : $\int_{\mathcal{C}} d\mu \ f_\mu(t) \ f_\mu^*(t') = \delta(t - t')$

Normalization : $\int_{-\infty}^{+\infty} dt \ f_\mu(t) \ f_\nu^*(t) = \delta_{\mu\nu}$

We will focus on the eigenfunctions $f_\mu(t)$

IR/UV components of the eigenfunctions

$$f_\mu(t) = \underbrace{f_\mu^<(t)\,\theta(t_0 - t)}_{IR} + \underbrace{f_\mu^>(t)\,\theta(t - t_0)}_{UV}$$

We get two 'coupled' integral equations

$$\begin{aligned} bt_0\omega f_\omega^<(t) &= \int_{-\infty}^{t} \mathcal{K}_N^<(t,t') f_\omega^<(t') dt' \\ &+ \int_t^{t_0} \mathcal{K}_N^>(t,t') f_\omega^<(t') dt' + \int_{t_0}^{\infty} \mathcal{K}_N^>(t,t') \frac{t_0}{t'} \underline{f_\omega^>(t')} dt' \end{aligned}$$

$$\begin{aligned} bt\omega f_\omega^>(t) &= \int_{t_0}^{t} \mathcal{K}_N^<(t,t') f_\omega^>(t') dt' \\ &+ \int_t^{\infty} \mathcal{K}_N^>(t,t') \frac{t}{t'} f_\omega^>(t') dt' + \int_{-\infty}^{t_0} \mathcal{K}_N^<(t,t') \underline{f_\omega^<(t')} dt' \end{aligned}$$

Decoupling IR/UV components

Applying

$$\mathcal{O}_<^N(t) = \frac{d}{dt}\left(\frac{d}{dt} - 1\right)\cdots\left(\frac{d}{dt} - N\right) e^{-t/2}$$

$$\mathcal{O}_>^N(t) = \frac{d}{dt}\left(\frac{d}{dt} + 1\right)\cdots\left(\frac{d}{dt} + N\right) e^{t/2}$$

Get homogeneous integro-differential equations.

Solutions will have <u>arbitrary constants</u>.
To fix them:

1. Original integral equations must be obeyed.
2. $f_\omega^<(t)$ must be square integrable as $t \to -\infty$.
3. $f_\omega^>(t)$ must be square integrable as $t \to \infty$.

IR component of the eigenfunction

$$f_\omega^<(t) \;=\; \sum_{j=1}^{N+2} a_j e^{\gamma_j^0\, t}$$

γ_j^0 is implicitly given by

$$b\,\omega\,t_0 \;=\; \chi_N^<(\gamma_j^0) + \chi_N^>(\gamma_j^0) \;\equiv\; \chi_N(\gamma_j^0)$$

$\Re e(\gamma_j^0) \geq 0$ ensures $t \to -\infty$ square integrability

There are $\underline{N+2}$ arbitrary constants.

UV component of the eigenfunction

$$f_\omega^>(t) \;=\; \frac{t}{2\pi i}\int_{\mathcal{C}} d\gamma\, \tilde{f}_\omega(\gamma) e^{\gamma t}$$

$$\omega\, b\, \frac{\partial^2}{\partial\gamma^2}\tilde{f}_\omega(\gamma) + \chi_N \frac{\partial}{\partial\gamma}\tilde{f}_\omega(\gamma) + \chi_N^{>\prime}\tilde{f}_\omega(\gamma) \;=\; 0$$

WKB approximation

$$\tilde{f}_\omega(\gamma) = \frac{\exp\left\{-\frac{1}{2\omega b}\int^\gamma d\gamma' \left[\chi_N(\gamma') + \sqrt{V(\gamma',\omega)}\right]\right\}}{V(\gamma,\omega)^{1/4}}$$

with $\qquad V(\gamma,\omega) \;=\; \chi_N^2(\gamma) - 4\,\omega\, b\, \chi_N^{>\prime}(\gamma)$

$\tilde{f}_\omega(\gamma)$ has branch points at

$$\gamma = -\frac{1}{2} - N, \ldots, -\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2} + N$$

UV component, the γ-contours

Choose cuts with $\tilde{f}_\omega(\gamma)$ analytic in

$$\gamma \in \left[n - \frac{1}{2}, n + \frac{1}{2}\right] \text{ with } \mathsf{n} = -\mathsf{N}, \ldots, \mathsf{N}$$

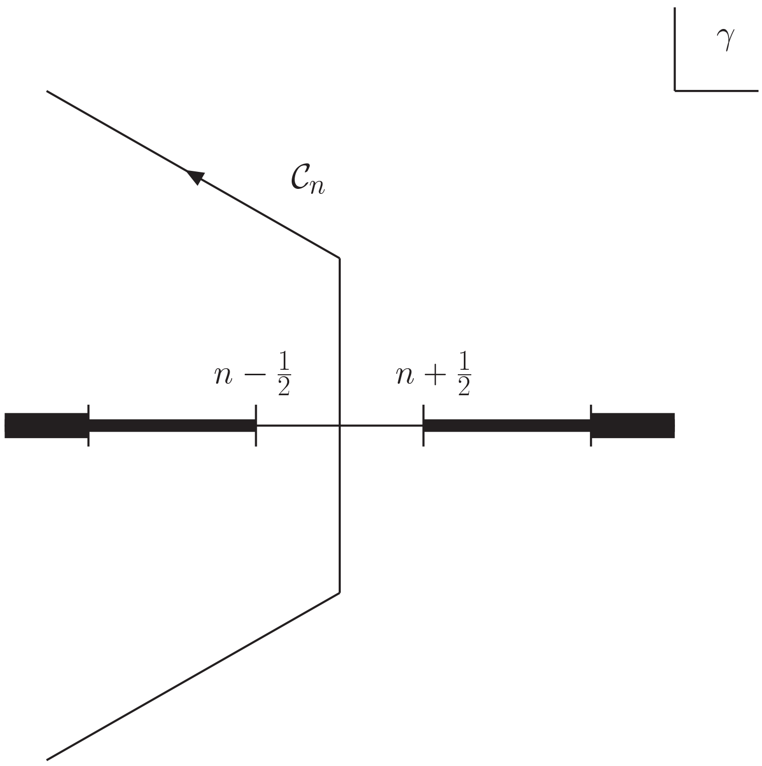
and make $\mathcal{C}$ cross this region.

Each of these choices for $\mathcal{C}$ is a solution.

$n = -N, \ldots, 0$ for $t \to \infty$ square integrability.

$$f_\mu^{>}(t) \;=\; \sum_{n=-N}^{0} b_n\, t \int_{\mathcal{C}_n} \frac{d\gamma}{V(\gamma,\mu)^{1/4}}$$
$$\exp\left\{\gamma t - \frac{1}{2\mu b}\int_0^\gamma d\gamma' \left(\chi_N(\gamma') + \sqrt{V(\gamma',\mu)}\right)\right\}$$

There are N+1 arbitrary constants.


γ
$\mathcal{C}_n$
$n - \frac{1}{2}$
$n + \frac{1}{2}$

Fixing the constants

$2N+3$ coefficients, a_j and b_n.

$f_\mu^<(t)$, $f_\mu^>(t)$ back to 'coupled' equations.

These equations are obeyed if for $r = 0, \cdots, N$

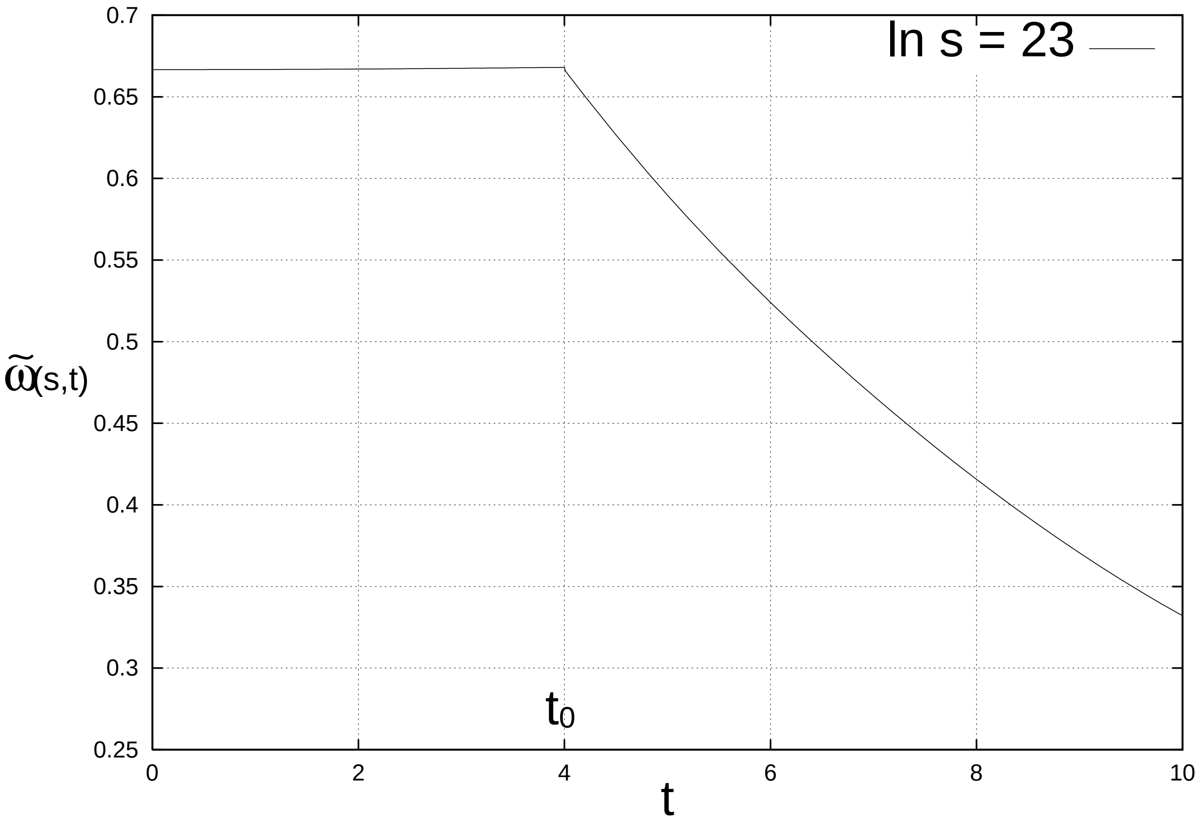
$$\sum_{j=1}^{N+2} a_j \frac{e^{\gamma_j^0 t_0}}{\left(\gamma_j^0 - r - \frac{1}{2}\right)} = t_0 \sum_{n=-N}^{0} b_n \int_{\mathcal{C}_n} d\gamma \frac{\tilde{f}_\omega(\gamma) e^{\gamma t_0}}{\left(\gamma - r - \frac{1}{2}\right)}$$

$$\sum_{j=1}^{N+2} a_j \frac{e^{\gamma_j^0 t_0}}{\left(\gamma_j^0 + r + \frac{1}{2}\right)} = -\sum_{n=-N}^{0} b_n \int_{\mathcal{C}_n} d\gamma \frac{\tilde{f}'_\omega(\gamma) e^{\gamma t_0}}{\left(\gamma + r + \frac{1}{2}\right)}$$

These give us $2N+2$ conditions.
The extra one comes from normalization.

Asymptotics for N=0

Extremely large s and $\exp(-t_0) \ll 1$
s-dependence of cross section is $\sim s^{\tilde{\omega}(s,t)}$ with



IR power-like behaviour becomes softer in UV.

Future work

- This is a formal solution at **LO**
- Exact analysis for any N?
- Valid for **NLO corrections**?

Idea: Bulk of higher order corrections resummed with

$$f_\omega^<(t) = \sum_{j=1}^{N+2} a_j e^{\gamma_j^0 t}$$

$$\text{with} \quad b\,\omega\,t_0 = \chi_N^<\left(\gamma_j^0 - \frac{\omega}{2}\right) + \chi_N^>\left(\gamma_j^0 + \frac{\omega}{2}\right)$$

$$f_\omega^>(t) = \sum_{n=-N}^{0} b_n\, t \int_{\mathcal{C}_n} \frac{d\gamma}{V(\gamma,\omega)^{1/4}}$$

$$\exp\left\{\gamma t - \frac{1}{2\omega b}\int_0^\gamma d\gamma' \left[\chi_N^<\left(\gamma' - \frac{\omega}{2}\right) + \chi_N^>\left(\gamma' + \frac{\omega}{2}\right) + \sqrt{V(\gamma',\omega)}\right]\right\}$$

$$\text{with} \quad V(\gamma,\omega) = \left[\chi_N^<\left(\gamma - \frac{\omega}{2}\right) + \chi_N^>\left(\gamma + \frac{\omega}{2}\right)\right]^2 - 4\omega b \chi_N^{>\,\prime}\left(\gamma + \frac{\omega}{2}\right)$$